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Endogenous uncertainty and market volatility*

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Summary. We advance the theory that the distribution of beliefs in the market is *the most important propagation mechanism of economic* volatility. Our model is based on the theory of Rational Beliefs (RB) and Rational Belief Equilibrium (RBE) developed by Kurz (1994, 1997). We argue that the diverse market puzzles which are examined, such as the equity premium puzzle, are all driven by the structure of market expectations. In support of our view, we present an RBE model with which we study financial markets. The model is able to simulate the correct order of magnitude of: (i) the long term mean and standard deviation of the price\dividend ratio; (ii) the long term mean and standard deviation of the risky rate of return on equities; (iii) the long term mean and standard deviation of the riskless rate; (iv) the long term mean equity premium. In addition, the model predicts (v) the GARCH property of risky asset returns; (vi) the observed pattern of the predictability of long returns on assets, and (vii) the Forward Discount Bias in foreign exchange markets.

The common economic explanation for these phenomena is the existence of heterogenous agents with diverse but correlated beliefs such that some agents are optimistic and some pessimistic about future capital gains. The model has a unique parameterization under which the model makes all the above predictions *simultaneously*. The parameterization requires the optimists to be in the majority but the rationality of belief conditions of the RBE require the pessimists to have a higher intensity level. In simple terms, the large equity premium and the low equilibrium riskless rate are the result of the fact that at any moment of time

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there are agents who hold extreme pessimistic beliefs and they have a relatively stronger impact on the market. The paper also studies the effect of correlation of beliefs among investors. It shows that the main effect of such correlation is on the dynamic patterns of asset prices and returns and is hence important for studying such phenomena as stochastic volatility.

Keywords and Phrases: Market volatility, Diversity of beliefs, Stable dynamical system, Empirical distribution, Stationary measure, Rational Belief, Rational Belief Equilibrium, State of beliefs, Regime variables, Assessment regime variables, Equity premium puzzle, Market optimism and pessimism, Garch, Forward discount bias in foreign exchange markets.

JEL Classification Numbers: D5, D84, G12.

Introduction

The theory of Rational Belief Equilibrium (in short, RBE; see Kurz 1994, 1997) was developed with the view of studying the effects of the beliefs of economic agents on the volatility of economic variables and on social risk. Application of the theory to various markets were reported by Kurz and Beltratti (1997), Kurz and Schneider (1996), Kurz (1997a,b), (1998), Nielsen (1997), Wu and Guo (1998) and ET Symposium, Kurz (1996). Some of these papers advanced the idea that the "equity premium puzzle" due to Mehra and Prescott (1985) (in short, M&P 1985), can be resolved by the theory of Rational Beliefs (in short, RB). This is in contrast with recent attempts to resolve the equity premium puzzle by the use of a "habit forming" utility function (see Abel, 1999; Campbell and Cochrane (1999); Constantinides, 1990).[1]

Most of the work on the equity premium concentrated on the analysis of the premium as an isolated phenomenon and in this context researchers usually examine the riskless rate, the risky rate and their second moments. The fact is that there are other volatility phenomena which have puzzled students of financial markets. It appears to us that participants in the equity premium debate have not stressed sufficiently that the question of excess stock price volatility raised by Shiller (1981) is intimately related to the equity premium puzzle. Also, standard models have not clarified the nature of other dynamical patterns exhibited in financial markets such as the GARCH phenomenon in asset returns, the "Forward Discount Bias" in foreign exchange markets and the various "smile curves" in derivative asset pricing. It is clear that the validity of any equilibrium theory should not be judged by its ability to match any specific market statistic but rather by the range and depth of market phenomena and "anomalies" that the theory is capable of explaining.

[1] Other approaches to the equity premium puzzle were reported by Brennan and Xia (1998), Epstein and Zin (1990), Cecchetti, Lam and Mark (1990, 1993), Heaton and Lucas (1996), Mankiw (1986), Reitz (1988), Weil (1989) and many others. For more details see Kocherlakota (1996).

This paper is not another study of the equity premium. In scope it is broader than previous papers on RBE and it presents a *unified* framework for the study of market volatility. It argues that *the distribution of beliefs is the central volatility propagation mechanism in the market*. It thus claims that most volatility in financial markets is expectationally generated and that many market "anomalies" such as the excess volatility of asset prices and foreign exchange rates, the equity premium puzzle, the GARCH pattern of asset returns and the Forward Discount Bias in foreign exchange markets *are all driven by the structure of heterogenous beliefs in the market.* In support of this unified view we present a single, relatively simple, market model and show by simulations that the RBE of the model is able to explain a wide range of phenomena.

First, it predicts the correct order of magnitude of (i) the first and second moments of the price/dividend ratio, (ii) the first and second moments of the risky return, (iii) the first and second moments of the riskless rate and hence of the equity premium. Second, the time series of stock returns exhibit a GARCH phenomenon and are consistent with the observed pattern of the predictability of long returns. Third, an extension of the model to a two countries model exhibits a high volatility of the foreign exchange rate and a "forward discount bias" in its foreign exchange market. Technically speaking our model drastically generalizes the approaches of Kurz and Beltratti (1997), Kurz and Schneider (1996) and Kurz (1997b) with a simplified parameter space which satisfies anonymity in accord with Kurz (1998). In addition, the paper provides an integrated economic interpretations of the results which are presented in Section 3. We start by discussing the merits of the heterogenous belief paradigm.

1 A paradigm of heterogenous beliefs

The theory of RBE is motivated by the observation that intelligent economic agents hold diverse beliefs even when there is no difference in the information at their disposal. Indeed, the center of their disagreement is the diverse *interpretations* of this information. By adopting axioms which allow rational agents to hold diverse beliefs, our theory does not lead, in general, to a Rational Expectations Equilibrium (in short, REE). However, an REE is also an RBE since the theory of RBE is an extension of the theory of REE.

The search for an extension of the theory of REE is motivated by the widespread dissatisfaction with the REE model (see Sargent, 1993). This results from the fact that central implications of the REE theory are contradicted by the empirical evidence. One recent line of research has focused on alternative choice criteria such as Robustness [see Anderson, Hansen and Sargent (1999) or Hansen, Sargent and Tallarini (1999)] even if such criteria imply behavior that is viewed as irrational by the REE or the RBE theories. Instead of adopting such a "Bounded Rationality" approach, we follow Kurz (1974, 1994, 1998), the papers included in Kurz (1996, 1997), Garmaise (1998), Motolese (1998), Nielsen (1997) and Wu and Guo (1998) in studying the heterogeneity of beliefs as the key propagation mechanism of market volatility. We note the existence of ample empirical evidence showing that equally informed agents interpret differently the same information (see Frankel and Froot, 1990; Frankel and Rose, 1995; Kandel and Pearson, 1995; Takagi, 1991 and others). Moreover, the heterogeneity of beliefs persists regardless of the amount of past information available implying that agents use different probability beliefs which they condition on *the same public information*. We thus discuss the common view that the observed diversity of beliefs is caused by heterogeneity of information.

1.1 Diversity of information or diversity of beliefs?

Starting with financial markets, there is a significant REE based literature which holds that the observed heterogeneity of beliefs does not arise from the heterogeneity of prior probabilities but, rather, from the diversity of private information (see, for example, Kyle, 1985; Wang, 1993, 1994 and references there). This explanation is unsatisfactory from both theoretical as well as empirical perspectives. Theoretical considerations lead to the information revelation of REE (e.g. Grossman, 1981; Radner, 1979) which implies that prices make public all private information and therefore the introduction of asymmetric information, by itself, is not sufficient. It simply transforms the problem into other paradoxes. These include the problem of explaining why under REE agents trade at all (e.g. Milgrom and Stokey, 1982); why asset prices fluctuate more than could be explained by “fundamentals” (e.g. Shiller, 1981), indirectly generating an equity premium puzzle (see M&P 1985); and why any resources are ever used for the production of information (see Grossman and Stiglitz, 1980). To explain the observed heterogeneity and avoid such paradoxes researchers had, therefore, to introduce some additional assumptions of market structure to remove the information revelation of REE. An example is the explicit introduction of uninformed noise traders or general “noise” which leads to a theory of “Noisy REE.” In our view the use of “noise” is an unsatisfactory approach and stands in contrast with the discipline of REE. It proposes to solve the problem of market volatility by introducing a central component which is not deduced from first principles and lacks a formal structure. Indeed, under some interpretations noise traders are simply irrational.

Empirical considerations also suggest that the assumption of asymmetric information in financial markets is unsatisfactory. Recall the ample empirical evidence supporting the view that equally informed agents interpret differently the same information. This leads to a direct question: is there an empirical evidence to support the assumption of widespread use of private information in financial markets? We think that the evidence is not there. Observe first that it is illegal to trade on inside private information. The majority of firms whose securities are traded on public exchanges are monitored by a professional community of regulators, brokers and financial managers. The evidence suggests that, on the whole, the majority of firms avoid letting market participants either obtain private information or trade on it when they have it. Moreover, since financial markets are dominated by large institutions, competitive behavior in the search for information would lead us to conclude that all will possess essentially the same information.

As a matter of history of economic thought, REE was introduced into macroeconomics as part of the critique of the Keynesian theory. However, under the classical assumptions of price and wage flexibility, an REE cannot explain observed cyclical correlations such as the positive correlation between the price level and aggregate output (the "inflation - output tradeoff"). To explain the data, the "New Classical Theory" introduced asymmetric information which became the driving force of the theory. More specifically, agents were assumed to be *unable to obtain information which is public in other parts of the economy*. This rigidity in the transmission of information lead to diverse models of Phelpsian or Lucasian "islands" (see Phelps, 1970; Lucas, 1973). The Lucas supply curve (Lucas, 1973) is deduced from the assumption that firms cannot observe the aggregate price level, which is normally an observable variable. The heterogeneity of beliefs is then caused by very strong informational assumptions. The decline in practice of this approach occurred, in part, due to the unsatisfactory informational assumptions.

We suggest that the implications of the common belief assumption in an REE – by itself – are counter factual. The crucial implications of these models are generated by *added assumptions* such as rigidity in transmission of information or irrational behavior of agents. These added components of the analysis introduce unrealistic assumptions which often drive the results.

The theory of RBE provides the foundation for the use of heterogenous beliefs *as a substitute* for the "additional" assumptions driving much of the REE based models. The RBE theory shows that the paradigm of diverse beliefs provides a powerful propagation mechanism of market fluctuations. It offers insights which have proved useful in the study of several important problems (e.g. see the papers in Kurz, 1997). We also note that up to the 1970's heterogenous beliefs played a central role in economic thought about the workings of financial markets. Diverse expectations are used often in Keynes' writings but the decline of the Keynesian system resulted from Keynes' failure to develop a formal theory of expectations. Rational expectations thus provided macroeconomics with a useful, missing, discipline. We believe, however, that something central was lost due to this particular discipline. The theory of RBE aims to restore the balance by developing a rigorous formal paradigm of market volatility based on diverse expectations.

1.2 Rational beliefs

The central assumption of the RB theory (due to Kurz, 1994) is that economic agents do not know the exact demand or supply functions, equilibrium map or true equilibrium probability laws. In Kurz's (1994) terminology, they do not possess "*structural knowledge*" (as distinct from "empirical knowledge" or "information"). Hence, in contrast to the standard REE view, agents do not deduce the probability of endogenous variables from the equilibrium map and the knowledge of the probability of exogenous variables. Lacking such knowledge, rational agents develop their own theories of the economy and use the available data to test the validity of these theories.

The second assumption which distinguishes the RB theory from a Bayesian perspective is that at each date t an economic agent has a vast amount of data about the past performance of the economy. Hence, instead of accepting Savage's (1954) axioms on preferences which imply an arbitrary prior belief at date 0, the agent's central point of reference at date t is the empirical distribution derived from the frequency at which events occurred in the past. The availability of a large amount of past data may lead one to speculate that updating by the agents may cause heterogeneity of beliefs to vanish. Work along this line lead to a debate under the heading of "Bayes Consistency" (see Diaconis and Freedman, 1986). The conclusion of the debate is that the convergence of the posterior to the true distribution is a rare occurrence. In two influential papers, Freedman (1963, 1965) shows that even when the statistician has a controlled experiment and the data is generated i.i.d., the convergence of the posterior is a rare event if the true distribution is complex. The problem is compounded in learning situations in markets where the data is generated by an unknown process which may be non-stationary and the convergence of the posterior to the true probability is even a less likely event (see Feldman, 1991).

The third component of the RB theory is the observation that the economic life of any agent is short relative to the clock at which new data arrives. Thus, let $x_t \in X \subseteq \mathbb{R}^N$ be a vector of the N observables in the economy and let $x = (x_0, x_1, x_2, \ldots)$ be the random data from 0 to infinity. Define the history from date t on by $x^t = (x_t, x_{t+1}, x_{t+2}, \ldots)$, hence $x^0 = x$. The history up to date t is defined by $I_t = (x_0, x_1, x_2, \ldots, x_t)$. To solve a dynamic programming problem an agent needs to form a belief at t about probabilities of events in the future. The theory assumes that t is very large so that an agent can construct the empirical distribution generated by the history. The agent's life L is the time span in which he makes decisions and L is very short relative to t. By this we mean that investors, fund managers, CEO of a corporations etc. make decisions over periods ranging from 10 to 20 years which is very short relative to t. An agent's belief may be correct or not but the little economic data – $(x_t, x_{t+1}, x_{t+2}, \ldots x_{t+L})$ – generated during his own economic life is too small to provide a reliable test of his theory since most economic data flow at very slow annual or quarterly rates. This is particularly true if the agent believes that the data is generated by a non-stationary process and he has little data on each regime which may be in place during an interval of time. One must then conclude that the rationality of a belief Q cannot be judged by the usual Bayesian learning criterion which insists on the compatibility of Q with the limit of the data in the *future*. Instead, the RB theory defines the rationality of belief in terms of its compatibility with the empirical distribution of *past data*.

To explain the rationality conditions of the RB theory we start with the definition of *Statistical Stability*. Let X^∞ be the space of infinite sequences x

and $\mathscr{B}(X^\infty)$ be the Borel σ-field of X^∞. For each finite dimensional set (cylinder set) $B \in \mathscr{B}(X^\infty)$ define the expression

$$m_n(B)(x) = \frac{1}{n}\sum_{k=0}^{n-1} 1_B(x^k) = \left\{ \begin{array}{l} \text{The relative frequency that } B \text{ occurred} \\ \text{among } n \text{ observations since date } 0 \end{array} \right\}$$

where

$$1_B(y) = \begin{cases} 1 & \text{if} \quad y \in B \\ 0 & \text{if} \quad y \notin B. \end{cases}$$

Although the set B is finite dimensional, it can be a very complicated intertemporal event.

Definition 1. *(Property 1) A stochastic process* $\{x_t, t = 0, 1, 2, ...\}$ *with true probability* Π *on* $(X^\infty, \mathscr{B}(X^\infty))$ *is said to be* Statistically Stable *if for each finite dimensional set* $B \in \mathscr{B}(X^\infty)$

$$\lim_{n\to\infty} m_n(B)(x) = \mathring{m}(B)(x) \quad \textit{exists } \Pi \text{ a.e.}$$

We assume that the data is generated by a stable process but for simplicity we assume that the process is *Ergodic*. To define this concept consider an event $B \in \mathscr{B}(X^\infty)$. Let $B = B^{(0)}$ and

$$B^{(k)} = \ \text{the event } B \text{ occurring } k \text{ periods later } \equiv \{x | x^k \in B\}.$$

A set B is said to be *invariant* if $B^{(k)} = B$ for all k. A process with probability Π is said to *ergodic* if $\Pi(B) = 0$ or 1 for each invariant set. The assumption of ergodicity implies that

$$\mathring{m}(B)(x) = \mathring{m}(B) \ \text{ independent of } x, \quad \Pi \text{ a.e.}$$

Agents do not know Π and start by computing the empirical frequencies. Although they have only finite data, we assume they actually know the limits $\mathring{m}(B)$ in Definition 1 for all cylinders. Again, this assumption is made for simplification.[2] It can be shown that the agents deduce from the data $\mathring{m}$ a full probability measure m on the space $(X^\infty, \mathscr{B}(X^\infty))$. Indeed, we know (see Kurz, 1994) that (i) m is unique; (ii) m is stationary and hence is called *"the stationary measure of* Π*."* Since m is obtained from the data, there is no disagreement among the agents about it. The probability m is their *common* empirical knowledge.

Agents who do not know the true probability Π discover from the data the probability m induced by Π. If an economy is stationary, $m = \Pi$ but agents could not know this fact. What are the restrictions which the knowledge of m places on the beliefs of rational agents? To explore this we introduce the concept of Weak Asymptotic Mean Stationary (WAMS) Dynamical System.

[2] The assumption that the limit in Definition 1 is known to the agents is made to avoid the complications of an approximation theory. Without this assumption the diversity of beliefs would be increased due to the diverse opinions about the approximation. The assumption of Ergodicity is also not needed and is not made in Kurz (1994).

Definition 2. *(Property 2) A system* $\{x_t, t = 1, 2, 3, \dots\}$ *with probability* Π *on* $(X^\infty, \mathscr{B}(X^\infty))$ *is said to be WAMS if for each finite dimensional (cylinder) event* $B \in \mathscr{B}(X^\infty)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(B^{(k)}) = \mathring{m}^{\Pi}(B) \quad \textit{exists.}$$

The collection of $\mathring{m}^{\Pi}(B)$ induces a unique probability m^{Π} on $(X^\infty, \mathscr{B}(X^\infty))$ which is stationary. For *any* WAMS probability Q, we shall then use the notation m^Q to denote probability on $(X^\infty, \mathscr{B}(X^\infty))$ induced by Property 2 which Q must satisfy. The central result of the RB theory can now be stated (see, Kurz, 1994, Proposition 2):

Theorem 1. *Properties 1 and 2 are equivalent and* $m(A) = m^{\Pi}(A)$ *for all events* $A \in \mathscr{B}(X^\infty)$.

Agents compute m from the data and Theorem 1 leads to a natural definition of what it means for a probability belief Q to be "compatible with the data", which m represents:

Definition 3. *A probability belief* Q *is said to be* compatible with the observed data m *if*

(i) Q is a WAMS probability on $(X^\infty, \mathscr{B}(X^\infty))$,
(ii) $m^Q(A) = m(A)$ for all events $A \in \mathscr{B}(X^\infty)$.

Equality (ii) is the key implication of Theorem 1. Now consider a rational agent with a stable belief Q. If $m^Q \neq m$, it would prove that Q is not the truth. Indeed, it would prove that the Q is not compatible with the data m. This leads to

Definition 4. *A probability belief* Q *is said to be* a Rational Belief (RB) relative to m *if* Q *is compatible with the known data* m.

An agent who holds a rational belief Q relative to m must then satisfy the rationality conditions

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} Q(B^{(k)}) = m^Q(B) = m(B) \text{ for all cylinder sets } B \in \mathscr{B}(X^\infty). \quad (1)$$

These conditions are the restrictions of the RB theory. For an example of these conditions, let t be the current period and I_t be the history up to t and consider the following random variables:

$Z^{(t+k)}$ = the annualized rate of return on the S&P500 stock index k periods after date t. Consider the expectation $E_Q[Z^{(t+k)}|I_t]$. The time average of $Z^{(t+k)}$ is approximately 8%, excess future returns are unpredictable under the probability m and $E_m[Z^{(t+k)}|I_t] = 8\%$ for all k. Under a rational belief Q such that $Q \neq m$, the rationality conditions require that

$$\frac{1}{n} \sum_{k=0}^{n-1} E_Q[Z^{(t+k)}|I_t] \cong 8\% \text{ for large } n.$$

It is a fact that in any experiment in which agents are asked to predict $Z^{(t+k)}$ there is a wide distribution of forecasts but there is little disagreement about the long term average.

The implication of (1) is that under RB agents may disagree about probabilities of short term events but not about long term averages. Observe that (i) Π is a Rational Belief and hence REE is an RBE; (ii) m is an RB although it is possible that $m \neq \Pi$; (iii) an RB Q and the true Π may disagree on timing or sequencing; (iv) an RB Q and the true Π may put different probabilities on important rare events; (v) an RB Q allows optimism/pessimism relative to m.

Three observations

(i) If all agents believe the economy is stationary, their beliefs would satisfy $Q^k = m$. Hence, in an RBE agents may believe that the economy is non-stationary and be uncertain about the underlying structure. Any disagreement at t revolves around *unknown* regime parameter (e.g. mean value function) of the stochastic process of prices and quantities which prevail in the economy at t.

(ii) When agents disagree, the distribution of beliefs affects excess demand functions and hence in an economy in which the beliefs of H agents matter, the equilibrium map has the general form

$$p_t = \Phi(I_t, Q_t^1, Q_t^2, \ldots, Q_t^H)$$

where $(Q_t^1, Q_t^2, \ldots, Q_t^H)$ are date t conditional probabilities of the H agents. In such equilibria the distribution of beliefs is a propagation mechanism of price volatility. More fundamentally, the RBE theory rejects the formulation of uncertainty as only an *exogenous phenomenon.* It proposes that uncertainty and fluctuations have an endogenous component which is propagated *within* the economy. Recognizing this, Kurz (1974) called it *Endogenous Uncertainty* and the present paper argues that this uncertainty is the dominant form of uncertainty in our economy. We thus define:

Definition 5. Endogenous Uncertainty *is that component of the volatility of quantities and prices in the economy which is generated by the distributions of beliefs.*

(iii) *Disagreement among agents in an RBE implies their conditional probabilities fluctuate over time.* For example, consider a finite state Markov economy where Q_t^k are time dependent selections from J different Markov matrices $\{G_1^k, G_2^k, \ldots, G_J^k\}$ which k believes are possible and m is represented by a single matrix Γ. If agents disagree, their beliefs are not represented by Γ implying $G_j^k \neq \Gamma$ except, perhaps for one j. Rationality conditions imply the average forecasts of k, where each forecast is made with some G_i^k, equals the forecast under Γ. It is irrational for k to use only one matrix, say $G_1^k \neq \Gamma$, since the forecasts under G_1^k and Γ are not equal. Hence in a world with disagreement, rational agents must use varying matrices over time.

The main challenge to applications is the simplification of the general rationality conditions in (1).

2 The RBE of an OLG stock market economy

2.1 The OLG economy

We aim to incorporate the RB theory in a simple model where endogenous volatility can be studied and whose equilibrium is computable. For this reason our stock market economy is an OLG economy with two *types* of agents and with a single, homogenous, consumption good.
Each agent lives two periods, the first when he is "young" and the second when he is "old." At the start of each date, an old agent has a young offspring who replicates him, where the term "replicates" refers to *utilities* and *beliefs*. Hence, this is a model of two infinitely lived "dynasties" denoted by $k = 1, 2$ and the index k identifies the two young and old agents of the dynasty at each date. We use the term "agent k" but the context makes it clear whether the agent is a young or an old member of dynasty k. Only young agents receive an endowment $\Omega_t^k, t = 1, 2, ...$ of the single consumption good. We view Ω_t^k as the labor income of agent k at date t and the stochastic processes $\{\Omega_t^k, t = 1, 2, ...\}$ for $k = 1, 2$ is exogenously specified. Additional net output is supplied by a firm which produces exogenously, as in Lucas (1978), the strictly positive profit process $\{D_t, t = 1, 2, ...\}$ with no input. These net outputs are paid out to the shareholders of the firm as dividends at the date at which the output is produced. The ownership shares are traded on a stock market and their aggregate supply is 1.

The economy has three markets: (i) a market for the consumption good with an aggregate supply equaling the total endowment plus total dividends, (ii) a stock market with a total supply of 1, and (iii) a market for a zero net supply, short term riskless debt instrument which we call a "bill". The financial sector is initiated at date 1 by distributing the unit supply of shares among the old of that date. Our notation is as follows: for $k = 1, 2$

C_t^{1k} – consumption of k when young at t;
C_{t+1}^{2k} – consumption of k when old at $t+1$ (implying that the agent was born at t);
θ_t^k – amount of stock purchases by young agent k at t;
B_t^k – amount of one period bill purchased by young agent k at t;
P_t^n – the price of the common stock at t;
q_t – the price of a one period bill at t. This is a discount price;
I_t – the history of all observable variables up to and including t.

$Q_t = Q(A|t, I_t)$ for all $A \in \mathscr{B}(X^\infty)$ – is a conditional probability measure at date t. For k we denote it Q_t^k. To condition on a random variable y we use the notation Q_y^k.

Using consumption as a numeraire the problem of a young agent, for $k = 1, 2$ is

$$\underset{(C_t^{1k}, \theta_t^k, B_t^k, C_{t+1}^{2k})}{\text{Max}} E_{Q_t^k}\{u^k(C_t^{1k}, C_{t+1}^{2k})| \; I_t\} \tag{2a}$$

subject to

$$C_t^{1k} + P_t^n \theta_t^k + q_t B_t^k = \Omega_t^k \tag{2b}$$

$$C_{t+1}^{2k} = \theta_t^k (P_{t+1}^n + D_{t+1}) + B_t^k . \tag{2c}$$

To enable us to compute equilibria we take the utility function agent k to be

$$u^k(C_t^{1k}, C_{t+1}^{2k}) = \frac{1}{1-\gamma_k}(C_t^{1k})^{1-\gamma_k} + \frac{\beta_k}{1-\gamma_k}(C_{t+1}^{2k})^{1-\gamma_k}, \quad \gamma_k > 0\,, \quad 0 < \beta_k < 1\,. \tag{3}$$

With this specification, the Euler equations for agent k are

$$-P_t^n (C_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((C_{t+1}^{2k})^{-\gamma_k}(P_{t+1}^n + D_{t+1})|I_t) = 0 \tag{4a}$$

$$-q_t (C_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((C_{t+1}^{2k})^{-\gamma_k}| \; I_t) = 0. \tag{4b}$$

Now define the random growth rate of dividends d_{t+1} by

$$D_{t+1} = D_t d_{t+1}. \tag{5}$$

$\{d_t, t = 1, 2, ...\}$ is a stochastic process on a dividend space $D \subseteq \mathbb{R}$ with probability μ_D hence $((D)^\infty, \mathscr{B}((D)^\infty), \mu_D)$ is the probability space of infinite sequence of growth rates d_t. In the simulations below we study growing economies with $E(d_t) > 1$ hence with a secular rise of total dividends. To assure stability we focus on the ratios of variables which are defined as follows:

$$\begin{aligned}
\omega_t^k &= \frac{\Omega_t^k}{D_t} && \text{is the endowment/dividend ratio of agent } k \text{ at date } t;\\
b_t^k &= \frac{B_t^k}{D_t} && \text{is the bill/dividend ratio of agent } k \text{ at date } t;\\
c_t^{1k} &= \frac{C_t^{1k}}{D_t} && \text{is the ratio of consumption when young to aggregate capital income;}\\
c_{t+1}^{2k} &= \frac{C_{t+1}^{2k}}{D_{t+1}} && \text{is the ratio of consumption when old to aggregate capital income;}\\
p_t &= \frac{P_t^n}{D_t} && \text{is the price/dividend ratio at } t;\\
I_t &= (I_{t-1}, p_t, q_t, d_t);\\
\tilde{I}_t &= (\tilde{I}_{t-1}, \theta_{t-1}^1, \theta_{t-1}^2, b_{t-1}^1, b_{t-1}^2, p_t, q_t, d_t).
\end{aligned}$$

We assume $\omega_t^k = \omega^k$ – constant for $k = 1, 2$, and $\nu = \omega^1 + \omega^2$ hence $(\Omega_t^1 + \Omega_t^2) = \nu D_t$ for all t. We do not consider ω^k in part because production and labor markets are not the focus of this paper and in part because of computational feasibility. Now divide (2b) by D_t, $(2c)$ by D_{t+1}, equation (4a) by $D_t^{1-\gamma_k}$ and equation (4b) by $D_t^{-\gamma_k}$ to obtain, for $k = 1, 2$

$$c_t^{1k} = -p_t \theta_t^k - q_t b_t^k + \omega^k, \tag{6a}$$

$$c_{t+1}^{2k} = \theta_t^k (p_{t+1} + 1) + \frac{b_t^k}{d_{t+1}}, \tag{6b}$$

$$-p_t (c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k} d_{t+1})^{-\gamma_k} (p_{t+1} + 1)\, d_{t+1} | \; I_t) = 0, \tag{6c}$$

$$-q_t (c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k} d_{t+1})^{-\gamma_k} | \; I_t) = 0. \tag{6d}$$

(6a) – (6d) imply demand functions which take the general time dependent form, for $k = 1, 2$

$$b_t^k = b^k(p_t, q_t, d_t, I_{t-1}, Q_t^k) \tag{7a}$$

$$\theta_t^k = \theta^k(p_t, q_t, d_t, I_{t-1}, Q_t^k)\ . \tag{7b}$$

Equilibrium requires markets to clear

$$\theta^1(p_t, q_t, d_t, I_{t-1}, Q_t^1) + \theta^2(p_t, q_t, d_t, I_{t-1}, Q_t^2) = 1 \tag{7c}$$

$$b^1(p_t, q_t, d_t, I_{t-1}, Q_t^1) + b^2(p_t, q_t, d_t, I_{1-1}, Q_t^2) = 0\ . \tag{7d}$$

The main simplification of the one-commodity OLG model is that portfolio demands and hence the equilibrium map do not depend upon the portfolios $(\theta_{t-1}^1, \theta_{t-1}^2, b_{1-1}^1, b_{t-1}^2)$ of the old which are offered to the market inelastically (we return to this issue below). Also, (7a)–(7d) depends upon the beliefs of the agents and the notation (Q_t^1, Q_t^2) highlights possible non-stationary beliefs and time dependency of (Q_t^1, Q_t^2). Now, since beliefs are not observable, an equivalent way of writing the market clearing conditions is

$$\theta_t^1(p_t, q_t, d_t, I_{t-1}) + \theta_t^2(p_t, q_t, d_t, I_{t-1}) = 1 \tag{7c'}$$

$$b_t^1(p_t, q_t, d_t, I_{t-1}) + b_t^2(p_t, q_t, d_t, I_{t-1}) = 0 \tag{7d'}$$

where the time dependency of the decision functions represent time variations due to (Q_t^1, Q_t^2).

(7a)–(7d) or (7c')–(7d') imply that the equilibrium price process $\{(p_t, q_t), t = 1, 2, ...\}$ is generated by an equilibrium selection sequence of maps which can be written as

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi(d_t, I_{t-1}, Q_t^1, Q_t^2). \tag{7e}$$

Equivalently, by incorporating (Q_t^1, Q_t^2) into the equilibrium map we can write

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi_t(d_t, I_{t-1}). \tag{7e'}$$

In a REE, $Q^1 = Q^2 = \Pi$ where Π is the true probability induced by μ_D and by (7a) – (7e).

For full generality note that when the old agents optimize (e.g. a two consumption goods model), portfolio demands and equilibrium maps depend upon the history of past portfolios. In such cases the OLG equilibrium map takes the more general form

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi(d_t, \theta_{t-1}^1, \theta_{t-1}^2, b_{t-1}^1, b_{t-1}^2, \tilde{I}_{t-1}, Q_t^1, Q_t^2) \tag{7f}$$

or

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi_t(d_t, \theta_{t-1}^1, \theta_{t-1}^2, b_{t-1}^1, b_{t-1}^2, \tilde{I}_{t-1}). \tag{7f'}$$

2.2 Agents' beliefs and the general definition of rational belief equilibrium (RBE) [3]

In this section we give a general definition of an RBE and compare it to other equilibrium concepts. To generalize, we observe that the definition which we give is applicable to economies in which agents are infinite horizon maximizers rather than OLG. To briefly explain this point we consider such an economy but hold the view that in this economy an "agent" is *an infinite sequence of family members each finitely lived (say, for L periods) with interdependent utility.*

The problem of agent $k (k = 1, 2)$ is then

$$\underset{(C^{1k}, \theta^k, B^k)}{\text{Max}} E_{Q_t^k} \left\{ \sum_{\tau=t}^{\infty} \beta_k^{\tau-1} u^k(C_\tau^k) | \tilde{I}_\tau \right\} \tag{8a}$$

subject to

$$C_t^k + P_t^n \theta_t^k + q_t B_t^k = \Omega_t^k + \theta_{t-1}^k (P_t^n + D_t) + B_{t-1}^k; \tag{8b}$$

$$(\theta_0^k, B_0^k) \quad \text{given and the price of consumer good normalized to 1;} \tag{8c}$$

[3] Section 2.2 is somewhat more technical than the rest of the paper. A first time reader may skip it, review Section 2.3 and continue to Section 2.4 where the RBE of the OLG economy is developed. This will enable the reader a more direct access to the simulation results in Section 3. The reader could then return to Section 2.2 in order to explore the more general definition of an RBE.

$$\text{some transversality conditions.} \tag{8d}$$

The differences between the OLG problem (2a)-(2c) and (8a)-(8c) are the transversality conditions, the budget constraints and the conditioning on past portfolios $(\theta^1_{t-1}, \theta^2_{t-1}, b^1_{t-1}, b^2_{t-1})$. We use the same definitions of the intensive variables and observe that portfolio demands (or selections from demand correspondences) are functions of prices (p_t, q_t), dividend d_t, date $t-1$ portfolios $(\theta^1_{t-1}, b^1_{t-1}, \theta^2_{t-1}, b^2_{t-1})$, states of belief (Q^1_t, Q^2_t) and, perhaps, the history I_{t-1} of all observables. Hence we can then write a selected equilibrium map as

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi(d_t, \theta^1_{t-1}, b^1_{t-1}, \theta^2_{t-1}, b^2_{t-1}, \tilde{I}_{t-1}, Q^1_t, Q^2_t) \tag{9}$$

or equivalently, by incorporating (Q^1_t, Q^2_t) into the equilibrium map, we can write

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi_t(d_t, \theta^1_{t-1}, b^1_{t-1}, \theta^2_{t-1}, b^2_{t-1}, \tilde{I}_{t-1}). \tag{9'}$$

(7f)–(7f') and (9)–(9') clearly have the same form. In the definition developed below we always refer to the OLG optimization problem (2a)–(6c) and to the portfolio demands and the equilibrium map in (7a)–(7e) or (7e'). However, all definitions below apply *verbatim* to OLG models with equilibrium maps like (7f)–(7f') or to infinite horizon economies with maps like (9)–(9').

2.2a The true equilibrium process

An equilibrium is a stochastic process of prices, dividends and portfolios such that at all t (i) *agents select optimal portfolios* and, (ii) *the market clearing conditions* $\theta^1_t + \theta^2_t = 1$ *and* $b^1_t + b^2_t = 0$ *are satisfied.* An RBE is an equilibrium in which agents hold Rational Beliefs. This informal observation shows that an RBE is *a recursive equilibrium of an infinite horizon economy which may be non-stationary*; we develop now a formal definition.

Denote by Π the *true* probability over sequences $\{(p_t, q_t, \theta^1_t, b^1_t, \theta^2_t, b^2_t, d_t), t = 1, 2, ...\}$ of observables. This *true* stochastic process is induced by maps (7e) or (7e'), by decision functions (7a)–(7b), by the *true* process of dividends and by time-varying, non-stationary, factors injected endogenously by the agents. To specify Π formally, define the relevant spaces:

(i) $\{(p_t, q_t), t = 1, 2, ...\}$ is the price process with $(p_t, q_t) \in P \subseteq \mathbb{R}^2_+$
(ii) $\{(\theta^1_t, b^1_t, \theta^2_t, b^2_t), t = 1, 2, ...\}$ is the portfolio process with $(\theta^1_t, b^1_t, \theta^2_t, b^2_t) \in W \subset \mathbb{R}^4$
(iii) $\{d_t, t = 1, 2, ...\}$ is the dividend process with $d_t \in D \subseteq \mathbb{R}$.

Let $X = (P \times W \times D)$ be the space of observables; $X^\infty = (P \times W \times D)^\infty$ the space of infinite sequences $\{x_t \equiv (p_t, q_t, \theta^1_t, b^1_t, \theta^2_t, b^2_t, d_t), t = 1, 2, ...\}$ and $\mathscr{B}((X)^\infty)$, the Borel σ-field of $(X)^\infty$.

Definition 6. *The true equilibrium dynamics of the economy is specified by the probability space* $((P \times W \times D)^{\infty}, \mathscr{B}((P \times W \times D)^{\infty}), \Pi)$ *with the following restrictions. Let* Π_Z *be the marginal measure of* Π *on a subspace* Z *then* Π *must satisfy the following marginality conditions:*

(i) $\Pi_D = \mu_D$ *the exogenously specified probability of the process of dividends;*
(ii) (Π_p, Π_W) *are probabilities of stochastic processes induced by (7a) – (7e).*

Definition 7. *The economic dynamical system* $(X^{\infty}, \mathscr{B}(X^{\infty}), \Pi)$ *is said to be Stable if under* Π *the process* $\{x_t, t = 1, 2...\}$ *is stable in the sense of Definition 1 with a stationary measure* m *over* $(X^{\infty}, \mathscr{B}(X^{\infty}))$. *It is said to be Ergodic if under* $\Pi, \{x_t, t = 1, 2, ...\}$ *is ergodic.*[4]

In a stable and ergodic economy a random realization $((p_t, q_t, \theta_t^1, b_t^1, \theta_t^2, b_t^2, d_t), t = 1, 2, ...)$ induces, with Π probability 1, a *unique* empirical measure m in accord with Definition 1. The stationary measure m is independent of the realized sequence. Although Π satisfies the marginality conditions (i) and (ii), the stationary measure m does *not* generally satisfy them and this is *the crucial reason for the emergence of Rational Beliefs.* To see why, note that non-stationarity of the dividend process and/or time variability of the distribution of beliefs (Q_t^1, Q_t^2) represents temporal parametric shifts in the non-stationary probability Π. However, these are averaged out over time and translated into moments of the empirical distribution. For example, if (Q_t^1, Q_t^2) move into a regime of bullish expectations, resulting in a regime of historically high asset prices, this is a non-stationary change in the true probability law Π. In the stationary measure m this regime is translated into the frequency and covariances related to high prices. Non-stationary regimes of the dividends due to technological changes have similar effect. If the dividend process is stationary[5] the marginal measure m_D satisfies the marginality condition (i).

2.2.b The individual perception models

In an RBE agents do not know the true dynamics $(X^{\infty}, \mathscr{B}(X^{\infty}), \Pi)$ and develop theories reflecting how they perceive it, given the known stationary measure m. In expressing these perceptions they treat all observables as random variables with probability beliefs (Q^1, Q^2) over infinite sequences of these variables. To describe the perceived model of agent k we define first the perceived spaces of the observables:

[4] In this paper we assume that the economy is ergodic and incorporate this condition into all the definitions in the text. This assumption is not needed but used to simplify the exposition. Kurz (1994) argues that assuming ergodicity does not limit the scope of the analysis since in a non-ergodic economy an RBE depends upon the history of the economy but the structure of the equilibrium remains the same.

[5] This is the case in the simulation model below where $D = \{d^H, d^L\}$ and the stochastic process of dividends is a stationary Markov process with transition matrix to be specified later. Hence, in that economy m satisfies the marginality condition (i). Here we give a general definition of an RBE and allow the dividend process to be a complex, non-stationary, process in order to provide a more complete motivation for the RBE theory.

(i) $\{(p_t^k, q_t^k),\ t = 1, 2, \ldots\}$ – the price process with $(p_t^k, q_t^k) \in P^k \subseteq \mathbb{R}_+^2$;
(ii) $\{(\theta_t^{k1}, b_t^{k1}, \theta_t^{k2}, b_t^{k2}),\ t = 1, 2, \ldots\}$ – the portfolio process with $((\theta_t^{k1}, b_t^{k1}, \theta_t^{k2}, b_t^{k2}) \in W^k \subset \mathbb{R}^4$;
(iii) $\{d_t^k, t = 1, 2, \ldots\}$ is the dividend process with $d_t^k \in D^k \subset \mathbb{R}$.

Let $X^k = (P^k \times W^k \times D^k)$ be k's perceived space of observables; $(X^k)^\infty = (P^k \times W^k \times D^k)^\infty$ k's perceived space of infinite sequences $\{x_t^k \equiv (p_t^k, q_t^k, \theta_t^{k1}, b_t^{k1}, \theta_t^{k2}, b_t^{k2}, d_t^k),\ t = 1, 2, \ldots\}$ of the observables and $\mathscr{B}((X^k)^\infty)$, the Borel σ-field of $(X^k)^\infty$.

Definition 8. Agent k's belief *is a probability* Q^k *on* $((X^k)^\infty, \mathscr{B}((X^k)^\infty))$. His perceived model *is a probability space* $((X^k)^\infty, \mathscr{B}((X^k)^\infty), Q^k)$ *and* Q^k *is* k*'s probability belief of the stochastic process* $\{x_t^k, t = 1, 2, \ldots\}$. The perceived model of k is a stable and ergodic *if under* $Q^k\{x_t^k, t = 1, 2, \ldots\}$ *is stable and ergodic with a stationary measure* m^{Q^k} *on* $((X^k)^\infty, \mathscr{B}((X^k)^\infty))$.

Definition 9. *A belief of agent* k *is a* Rational Belief relative to a stationary measure m *if*

(i) His perceived model $((X^k)^\infty, \mathscr{B}((X^k)^\infty), Q^k)$ *is stable and ergodic;*
(ii) $m^{Q^k} = m$.

2.2.c Rational belief equilibrium

We are now ready for a formal definition of an RBE.

Definition 10. *A sequence of decisions* $\{(\theta_t^1(I_t), b_t^1(I_t), \theta_t^2(I_t), b_t^2(I_t)), t = 1, 2, \ldots\}$, *an implied true equilibrium dynamics* $((X)^\infty, \mathscr{B}((X)^\infty), \Pi)$ *and perceived models* $((X^k)^\infty, \mathscr{B}((X^k)^\infty), Q^k)$ *for* $k = 1, 2$, *constitute an RBE if*

(i) $\{(\theta_t^k(I_t), b_t^k(I_t)), t = 1, 2, \ldots\}$ *are optimal portfolios of* $k = 1, 2$ *satisfying market clearing*[6]
(ii) $((X)^\infty, \mathscr{B}((X)^\infty), \Pi)$ *is stable and ergodic with stationary measure* m*;*
(iii) Q^k *are rational beliefs relative to* m.

Note that in an RBE the support of Q^k need not equal the support of Π. Kurz (1994) shows that Q^k may even be orthogonal to Π and hence an agent may, at times, put positive probabilities on prices that do not occur in equilibrium. We have already observed that a central feature of an RBE is the potential discrepancy between the true probability Π and the stationary measure m. The

[6] Although the definition of RBE does not address the issue of multiple equilibria, we model the economy as a stochastic dynamical system in which every infinite random draws is associated with a *definite* sequences of prices and allocations. If at any date the economy has multiple market clearing outcomes, then *as part of the dynamics postulated* there is some procedure of selecting a particular one which generates the data observed. This procedure is then part of the equilibrium map. Thus, if at two different dates the economy reaches the same recursive state and the same state of belief then equilibrium prices should be the same at both dates. This could be weakened to a probability 1 requirement. In each simulation model developed later the equilibrium is unique and hence the issue does not arise.

true dynamics $((X)^\infty, \mathscr{B}((X)^\infty), \Pi)$ exhibits both exogenous shocks as well as unobserved parametric shifts in the dividend process and in the distribution of beliefs. Structural changes may be deterministic but their effects are transformed into empirical frequencies of the stationary measure. Endogenous uncertainty arises from endogenous variations of the states of beliefs which, in turn, are induced by agents holding diverse interpretations of public information.

Variations in the states of belief affect market volatility via the discrepancy between the reality $((X)^\infty, \mathscr{B}((X)^\infty), \Pi)$ of the true dynamics and the perceptions $((X^k)^\infty, \mathscr{B}((X^k)^\infty), Q^k)$ of this reality by the agents. *Diversity of beliefs is the volatility propagation mechanism of an RBE.*

The idea that the equilibrium reality is, in part, determined by the diverse perceptions of the agents is not new to economics. Expectations in financial markets are a central component of Thornton's (1802) view of paper money and financial markets; we have mentioned the importance of expectations in Keynes (1936) and add that expectations are key to "cumulative movements" in Pigou [see, Pigou's (1941), Chapter VI]. Expectations are basic to the process of deviations from a stationary equilibrium in the Swedish school [e.g. see Myrdal's 1939 view of money in Myrdal (1962, Chapter III)]. Also, the concept of "subjective values" based on diverse expectations is a cornerstone of Lindahl (1939) theory of money and capital. The effects of perceptions on equilibrium has been discussed in the learning literature and it is one of the objects of the literature on sunspot REE. The RBE theory provides a vocabulary for an analytical expression of these ideas to enable useful applications. It is based on a rigorous theory of diverse but rational beliefs and offers a unified theory of market volatility propagated by this diversity.

2.2.d Comparison of RBE with other equilibrium concepts

An RBE is a recursive equilibrium of an infinite horizon economy in which the agents hold rational beliefs about the equilibrium dynamics. Variations in technology parameters and of states of beliefs in an RBE are unobserved components of the state and are sources of economic volatility. As a result, an RBE is an equilibrium with unobserved, endogenous, states of belief and incomplete market structure.

A stable Rational Expectations Equilibrium (REE) is defined in this paper to be an equilibrium in which (1) *agents know the true equilibrium map*, and (2) *agents know the true equilibrium probability of all variables and adopt it as their own beliefs hence $Q^k = \Pi$ all k*. We ignore unstable REE (e.g. divergent bubbles). Hence when agents have correct perceptions $((D \times P \times W)^\infty, \mathscr{B}((D \times P \times W)^\infty), \Pi) = ((P^k \times W^k \times D^k)^\infty), \mathscr{B}((P^k \times W^k \times D^k)^\infty, Q^k)$, RBE is an REE.

Is an RBE a Radner Equilibrium in the sense of Radner (1972) or (1979)? Note first that a Radner (1972) equilibrium is *different* from Radner (1979). In Radner (1972) or in infinite horizon extensions such as Magill and Quinzii (1994, 1996), the equilibrium does not require agents to have probability beliefs: they only need preferences over risky prospects. They assume that (i) the state is observable and exogenous in being a primitive part of the description of the economy and, (ii) agents know the equilibrium map and hence the price that would be realized in each state. These conditions do not hold in an RBE and for each observable exogenous state, an agent has a probability distribution over prices. Also, given an observed exogenous state, agents in an RBE may make wrong forecasts and place positive probabilities on prices which cannot occur in equilibrium. Such RBE are not Radner Equilibria. In the special case of a single good economy a Radner equilibrium is less demanding, requiring agents only to place positive probabilities on prices realized in equilibrium. Relative to the exogenous Radner state space an RBE has an expanded state space which includes unobserved states of belief. Even if this component of the state is unobserved, in the expanded state space an RBE is an incomplete Radner equilibrium if at all dates and given any state, the support of each Q_t^k equals the support of Π_t. A non-stationary belief implies a time-dependent utility but from a purely mathematical perspective an agent's utility in an RBE can be considered *state dependent* as date t beliefs are part of the state. This has no bearing on whether an RBE is a Radner Equilibrium.

Focusing on signal extraction of the equilibrium, Radner's (1979) model introduces probability beliefs and postulates that agents have private information. In an REE, market prices reveal all private information. If, for comparison, a state of belief of agent k is viewed as private "information," an RBE is not a Radner (1979) REE since in this equilibrium agents do not know the equilibrium map and hence an RBE does not reveal the states of beliefs of other agents. Since knowing the equilibrium map essentially entails each agent knowing other's beliefs, most papers that use Radner's construction, define an REE as requiring agents to have the same true beliefs (e.g. Duffie, Geanakoplos, Mas-Colell and McLennan, 1994) or, to satisfy the *common prior assumption* (e.g. Jackson and Peck, 1991). Neither one of these are imposed in an RBE.

Definition 10 shows the general concept of an RBE does not entail any sunspots although sunspot REE are particular RBE. However, we revisit this issue since the RBE we develop next uses "assessment variables" with a mathematical structure which may appear like private sunspots.

2.3 Defining a rational belief with perceived regime (or assessment) variables

A general non-stationary RBE is rather abstract and we need a way to implement it in applied setting. The tool developed for simplification is the Conditional Stability Theorem (Kurz and Schneider, 1996; Nielsen, 1996). Let X be the space of observables and suppose we want to describe, in a tractable manner, the non-stationarity of a dynamical system on the space $((X)^\infty, \mathscr{B}((X)^\infty)$. The Conditional Stability Theorem aims to describe all the non-stationarity via artificial variables $y_t \in Y$ with a marginal probability space $((Y)^\infty, \mathscr{B}((Y)^\infty, \mu)$.

This is done by postulating $(X \times Y)$ to be the state space, introducing a universal probability measure Q and space $((X \times Y)^\infty, \mathscr{B}((X \times Y)^\infty, Q)$ and, by defining the desired non-stationary probability to be Q_y, the conditional probability of Q with respect to the sequence y. Thus Q must satisfy the requirement that for all $A \in \mathscr{B}(X^\infty)$ and $B \in \mathscr{B}(Y^\infty)$

$$Q(A \times B) = \int_B Q_y(A)\mu(dy).$$

The conditional probability space $((X)^\infty, \mathscr{B}((X)^\infty, Q_y)$ is non-stationary since probabilities of events in $\mathscr{B}((X)^\infty)$ are not time independent: they change with the parameters y which are time dependent. If Q_y describes all the non-stationarity of the given system, then we interpret the sequence $\{y_t, t = 1, 2, ...\}$ as a *mathematical description of that non-stationarity.* This approach is common in econometric (e.g. Hamilton, 1989) where Y is the set of possible "regimes" and identifies the regime at t. Generally, when used to describe the non-stationarity of exogenous processes, y_t should be viewed as *"regime variables."* In the present paper we use this method to describe the perceived non stationarity of an agent. Thus, $Q^k_{y_t}$ is the date t probability belief of future observables by k and y_t is interpreted as *"perceived regime variables"* or an *"assessment"* variables. It is important to see that assessment variables $\{y^k_t, t = 1, 2, ...\}$ are generated by the agent himself, providing a vocabulary to describe his belief in the non-stationarity of the observables. From an informational perspective, assessment variables are privately perceived parameters indicating how an agent interprets current information. These variables have purely subjective meaning and should not be taken to be objective and transferable "information".

Nielsen (1996), who developed this method, defined a Simple, Independently Distributed Stable measure Q_y by a sequence of independent random variables with densities at date t fixed by a deterministic sequence $(y^*_t, y^*_t, ...)$. This leads to technical difficulties which Kurz and Schneider (1996) resolved by defining the *joint system* $((X^k \times Y^k)^\infty, \mathscr{B}((X^k \times Y^k)^\infty, Q^k)$ as the primitive and the *conditional system* $((X^k)^\infty, \mathscr{B}((X^k)^\infty, Q^k_y)$ as the effective belief of k. In summary:

(i) y^k_t are privately generated, representing parameters of the dynamics as perceived by k. Since k lives for L periods, at t he does not know y^k_τ for dates τ not in his own lifetime;
(ii) decision functions of agent i are not measurable with respect to y^j_t for $i \neq j$: agent i would not know at t how to interpret y^j_t even if he could "know" it;
(iii) (y^1_t, y^2_t) may be correlated and jointly distributed with observables. Agents do not know these distributions and cannot learn them from data since k knows only y^k_t in his own lifetime.

This approach leads to a technical difficulty. The proposed method of "regime" variables offers a tractable way of defining a rational belief but then, under what conditions is a conditional system $((X^k)^\infty, \mathscr{B}((X^k)^\infty, Q^k_y)$ stable and ergodic? And how do we compute the stationary measure of such a conditional system? For an answer, note that since the conditional system is intended to describe *all* the non-stationarity, we can assume the joint system to be stationary. Now, for a joint system $((X \times Y)^\infty, \mathscr{B}((X \times Y)^\infty), Q)$, the marginal measure Q_{X^∞} is defined by

$$Q_{X^\infty}(A) = Q(A \times Y^\infty) = \int_{Y^\infty} Q_y(A)\mu(dy) \quad \text{for all } A \in \mathscr{B}(X^\infty).$$

Theorem 2. *(Conditional Stability Theorem, Kurz and Schneider (1996) Theorem 2) Let $((X \times Y)^\infty, \mathscr{B}((X \times Y)^\infty), Q)$ be stationary and ergodic and let Y be countable then*

(a) $(X^\infty, \mathscr{B}(X^\infty), Q_y, T)$ is stable and ergodic for Q a.a. y;
(b) The stationary measure of Q_y is independent of y for Q a.a. y and if we denote it by m^{Q_y} then it satisfies the condition $m^{Q_y} = Q_{X^\infty}$.

Consider the simple example when $y_t^k \in Y = \{0, 1\}, Q^k$ is a probability belief on the *joint* process $\{(x_t, y_t^k), t = 1, 2, \ldots\}$ which is a Markov process and, conditionally on y_{t-1}^k, the distribution of y_t^k is independent of other observables. Then the *effective belief* $Q_{y^k}^k$ is defined by two transition functions F_1^k and F_2^k as follows

$$Q_{y_t^k} = \begin{cases} F_1^k & \text{if} \quad y_t^k = 1 \\ F_2^k & \text{if} \quad y_t^k = 0. \end{cases}$$

The *marginal* measure $Q_{X^\infty}^k$ is the probability of a stationary Markov process uniquely defined by a transition function F^k, computed by the simple expression (which we use again later):

$$F^k = F_1^k \mu^k(y_t^k = 1) + F_2^k \mu^k(y_t^k = 0).$$

Comparison with sunspot equilibria. A definition of an RBE with assessment variables may appear to some as *mathematically* similar to sunspot equilibria. Extrinsic sunspot signals are used in the standard sunspot REE (e.g. Azariadis, 1981; Azariadis and Guesnerie, 1986; Cass and Shell, 1983; Farmer, 1993; Peck, 1988; Woodford, 1986) as follows: (i) as exogenous, publicly observed signals with probabilities known to all agents and uncorrelated with any fundamental variables, (ii) as a coordination device to select among multiple equilibria; (iii) to have no direct effect on the fundamentals of the economy, and (vi) to construct sunspot equilibria which are REE.

Assessment variables are used to describe the agents' perceived regime parameters. However suppose, for the sake of discussion, we consider (y_t^1, y_t^2) as private "information" and in order to examine them in relation to sunspots we summarize their properties as follows:

(i) they are not exogenous as they are generated by the agents; they are not publicly observed; the distribution of y_t^k is known to k but the joint distribution of (y_t^1, y_t^2) is not known; in determining their "signals" agents may condition on observable variables hence (y_t^1, y_t^2) may be correlated with observables (in later simulations we exclude such correlation); as "information" they are not only privately observed, they must also satisfy the condition that the decision functions of i are not measurable with respect to $y_t^j, i \neq j$ hence agent i would not know what $y_t^j, i \neq j$ means even if he could "observe" it.

(ii) they do not provide a randomization over multiple equilibria since an RBE may not have multiple equilibria and the "signals" y_t^k may be statistically independent across agents.

(iii) they are as fundamental as preferences in that they determine the von-Neumann Morgenstern utility of agents over infinite sequences.

(vi) they define RBE with diverse beliefs of agents while sunspot equilibria are REE.

These properties imply that, in contrast to a sunspot interpretation, from a purely *mathematical* perspective it is more useful to think of a rational belief as employing state dependent preferences where the dependence arises through the state dependence of the agents' beliefs.

Jackson and Peck (1991) is one paper which develops extrinsic REE of an OLG model with private information signals. These authors postulate that (i) price formation is a Vickrey auction, (ii) the structure of the auction game is known to all, (iii) the players have a correct *common prior* about the information signals they jointly receive. In the resulting correlated equilibrium agents deduce from prices the only information they need, i.e. the private signal received by the price setting agent in the auction at each date. Thus, in equilibrium the joint private signal becomes, in effect, public information. The differences between our RBE and the extrinsic REE of Jackson and Peck (1991) should then be clear: in an RBE decision functions of an agent are not measurable with respect to the "signals" of others, agents do not know the price map and do not deduce private "signals" (i.e. assessments) from prices. Finally, in this RBE agents do not have a correct common prior of the joint distribution of "signals" and prices.

We note in conclusion that artificial variables are used extensively in economics. It is not useful to think of *any* artificial variable as sunspot-like. It is more constructive to determine the nature of such a variable, the way it is used, and the assumptions which it needs to satisfy.

2.4 Constructing a Markov RBE for the OLG economy

We now return to (6a)–(6d), (7a)–(7e) of the OLG economy and, using the technique of assessment variables, study a particular family of RBE. Our method is to construct a Markov RBE in which only a finite number of prices and portfolios are ever observed. Parameters of the dividend process and of preferences are selected to equal realistic empirical estimates, and belief parameters are simple and intuitive. We thus aim to calibrate the model and test its ability to explain the volatility characteristics of U.S. financial markets.

Starting with the dividend process, we assume the exogenous process of dividends to be as specified in M&P (1985). Hence, the dividend growth rate $\{d, t = 1, 2, ...\}$ is a stationary and ergodic Markov process. The state space of the process is $D = \{d^H, d^L\}$ with $d^H = 1.054$ and $d^L = 0.982$. M&P (1985) estimate the transition matrix of dividend growth rates to be

$$\begin{bmatrix} \phi, 1-\phi \\ 1-\phi, \phi \end{bmatrix} \tag{10a}$$

with $\phi = 0.43$. Since the stochastic growth rate of dividends is Markovian with two states, the economy has a dynamically complete market structure in the narrow sense that the number of financial instruments equals the number of exogenous states. *A Markov Competitive Equilibrium* is then characterized by the condition that (i) the dividend process is Markov and, (ii) the beliefs of agents are Markov, independent of the scale variable D_t and of the hisory I_{t-1}. Hence in comparison with (7e)–(7e') the equilibrium map becomes

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi(d_t, Q_t^1, Q_t^2). \tag{10b}$$

The constructed RBE is a finite state, non-stationary, Markov process in which agent k adopts an RB Q^k, which differs from the stationary measure m. Any belief of k which is non-stationary Markov with a finite number of states is fully characterized by a time varying sequence of Markov matrices $(M_1^k, M_2^k, \ldots)$ where M_t^k is the Markov matrix k uses at time t. This RB captures the idea that at any date agents are either "bulls" or "bears." We thus assume that each agent believes there are only two Markov matrices which are possible: $\{F_1, F_2\}$ for agent 1 and $\{G_1, G_2\}$ for agent 2. The non-stationary beliefs Q^1 and Q^2 are represented by two time functions (g_t^1, g_t^2) taking values in $\{1, 2\}$ and defining the sequence of matrices $(M_t^1 = F_{g_t^1}, M_t^2 = G_{g_t^2})$ for all t. The main problem at hand is how to implement the equilibrium map (10b). The method of assessment variables offers a tractable way of accomplishing this modeling task. Our construction requires seven steps and we regret if a full understanding of the model is attained only when the construction is completed.

2.4.a Step 1: Assessment variables and the equilibrium map

$\{y_t^k, t = 1, 2, \ldots\}$ for $k = 1, 2$ are assessment variables of agent k and we *assume* $y_t^k \in Y = \{0, 1\}$. The belief Q^k is then defined on the *joint* process $\{(p_t, q_t, d_t, y_t^k), t = 1, 2, \ldots\}$ which is a Markov process, and the effective belief is $Q_{y^k}^k$, the conditional probability of Q^k given the sequence y^k. Now, in (6c)–(6d) agent k uses the probability of $(p_{t+1}, q_{t+1}, d_{t+1}, y_{t+1}^k)$ conditional on (p_t, q_t, d_t, y_t^k). The Markov assumptions imply that the demands of agent k for stocks and bills have the time-independent form

$$b_t^k = b^k(p_t, q_t, d_t, y_t^k) \tag{11a}$$

$$\theta_t^k = \theta^k(p_t, q_t, d_t, y_t^k). \tag{11b}$$

Consequently we can write the market clearing conditions as

$$b^1(p_t, q_t, d_t, y_t^1) + b^2(p_t, q_t, d_t, y_t^2) = 0 \tag{11c}$$

$$\theta^1(p_t, q_t, d_t, y_t^1) + \theta^2(p_t, q_t, d_t, y_t^2) = 1 \tag{11d}$$

The system (11c)–(11d) implies that the equilibrium map of this economy has the form

$$\begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi^*(d_t, y_t^1, y_t^2). \tag{11e}$$

The equilibrium map (11e) shows that prices are determined by the exogenous shock d_t and by the "state of belief" (y_t^1, y_t^2). This terminology is justified since (y_t^1, y_t^2) *completely* determines the conditional probabilities (Q_t^1, Q_t^2). (11e) implies that there are at most 8 distinct price vectors (p_t, q_t) that may be observed, corresponding to the 8 combinations of (d_t, y_t^1, y_t^2). This follows from the fact that for each pair of rational beliefs, the RBE of this OLG model is unique. In a corresponding stable REE (excluding, for example, rational bubbles) beliefs do not matter and, as in M&P (1985), the model exhibits only two prices.

The Markov assumption implies that the *true* equilibrium transition probability from (p_t, q_t) to (p_{t+1}, q_{t+1}) is determined by the transition probabilities from (d_t, y_t^1, y_t^2) to $(d_{t+1}, y_{t+1}^1, y_{t+1}^2)$. For simplicity we select the joint process $\{d_t, y_t^1, y_t^2), t = 1, 2, \ldots\}$ to be a stationary Markov process with an 8×8 transition matrix Γ. This implies[7] that the *true* equilibrium process of prices has a fixed transition probability from (p_t, q_t) to (P_{t+1}, q_{t+1}) denoted Γ. Agents discover Γ from the data and use it to construct the stationary measure but they do not know that Γ is the true equilibrium probability and form RB relative to Γ. Indeed, the fact that they form RB using (y_t^1, y_t^2) *is what rationalizes* Γ *to be the equilibrium probability of the implied RBE.*

2.4.b Step 2: The state space of prices

The state space is $(D \times Y \times Y)$ but we can consider this space to be the index set $S = \{1, 2, \ldots, 8\}$. We thus define a new equilibrium map Φ between the *indices* of prices and the states of dividends and of assessment variables by

[7] The choice of the equilibrium dynamics being generated by a fixed, stationary, matrix is a matter of convenience and simplicity in this paper. In general the process $\{d_t, y_t^1, y_t^2), t = 1, 2, \ldots\}$ could have been selected to be a stable process with a Markov stationary measure induced by the empirical distribution. In such a case the fixed transition matrix Γ would characterize only the stationary measure of the equilibrium dynamics rather than be the matrix of the true probability of the equilibrium dynamics of prices. For simplicity we avoid this additional complication.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \Phi \begin{bmatrix} d_1 = d^H, & y_1^1 = 1, & y_1^2 = 1 \\ d_2 = d^H, & y_2^1 = 1, & y_2^2 = 0 \\ d_3 = d^H, & y_3^1 = 0, & y_3^2 = 1 \\ d_4 = d^H, & y_4^1 = 0, & y_4^2 = 0 \\ d_5 = d^L, & y_5^1 = 1, & y_5^2 = 1 \\ d_6 = d^L, & y_6^1 = 1, & y_6^2 = 0 \\ d_7 = d^L, & y_7^1 = 0, & y_7^2 = 1 \\ d_8 = d^L, & y_8^1 = 0, & y_8^2 = 0 \end{bmatrix}. \tag{12}$$

d^H is the "high dividends" and d^L is the "low dividends" states.

2.4.c Step 3: The equation system of a stable Markov RBE

We study stable equilibria in which the variables map into an invariant set. We do so not because these are the only stable equilibria but because these are simple and computable equilibria which bring out the essential phenomena of interest. Now, observe that (6c)–(6d) require an agent to forecast prices (p_{t+1}, q_{t+1}). But if prices live in an invariant set, there is a set of 8 prices that can occur at $t+1$ and all agents have observed them in past data. For simplicity we assume *agents in all generations place positive probabilities only on prices which occur in equilibrium*[8]. We use the set S to define equilibrium consumptions, portfolios and prices in terms of transitions from states s to j in S. To do that denote by $(Q^k(j\,|s, y_s^k)k$'s probability of state j given state s and the value of y_s^k which k perceives at state s but under the competitive assumption that k *knows neither the map (12) nor the fact that he influences prices.* Now restate equations (6)–(7) for $k = 1, 2$ and $j, s = 1, 2, ..., 8$:

$$c_s^{1k} = \omega^k - \theta_s^k p_s - b_s^k q_s \tag{13a}$$

$$c_{sj}^{2k} = \theta_s^k (p_j + 1) + \frac{b_s^k}{d_j} \tag{13b}$$

$$-(c_s^{1k})^{-\gamma_k} p_s + \beta_k \sum_{j=1}^{8} (c_{sj}^{2k} d_j)^{-\gamma_k} (p_j + 1) d_j Q^k(j\,|s, y_s^k) = 0 \tag{13c}$$

$$-(c_s^{1k})^{-\gamma_k} q_s + \beta_k \sum_{j=1}^{8} (c_{sj}^{2k} d_j)^{-\gamma_k} Q^k(j\,|s, y_s^k) = 0. \tag{13d}$$

[8] Hence, given *this structure of beliefs* our RBE is a dynamically incomplete Radner equilibrium. It is distinguished from Radner (1972) in that (i) the state space is endogenously expanded to include the unobservable states of belief, a component not in Radner, (ii) agents do not know the equilibrium map and, (iii) agents hold Rational Beliefs. Since alternate rational beliefs with different probability spaces are compatible with the same primitive exogenous specifications of the economy, some RBE are Radner equilibria and others are not.

$$\theta_s^1 + \theta_s^2 = 1 \ \text{ for all } \ s \tag{13e}$$

$$b_s^1 + b_s^2 = 0 \ \text{ for all } \ s. \tag{13f}$$

Our Markov RBE is then a solution of the 192 equations (13a)–(13f)(note: (13b) consists of 128 equations) for feasible parameters and for specifications of $Q^k(j\,|s,y_s^k)$ which constitute rational belief with respect to the empirical distribution implied by the dynamics of (13a)–(13f).

2.4.d Step 4: Defining the rationality conditions which (Q^1, Q^2) must satisfy

Since the stationary measure of price sequences is Markov, it is fully specified by the matrix Γ to be defined in Step 5. We know that the rationality conditions require the beliefs of the agents to imply the same stationary measure implied by Γ. Now, the probabilities used in (6c)–(6d) are $Q^k((\bullet)|y^k)$, and hence the rationality of belief conditions must apply to $Q^k((\bullet)|y^k)$. These conditions require:

(i) $Q^k((\bullet)|y^k)$ is a stable measure for Q^k almost all y^k sequences;
(ii) the stationary measure of $Q^k((\bullet)|y^k)$ equals the probability on sequences induced by Γ.

Since $Q^k((\bullet)|y^k)$ is represented by two Markov matrices, we need to specify the joint distribution of (p_t, q_t, y_t^k) and the rationality conditions which are consistent with these matrices. To that end we use Theorem 2 (the Conditional Stability Theorem). It says that if the probability Q^k of the *joint process* $\{(p_t, q_t, y_t^k), t = 1, 2, ...\}$ is stable, then a conditional probability $Q^k((\bullet)|y^k)$ is a stable probability on $\{(p_t, q_t,), t = 1, 2, ...\}$ and the stationary measure of $Q^k((\bullet)|y^k)$ is the *marginal* of Q^k on (p_t, q_t) obtained by integrating on y^k.

To simplify this procedure we *assume* that under Q^k, the marginal distribution of y_t^k is i.i.d. with $Q^k\{y_t^k = 1\} = \alpha_k$ for $k = 1, 2$. By Theorem 2, Q^1 and Q^2 are characterized by two pairs of (yet unspecified) matrices, (F_1, F_2) for agent 1 and (G_1, G_2) for agent 2, such that:

Q^1*for agent 1:* adopt F_1 if $y_t^1 = 1$ $\quad$ Q^2*for agent 2:* adopt G_1 if $y_t^2 = 1$ (14a)
adopt F_2 if $y_t^1 = 0$ $\quad$ adopt G_2 if $y_t^2 = 0$. (14a)

The rationality of belief conditions are then

$$\alpha_1 F_1 + (1 - \alpha_1)F_2 = \Gamma\ , \quad \alpha_2 G_1 + (1 - \alpha_2)G_2 = \Gamma. \tag{14b}$$

For an intuitive interpretation note that these rational agents believe that the price process is not stationary and their beliefs are parameterized by (y_t^1, y_t^2). (14b) implies that the sequence of matrices which they adopt generates the same empirical distribution as the Markov transition Γ. α_1 is the frequency at which agent 1 uses matrix F_1 and α_2 is the frequency at which agent 2 uses matrix G_1. We thus summarize the key properties of the RBE which we are constructing:

(i) It is a Markov Competitive Equilibrium which is a solution of (13a)–(13f);
(ii) The beliefs (Q^1, Q^2) are defined by (14a) and satisfy the rationality conditions (14b).

Clarification: The exogenous dividend process. To further clarify the properties of the RBE note that a belief Q^k is a probability on sequences $\{(p_t, q_t, d_t, y_t^k), t = 1, 2, ...\}$ but Q^k was also defined by a selection of transition matrices from (p_t, q_t) to (p_{t+1}, q_{t+1}). This appears to ignore the exogenous variable d_t. This is not so since the map Φ in (12) implies that the probability of d^H equals the probability of prices $\{1, 2, 3, 4\}$ and the probability of d^L equals the probability of prices $\{5, 6, 7, 8\}$. Thus, the distribution of d_t is defined by the partition of the state space. Agents discover this partition and for simplicity we have assumed that they believe it to be the truth.[9] We could have assumed the dividend process is non-stationary, represented by a sequence of Markov regimes which *average out* to the empirical matrix (10a). This can be done by using the techniques of assessment variables for the true dividend process. However, true technological regimes have no impact on prices unless agents perceive dividends to be non-stationary; prices do not respond to unobserved reality, they respond only to perceptions. Our simplified assumption means agents believe price maps are non-stationary. Hence, given a dividend state, they may be *bullish* or *bearish* about prices. Step 6 incorporates these ideas in matrices (F_1, F_2) and (G_1, G_2).

2.4e Step 5: Specifying the stationary measure

We assemble the conditions Γ must satisfy. Recall that (10a) specified the dividend process and since prices are functions of (d_t, y_t^1, y_t^2), the marginal of Γ with respect to d_t must equal the dividend matrix in (10a). Similarly with respect to (y_t^1, y_t^2): the marginal of Γ with respect to each of the y_t^k must be i.i.d. with probability α_k.

Each agent has a marginal i.i.d. distribution on his own assessment, hence the i.i.d. requirement on the marginals of Γ with respect to each y_t^k is a consistency condition between the market statistics and what each agent perceives. No such conditions apply to the *joint* distribution of the assessments. The joint effect, as distinct from the individually perceived effect of the assessment variables, is that part of Γ which describes the externalities of beliefs in the market equilibrium. These externalities reflect the interaction among the agents which result from communication in society, and how real variables (e.g. dividends) affect this interaction.

[9] By studying the relationship between prices and d agents discover the partition in the long run data. This happens to be the truth at all dates but an agent may not believe it. Instead he may form a rational belief about this variable. To avoid complicating the model we chose the simpler assumption.

In sum, the matrix Γ must satisfy the following:

$$\text{the marginal on } y_t^k \text{ is i.i.d. with} \quad P\{y_t^k = 1\} = \alpha_k \quad \text{for} \quad k = 1, 2; \tag{15a}$$

$$\text{the marginal on } d_t \text{ is Markov as specified by the dividend process (10a);} \tag{15b}$$

The family of matrices which satisfy these conditions is *very limited.* Our criterion for selecting the following matrix Γ from this family is simplicity and flexibility in parameterization:

$$\Gamma = \begin{bmatrix} \phi A, (1-\phi)A \\ (1-\phi)B, \phi B \end{bmatrix} \tag{16}$$

where A and B are 4×4 matrices which are characterized by the 10 parameters α_1, α_2, and (a, b) where $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4)$:

$$\begin{aligned} A &= \begin{bmatrix} a_1, \alpha_1 - a_1, \alpha_2 - a_1, 1 + a_1 - \alpha_1 - \alpha_2 \\ a_2, \alpha_1 - a_2, \alpha_2 - a_2, 1 + a_2 - \alpha_1 - \alpha_2 \\ a_3, \alpha_1 - a_3, \alpha_2 - a_3, 1 + a_3 - \alpha_1 - \alpha_2 \\ a_4, \alpha_1 - a_4, \alpha_2 - a_4, 1 + a_4 - \alpha_1 - \alpha_2 \end{bmatrix}, \\ B &= \begin{bmatrix} b_1, \alpha_1 - b_1, \alpha_2 - b_1, 1 + b_1 - \alpha_1 - \alpha_2 \\ b_2, \alpha_1 - b_2, \alpha_2 - b_2, 1 + b_2 - \alpha_1 - \alpha_2 \\ b_3, \alpha_1 - b_3, \alpha_2 - b_3, 1 + b_3 - \alpha_1 - \alpha_2 \\ b_4, \alpha_1 - b_4, \alpha_2 - b_4, 1 + b_4 - \alpha_1 - \alpha_2 \end{bmatrix}. \end{aligned} \tag{17}$$

If $A \neq B$ then the distribution of (y_{t+1}^1, y_{t+1}^2) depends upon d_t. (17) implies that $P\{y_t^k = 1\} = \alpha_k$ for $k = 1, 2$ as required in (15a). Note, however, that although each process $\{y_t^k, t = 1, 2, ...\}$ for $k = 1, 2$ is very simple, the joint process $\{(d_t, y_t^1, y_t^2), t = 1, 2, ...\}$ may be complex: it permits correlation among the three central variables and these effects are important. If we set $\alpha_1 = \alpha_2 = 0.5$ and $a_1 = b_1 = 0.25$ for $i = 1, 2, 3, 4$ then all correlations are eliminated. In this case the stationary distribution $(\pi_1, \pi_2, ..., \pi_8)$ implied in (17) is $\pi_i = 0.125$ for all i. If, in addition, the agents adopt the stationary measure as their belief (i.e. $F_1 = G_1 = \Gamma$), then we have exactly an REE.

For simplicity of parameterization, we set in almost all simulations the parameter values $\alpha_1 = \alpha_2 = 0.57, a = (a_1 \neq a_2 = a_3 = a_4)$ and $b = (b_1 \neq b_2 = b_3 = b_4)$. There are natural feasibility conditions which the parameters must satisfy and these are discussed later. We specify now the matrices (F_1, F_2) and (G_1, G_2):

2.4f Step 6: Specifying (F_1, F_2) and (G_1, G_2), a family of bullish/bearish beliefs

We use two parameters λ and μ to specify the matrices (F_1, F_2) of agent 1 and (G_1, G_2) of agent 2 satisfying the rationality conditions (14b). To do that denote the row vectors of A and B by:

$$A^j = (a_j, \alpha_1 - a_j, \alpha_2 - a_j, 1 + a_j - (\alpha_1 + \alpha_2)) \quad j = 1, 2, 3, 4$$

$$B^j = (b_j, \alpha_1 - b_j, \alpha_2 - b_j, 1 + b_j - (\alpha_1 + \alpha_2)) \quad j = 1, 2, 3, 4.$$

With this notation we define the 4 matrix functions of a real number z as follows:

$$A_1(z) = \begin{bmatrix} zA^1 \\ zA^2 \\ zA^3 \\ zA^4 \end{bmatrix}, \quad A_2(z) = \begin{bmatrix} (1-\phi z)A^1 \\ (1-\phi z)A^2 \\ (1-\phi z)A^3 \\ (1-\phi z)A^4 \end{bmatrix},$$

$$B_1(z) = \begin{bmatrix} zB^1 \\ zB^2 \\ zB^3 \\ zB^4 \end{bmatrix}, \quad B_2(z) = \begin{bmatrix} (1-(1-\phi)z)B^1 \\ (1-(1-\phi)z)B^2 \\ (1-(1-\phi)z)B^3 \\ (1-(1-\phi)z)B^4 \end{bmatrix}. \qquad (18)$$

Finally we define

$$F_1 = \begin{bmatrix} \phi A_1(\lambda), & A_2(\lambda) \\ (1-\phi)B_1(\lambda), & B_2(\lambda) \end{bmatrix} \quad G_1 = \begin{bmatrix} \phi A_1(\mu), & A_2(\mu) \\ (1-\phi)B_1(\mu), & B_2(\mu) \end{bmatrix}. \qquad (19)$$

By the rationality conditions (14b), $F_2 = \frac{1}{1-\alpha_1}(\Gamma - \alpha_1 F_1), G_2 = \frac{1}{1-\alpha_2}(\Gamma - \alpha_2 G_1)$.

To motivate this construction, note that the parameters λ and μ are proportional revisions of the conditional probabilities of states $(1, 2, 3, 4)$ and $(5, 6, 7, 8)$ relative to Γ. $\lambda > 1$ and $\mu > 1$ imply increased probabilities of $(1, 2, 3, 4)$ in matrix F_1 and matrix G_1 where the first four prices are associated with the $d_t = d^H$ states. Since these are the states of high prices, $\lambda > 1$ implies 1 is optimistic about high prices at $t + 1$. Similarly for $\mu > 1$. In all simulations $\lambda \geq 1$ and $\mu \geq 1$ and hence the interpretation of y_t^k is simple: when $y_t^k = 1$ agent k is optimistic (*relative to* Γ) at t about high prices at $t + 1$. The case $\lambda = 1, \mu = 1$ and $a_i = b_i = 0.25$ defines an REE. Finally, it turns out that the concepts of "agreement" and "disagreement" between the agents are useful. We say that *the agents agree if* $y_t^1 = y_t^2$ *and disagree if* $y_t^1 \neq y_t^2$.

Clarification: The assumption of competitive behavior. The assumption of competitive behavior in (13a)–(13f) is subtle and needs clarification. From the equilibrium map (12) it is clear that when we have a finite number of agents, the belief of each agent has an effect on equilibrium prices.

Competitive behavior means that an agent is required to disregard his effect on prices. To see what this entails observe from (12) that agent 1 uses matrix F_1 when $y_t^1 = 1$ but in those dates only prices $\{1, 2, 5, 6\}$ are realized contrary to his belief that all prices could be realized. If the agent takes into account his effect on prices, he would use this information in formulating his belief. This is what he is not allowed to do. The simple way we impose competitive behavior is to adopt the strong condition which Kurz (1998) calls "anonymity". It requires that λ and μ be constant and not vary with prices. This assumption also simplifies the parameter space.

2.4g Step 7: The parameter space and the feasibility conditions on the model parameters

To ensure non-negative probabilities the parameters need to satisfy a large number of feasibility conditions. The parameters $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), \alpha_1, \alpha_2$ and ϕ must satisfy.

$$\begin{aligned} a_i, b_i &\leq \alpha_1 < 1 \quad \text{for} \quad i = 1, 2, 3, 4 \\ a_i, b_i &\leq \alpha_2 < 1 \quad \text{for} \quad i = 1, 2, 3, 4 \\ 0 &\leq \phi \leq 1. \end{aligned} \tag{20}$$

Similarly, the selection of (λ, μ) is restricted by 10 inequality constraints:

$$\begin{gathered} \lambda \leq \frac{1}{\phi} \quad \mu \leq \frac{1}{\phi} \quad \lambda \leq \frac{1}{1-\phi} \quad \mu \leq \frac{1}{1-\phi} \\ \lambda \leq \frac{1}{\alpha_1} \quad \mu \leq \frac{1}{\alpha_2} \quad \lambda \geq \frac{\alpha_1 + \phi - 1}{\phi\alpha_1} \quad \mu \geq \frac{\alpha_2 + \phi - 1}{\phi\alpha_2} \\ \lambda \geq \frac{\alpha_1 - \phi}{(1-\phi)\alpha_1} \quad \mu \geq \frac{\alpha_2 - \phi}{(1-\phi)\alpha_2}. \end{gathered} \tag{21}$$

The family of RBE which we study in the simulations is drastically simplified as follows:

(i) a single intensity parameter $\lambda = \mu$;
(ii) a single frequency parameter $\alpha = \alpha_1 = \alpha_2$;
(iii) $a = (c_1, c_2, c_2, c_2) = b$ for two parameters (c_1, c_2);
(iv) γ_1 and γ_2 in the realistic interval [2.5, 3.5] (see Kurz and Beltratti 1997, pp. 290–294); β_1 and β_2 are in the empirically estimated interval [0.85, 0.95].

For realistic values for $\gamma_1, \gamma_2, \beta_1$ and β_2, equilibrium is uniquely determined by λ, α, c_1 and c_2.
Two comments:

(i) *The model unit of time.* β_1 and β_2 being around 0.9 means that the model is calibrated to annual data. Hence, fluctuations of the distribution of beliefs are slow and the model captures the effects of beliefs on relatively long swings of bull and bear markets.
(ii) *The large number of free parameters.* The constructed RBE could have a large number of free parameters and some may view this as a weakness of the RBE theory. We imposed strong restrictions to reduce the number of free parameters to four: $\lambda, \alpha, c_1, c_2$ and the parameter values selected have a clear economic meaning. With four parameters we (i) explain 5 moments of the time series of returns in the U.S. and, (ii) explain the GARCH property of asset returns, the predictability of long term returns on assets and the forward discount bias in foreign exchange. This set of parameter values is unique: any significant deviations from the specified values lead to the failure of the model to explain some component of the record. Hence, the economic

meaning of the belief parameters is offered as *a theoretical* explanation of the characteristics of market volatility. We hope the ideas developed here are tested empirically using data on the distribution of beliefs (see Chernozhukov and Morozov, 1999). In addition, the success of our simulation results do not prove that the theory is correct. However, they are encouraging and suggest that the model offers a first step in the development of a new and alternative view of market volatility.

3 Endogenous uncertainty and volatility: simulation results

We now examine the model's ability to simulate the real economy. To do that we first review the empirical averages of the seven key model variables in the U.S.:

p – the long term price/dividend ratio. M&P (1985) used Shiller's (1981) data for 1889–1978. We updated Shiller's data for 1889–1998 and estimated this variable to be 22.84;

σ_P – the standard deviation of the price/dividend ratio p. For the period 1889–1998 we estimated it to be 6.48 using the updated version of Shiller's (1981) data base;

R – the average risky return on equities was estimated by M&P (1985) to be 6.98%. Using the updated Shiller (1981) data for 1889–1998 our estimate is 8.34% suggesting that 6.98% is on the low side. We thus record the mean risky rate to be around 8.00%;

σ_R – the standard deviation of R was estimated by M&P (1985) to be 16.67%. Using the updated data for 1889–1998 our estimate is 18.08%;

r^F – M&P (1985) estimated the mean riskless rate to be 0.80% for 1889–1978 based on the 90 day treasury bill rate for 1931–1978. For 1889–1931 one may use alternate securities. We offer no independent estimate and accept the view that the evidence places the mean riskless rate around 1.00%. Some evidence suggests that a low rate has prevailed mostly since the Great Depression and that prior to 1931 the rate was higher (see Siegel, 1994);

σ_{r^F} – the standard deviation of r^F was estimated by M&P (1985) to have an average of 5.67%;

ρ – the premium of equity return over the riskless rate. With the mean of r^F set around 1.00% and the mean of R estimated at 8.00%, the approximate mean equity premium is 7.00%.

3.1 The scaling problem of OLG models

Before proceeding we resolve the issue of scaling an RBE. In an OLG economy agents live two periods and the young purchase from the old the capital stock of the economy. Hence, equilibrium price $p_t = \frac{P_t}{D_t}$ depends upon the labor endowment of the young. In the real economy it takes a generation for the capital stock to change ownership from old to young, hence an OLG model is problematic. If labor income of the young is – as in the U.S. – approximately twice the magnitude of capital income, the model will not generate a price $p = 23$. Hence, the young's labor endowment must be a multiple of D_t in any one year in order to attain an equilibrium p equal to the average of about 23. Table 1 below presents the simulated equilibrium mean values of (p, r^F, R, ρ) in a sequence of REEs in which $\omega = \omega^1 = \omega^2$ take different values. Other parameters values are: $a_i = b_i = 0.25$ for all i; $\lambda = \mu = 1$; $\alpha_1 = \alpha_2 = 0.5$; $\gamma_1 = \gamma_2 = 3.25$; $\beta_1 = \beta_2 = 0.90$.

Table 1. REE solutions for varying values of $\omega = \omega^1 = \omega^2$

	$\omega = 12$	$\omega = 14$	$\omega = 18$	$\omega = 22$	$\omega = 23$	$\omega = 24$	$\omega = 25$	$\omega = 26$
p	11.39	13.35	17.26	21.17	22.15	23.13	24.11	25.09
r^F	10.24%	8.93%	7.21%	6.13%	5.92%	5.72%	5.54%	5.38%
R	10.75%	9.44%	7.71%	6.62%	6.41%	6.21%	6.04%	5.87%
ρ	0.51%	0.51%	0.50%	0.49%	0.49%	.49%	.49%	.49%

Table 1 shows that ω acts as a scaling factor determining the levels of prices. For $\omega = 24$ we have $p = 23.13, R = 6.21\%$ and both are close to the historical average. We thus select ω so that the model predicts p around 23. *For the REE we use* $\omega = 24$ *and for the RBE,* $\omega = 26$. Hence the model does not reproduce the empirical mean of $p = 23$, *it is scaled to that level.*

The problem of scaling the OLG model raises a deeper question: why should an unrealistic OLG model be a useful tool for the study of market volatility? The model is clearly simplistic and the results of this paper may not generalize to more complex, infinite horizon models. We think, however, that there are reasons to hope that our model is a useful first approximation, suggesting the kind of analysis one would carry out in more realistic models. The first indication of this are the results to be reported in Table 2: once the model is scaled, the prediction of the REE closely reproduce all the predictions of the infinite horizon M&P (1985) model.

The second reason is analytical. The Euler equations (6c)–(6d) are *exactly* the same for the OLG and for the infinite horizon models. The differences are in the budget constraint and in the formulation of the transversality conditions. Also, the growth rates of dividends and aggregate consumption are the same in the OLG as in the M&P (1985) infinite horizon model. Hence, the crucial difference between the date t portfolio demands of the two models are the effects of date $t-1$ portfolio distribution. Given this, why should the two models generate different predictions? If the equity premium and other volatility characteristics are determined by real factors such as the horizon of the agents' optimization or by the long horizon life cycle saving patterns, then the OLG and the infinite horizon models would yield different results. However, if the characteristics of price volatility are essentially driven by expectations, then it would make little difference whether agents trade many times over their own life-time or only once: their expectations for each date at a time will drive the results. In that case the OLG model would be a useful tool for the study of market volatility.

3.2 REE simulations: matching the M&P (1985) results

Table 2. REE results

variable	REE	Empirical record
p	23.13	23
σ_p	0.069	6.48
R	6.21%	8.00%
σ_R	4.12%	18.08%
r^F	5.72%	1.00%
σ_{r^F}	0.88%	5.67%
ρ	0.49%	7.00%

Assuming $\omega^1 = \omega^2 = 24$ we study the REE defined by the parameter values: $a_i = b_i = 0.25$ for all $i; \lambda = \mu = 1; \alpha_1 = \alpha_2 = 0.5; \gamma_1 = \gamma_2 = 3.25$ and $\beta_1 = \beta_2 = 0.90$. The results in Table 2 represent what M&P (1985) call "the equity premium puzzle," which is the observation that the model predicts $\rho = 0.49\%$ while the historical average is 7.00%. Exactly as in M&P (1985), this REE approximates well the mean rate of return on equities but errs in predicting a riskless rate of 5.72% when the empirical record is 1.00%. Table 2 shows that our OLG model reproduces very well the M&P (1985) results. Note, however, that the equity premium is not the only problem which the REE presents; all REE volatility measures are low relative to the historical record. The empirical estimates of σ_p is 94 times larger than the REE prediction, of σ_R is more than 4 times larger than the REE prediction and of σ_{r^F} is over 6 times larger than the model prediction. Before exploring the volatility of the RBE we make two additional observations about Table 2:

(i) The model prediction of σ_p is downward biased (in both the REE and RBE) since we assume, with M&P (1985), that dividends and GNP are proportional. Under the realistic assumption that profits are more volatile than GNP, the model predictions of σ_p would become larger, but not large enough to alter the general result for the REE in Table 2.

(ii) The 5.67% historical estimate of σ_{r^F} is downward biased relative to the model since during the second half of the 20th century, monetary policy tended to stabilize short term rates. Such a policy is not in the model. Indeed, there is some evidence that before the Great Depression σ_{r^F} was substantially higher than 5.67% (see Siegel, 1994).

3.3 A family of RBE with bulls/bears

We study a family of "optimists / pessimists" RBE. For this family we scale the model by selecting $\omega^1 = \omega^2 = 26$ and the four parameters which characterize this family are as follows:

(i) *The degree of optimism is* $\lambda = \mu = 1.7542$. When $y_t^k = 1, k$ is optimistic and he adjusts the probabilities of high prices at $t + 1$ by 1.7542 which is the maximal feasible value $\lambda = \mu \simeq \frac{1}{\alpha_i}$.

(ii) *Frequency of optimism is* $\alpha = \alpha_1 = \alpha_2 = 0.57$. In 57% of the dates an agent is optimistic but only at 43% of the dates he is pessimistic. In a large economy this means the optimists are always in the *majority*. We shall see that this also means that pessimists are more intense than optimists;

(iii) *Correlation of belief:* $a = b$ with $a_1 = b_1 = c_1 = 0.50$ and $a_i = b_i = c_2 = 0.14$ for $i = 2, 3, 4$. To understand the nature of this correlation we observe that a random variable which sums up this effect is L_t : (i) $L_t = 1$ if $y_t^1 = y_t^2 = 1$ denoted OO (both agents are optimistic); (ii) $L_t = 0$ if $y_t^1 = y_t^2 = 0$ denoted PP (both agents are pessimistic) and (iii) $L_t = 2$ if $y_t^1 \neq y_t^2$ denoted DIS (the agents disagree). We know that $\{L_t, t = 1, 2, ...\}$ is a Markov process with transition matrix:

	$(\text{OO})_{t+1}$	$(\text{PP})_{t+1}$	$(\text{DIS})_{T+1}$
$(\text{OO})_t$	0.50	0.36	0.14
$(\text{PP})_t$	0.14	0.00	0.86
$(\text{DIS})_t$	0.14	0.00	0.86.

(22)

The values $a_i = b_i = c_2 = 0.14$ for $i = 2, 3, 4$ imply that if at t the state is PP or DIS, then at $t + 1$ the state must be OO or DIS; PP *cannot occur* at $t + 1$. The values $a_1 = b_1 = c_1 = 0.50$ imply that total optimism at t can be followed by any state at $t + 1$. Hence, the correlation takes the form:

(i) unanimous optimism at t may lead to any state of belief at $t + 1$;
(ii) unanimous pessimism or disagreement at t *prevents total pessimism* at $t + 1$.

The emergence of asymmetries in an otherwise symmetric economy affects volatility since price movements are caused by the joint movement of d_t and (y_t^1, y_t^2). Asymmetry in the transition t of (y_t^1, y_t^2) translates into asymmetry in the dynamics of prices. We explore the pattern later.

We now report the simulation results for $\gamma = \gamma_1 = \gamma_2$ from 2.5 to 3.5 and $\beta = \beta_1 = \beta_2$ from 0.85 to 0.95. Table 3 shows that for this parameterization, the model predicts well the historical record. The mean risky return R is close to the average of 8.00% and its standard deviation σ_R is close to 18.08%; the riskless rate is within range of the average of 1.00%, and the equity premium is close to the average of 7.00%. The moments σ_p and σ_{r^F} deviate somewhat from the historical record: (i) the record of σ_p is 6.48% while the model predictions are smaller, around 2.5%–3.4%, and (ii) the record of σ_{r^F} is 5.67% while the model predictions are higher, around 14.2%–19.4%. Both are of the correct order of magnitudes of the record, and the sizes and signs of the deviations are explained by the two model biases noted at the end of Section 3.2.

3.4 Interpreting the propagation mechanism of the RBE

Why is the RBE able to explain the data? Also, since an RBE has a propagation mechanism for market volatility, what is the economic interpretation of the parameter choices and why do they enable the model to explain the record? A

Table 3. Results for RBE with optimists\pessimists

		γ = 2.5	γ = 2.75	γ = 3.00	γ = 3.25	γ = 3.50
β = 0.85	p	23.06	23.12	23.19	23.26	23.34
	σ_p	2.53	2.78	3.00	3.20	3.36
	R	7.85%	8.19%	8.51%	8.80%	9.05%
	σ_R	18.76%	20.63%	22.27%	23.69%	24.89%
	r^F	2.36%	1.79%	1.22%	0.66%	0.12%
	σ_{r^F}	14.62%	16.12%	17.41%	18.48%	19.35%
	ρ	5.49%	6.40%	7.29%	8.14%	8.93%
β = 0.90	p	23.36	23.38	23.43	23.48	23.54
	σ_p	2.52	2.77	2.99	3.18	3.34
	R	7.75%	8.08%	8.39%	8.68%	8.93%
	σ_R	18.48%	20.32%	21.94%	23.35%	24.55%
	r^F	2.37%	1.81%	1.25%	0.71%	0.18%
	σ_{r^F}	14.40%	15.89%	17.17%	18.24%	19.11%
	ρ	5.38%	6.27%	7.14%	7.97%	8.75%
β = 0.95	p	23.64	23.63	23.66	23.69	23.74
	σ_p	2.51	2.76	2.97	3.16	3.28
	R	7.65%	7.98%	8.29%	8.57%	8.61%
	σ_R	18.22%	20.03%	21.64%	23.03%	23.40%
	r^F	2.37%	1.83%	1.29%	0.75%	0.04%
	σ_{r^F}	14.20%	15.67%	16.95%	18.02%	19.04%
	ρ	5.28%	6.15%	7.00%	7.82%	8.57%

skeptical view could suggest that even the tight space of parameters specified in Section 2.4g is sufficiently large for the success of the model to be a chance event. Here we focus on three main facts:

(A) Only a very small neighborhood of parameters enables the model to match the empirical record. Also, only the two parameters (α, λ) are really used to explain the five moments while the other two parameters contribute to explain diverse price dynamics phenomena.
(B) There is no other neighborhood of feasible parameters defining RBE which match the data;
(C) The values ($\alpha = \alpha_1 = \alpha_2 = 0.57, \lambda = \mu = 1.7542$) entails a simple economic interpretation: *in this RBE optimists are in the majority but the pessimists are more intense than the optimists.*

We start by an examination of the small neighborhood mentioned in fact (A). Table 4 reports the results of varying the parameters α_1 and α_2 over the range of 0.56–0.58. The results are sensitive to variations of (α_1, α_2) and of $\lambda = \mu$. The feasibility conditions in (20)–(21) show that small changes in α_1 and α_2, require changes of other parameters. For example, if α_1 is changed from 0.57 to 0.58, the feasible values of $\lambda \simeq \frac{1}{\alpha_1}$ changes to 1.7241, of c_2 to 0.15 but the value of c_1 remains equal to 0.50. In all cases $\beta = 0.90$ and $\gamma = 3.25$.

Moving on to fact (B) we observe that no other parameter configuration yields predictions which are *simultaneously* close to the empirical record. Many feasible model parameters generate volatility of prices and returns. However, as we move away from the small neighborhood under discussion, the model fails to

Table 4. Results for the parameter neighborhood

		$\alpha_1 = 0.56$	$\alpha_1 = 0.57$	$\alpha_1 = 0.58$
$\alpha_2 = 0.56$	p	23.56	23.59	23.97
	σ_p	2.69	2.79	2.11
	R	7.95%	8.09%	7.19%
	σ_R	19.86%	20.60%	15.84%
	r^F	3.32%	1.56%	1.41%
	σ_{r^F}	17.09%	16.42%	12.12%
	ρ	4.63%	6.53%	5.78%
$\alpha_2 = 0.57$	p	23.59	23.48	23.90
	σ_p	2.79	3.18	2.22
	R	8.09%	8.68%	7.31%
	σ_R	20.60%	23.35%	16.52%
	r^F	1.56%	0.71%	0.92%
	σ_{r^F}	16.42%	18.24%	12.70%
	ρ	6.53%	7.97%	6.39%
$\alpha_2 = 0.58$	p	23.97	23.90	23.87
	σ_p	2.11	2.22	1.94
	R	7.19%	7.31%	7.00%
	σ_R	15.84%	16.52%	14.35%
	r^F	1.41%	0.92%	1.89%
	σ_{r^F}	12.12%	12.70%	10.96%
	ρ	5.78%	6.39%	5.11%

generate some essential components of the empirical record, mostly the riskless rate and the premium. The reason is that this parameter configuration implies unique asymmetries, to be discussed, which provide the economic reasoning for the behavior of the model. That is, given realistic values of (β_k, γ_k) for $k = 1, 2$, the RBE offers a unique explanation of the historical record which we now explore. We thus turn to (C), the economic interpretation of the family of RBE reported in Tables 3 and 4.

Recall that $\alpha_1 = \alpha_2 = 0.57$ means that both agents are optimistic in 57% of the dates and that $\lambda = 1.7542$ is the maximal ratio by which an optimist at t adjusts the probability of the high prices $((p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4))$ at $t + 1$. To interpret this recall the feasibility conditions (21). Since we assume $\alpha = 1 - \phi = .57$, around $\alpha = 0.57$ the binding feasibility constraints are $\lambda \leq \frac{1}{1-\phi}, \lambda \leq \frac{1}{\alpha}$. Suppose agent 1 is an optimist using F_1. As λ in F_1 rises, the rationality conditions $\alpha F_1 + (1-\alpha)F_2 = \Gamma$ require a downward adjustment of the probability of high prices in the pessimistic matrix F_2. Although the changes of the probabilities in F_2 are linearly adjusted to changes of probabilities in F_1, the rationality conditions (which fix the relation between them) induce an *asymmetry* between the intensities of the two. The term "*intensity*" is defined to measure *the number of states in which the agent holds extreme beliefs so that his transition probability has a value close to 1 at some entry.*

To understand the asymmetry in intensities, note that the matrix (10a) implies that λ reaches its maximal feasible value at about $1.7542 \simeq \frac{1}{\alpha} = \frac{1}{0.57}$. When λ hits the boundary, some entries in F_2 are close to 0. Symmetry appears to dictate a correspondence between the 0 entries in F_2 and the entries of 1 in F_1. At

$(\alpha = 0.57, \lambda = 1.7542)$ this symmetry does not hold. Let f^v_{ij} be the (ij) entry of F_v for $v = 1, 2$. Then $f^1_{ij} = \lambda \Gamma_{ij}$ and $f^2_{ij} = \frac{1}{1-\alpha}[\Gamma_{ij} - \alpha\lambda\Gamma_{ij}]$ for $j = 1, 2, 3, 4$ all i. One can check that round $\alpha = 0.57$ and $\lambda = 1.7542$ we have the following *asymmetry:*

$$\text{For all} \quad i = 1, 2, ..., 8, \quad f^2_{ij} \approx 0 \quad \text{for} \quad j = 1, 2, 3, 4. \tag{23a}$$

$$\text{Only for} \quad i = 5, 6, 7, 8, \quad f^1_{ij} \approx 0 \quad \text{for} \quad j = 5, 6, 7, 8. \tag{23b}$$

Hence we can explain the asymmetry in the following way:

(i) (23a) says that *in all 8 states of the economy* pessimistic agents at t are almost certain that they will experience low prices at $t + 1$;
(ii) (23b) says that optimistic agents at t are almost certain that they will have capital gains at $t + 1$ *only if at t the economy is in states 5, 6, 7, and 8,* the recession states of the economy.
(iii) If at t the economy is in an expansion phase (states 1, 2, 3, 4), the optimistic agent thinks that the probability of a recession at $t + 1$ is about 25%.

In short, the pessimistic agents hold extreme beliefs and adopt a strategy of *capital preservation* in all states. Optimistic agents hold extreme beliefs only when the economy is in a recession and then act as *value investors.* By our definition, the pessimists hold extreme beliefs in more states and hence are more intense than the optimists. Their impact on the bills market is then stronger because *in all states* they rush to sell the stock and buy the bills while the optimists are happy to sell the bills to them only in the recession states of (5,6,7,8). Observe that this asymmetry results from the rationality of belief conditions which are the essence of an RBE.

We finally turn to the interpretation of the parameters $a = b = (0.50, 0.14, 0.14, 0.14)$. These regulate the correlation between y^1_t and y^2_t. This correlation impacts the dynamics of prices and the values $a = b = (0.50, 0.14, 0.14, 0.14)$ imply that bull and bear markets are asymmetric. For the market to transit from the lowest price of the crash states (in a recession $d = d^L$ and the belief state of DIS) to the states of the highest prices (at PP) it must take several steps: it cannot go *directly* from the low to the high prices. The opposite is possible since at the bull market states there is a positive probability of reaching the crash states in one step. *This implies that a bull markets which reaches the high prices must evolve in several steps but a crash can occur in one step.* Substantial empirical evidence suggests that this implication of the parameters is very realistic.

To sum up, we offer a simple and intuitive reason why the RBE generates a low riskless rate and a high equity premium. It proposes that relative to Γ there are, at any time, optimists and pessimists in the population of investors but on average there are more optimists than pessimists. The rationality of belief conditions imply that the intensity level of the pessimists dominates and their high demand for the riskless asset raises its price, leading to a low equilibrium riskless rate and high equity premium. The rationality of belief conditions are essential for this explanation.

3.5 The dynamics of asset prices and returns

We examine now some dynamic characteristics of asset prices under the RBE theory.

(i) The structure of asset price volatility. Figure 1a,b presents time series of model simulations. Each contains 200 realized price\dividend ratios (which we call “the” price) generated by the REE of Table 2 and the RBE of Table 3 with $\beta_1 = \beta_2 = 0.90$ and $\gamma_1 = \gamma_2 = 3.25$. The standard deviation of the price\dividend ratio is 0.069 in the REE and 3.18 in the RBE. There are two distinct prices in the REE: 23.20 and 23.06 with a mean of 23.13. In the RBE there are 6 distinct prices with a conditional mean of 25.82 given d^H, with a conditional mean of 21.14 given d^L and with an unconditional mean of 23.48. We decompose the standard deviation of prices in the RBE into two components. The first component, which is *overshooting*, or an *amplification* of the effect of d_t on prices, is measured by the standard deviation of a random variable which takes the values of 25.82 when $d_t = d^H$ and 21.14 when $d_t = d^L$. Hence, keeping the REE functional relation between prices and exogenous variables, amplification or overshooting increases the impact of exogenous variables on prices.

The second component of volatility is the *pure* effect which the states of belief have on price volatility. This component is uncorrelated with the exogenous dividend process and represents pure Endogenous Uncertainty which takes the form of additional prices induced by the states of beliefs and by the variability of the states of beliefs over time. To define this effect let $z_t^1 = 1$ when $d_t = d^H$ and 0 otherwise, and let $z_t^2 = 1$ when $d_t = d^L$ and 0 otherwise. Now define $e_t = p_t - 25.82z_t^1 - 21.14z_t^2$. In Figure 2 we exhibit 200 values of e_t computed from the simulated values of the RBE in Figure 1b. What is interesting about Figure 2 is the asymmetry in the distribution of e_t which is generated by the basic asymmetry in the causal structure of volatility in this model. We conclude by noting that if we take the volatility of the price\dividend ratio in the REE to be approximately the volatility that can be justified by the dividends, our analysis demonstrates that *most of the volatility of stock prices is generated by the beliefs of the agents* either in the form of price amplification or in the form of pure endogenous volatility. Thus, most of the volatility of asset prices is endogenously generated. However, an examination of the relative contribution of these two components of endogenous volatility shows that *price amplification* or *overshooting* is the more important of the two. We return to this conclusion in (iv) when we discuss the issue of correlation of beliefs. Here we note that our result is consistent with the empirical evidence studied by Campbell and Shiller (1988).

(ii) The GARCH property of asset returns. In Figure 3 we exhibit R_t^2 – the square of the risky returns – associated with the prices generated by the RBE of Figure

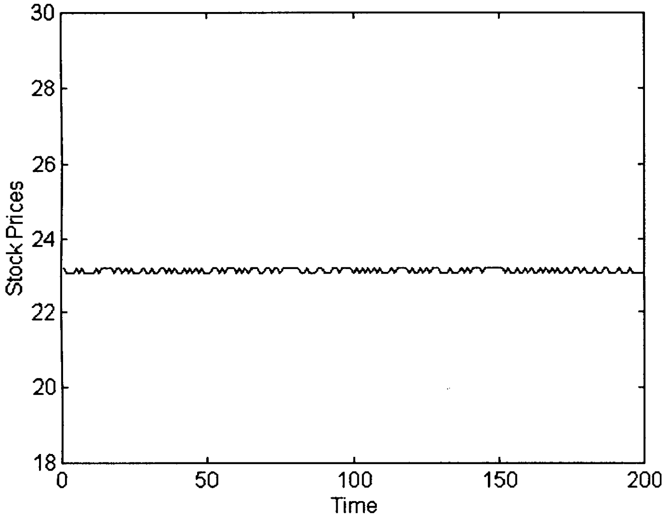

Figure 1a. REE simulation

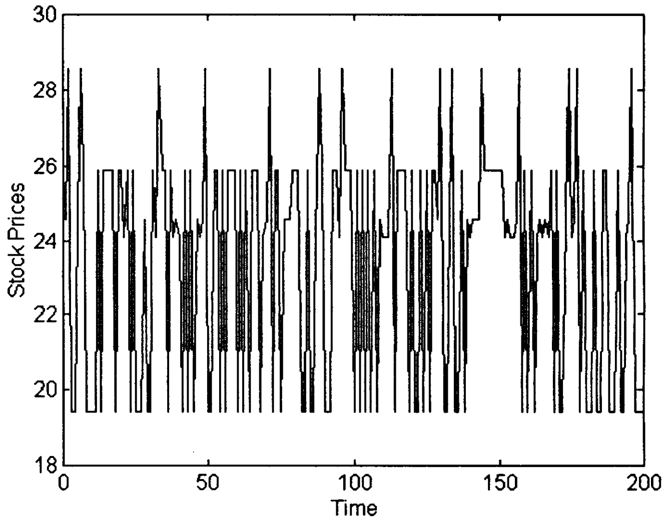

Figure 1b. RBE simulation

1b. Note that the bursts of price volatility in Figure 1b reappear as a GARCH property of asset returns. That is, Figure 3 shows that the variance of the risky rates of return is stochastic. Since the growth of dividends is a stationary Markov process, the stochastic volatility of the risky return is the result of the dynamical properties of the states of belief in the market. What is the cause for the GARCH property of the risky return? To answer this question recall the transition matrix (22) of the state of beliefs. We observe first that a regime of "agreement" (when $y_t^1 = y_t^2$ in states OO or PP) generates price variability which is sharply different from the price variability in the regime of disagreement (when $y_t^1 \neq y_t^2$ in state

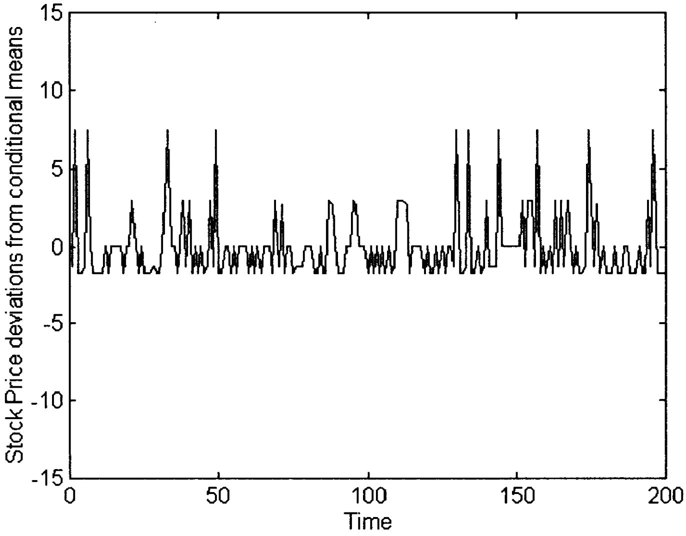

Figure 2. RBE simulation

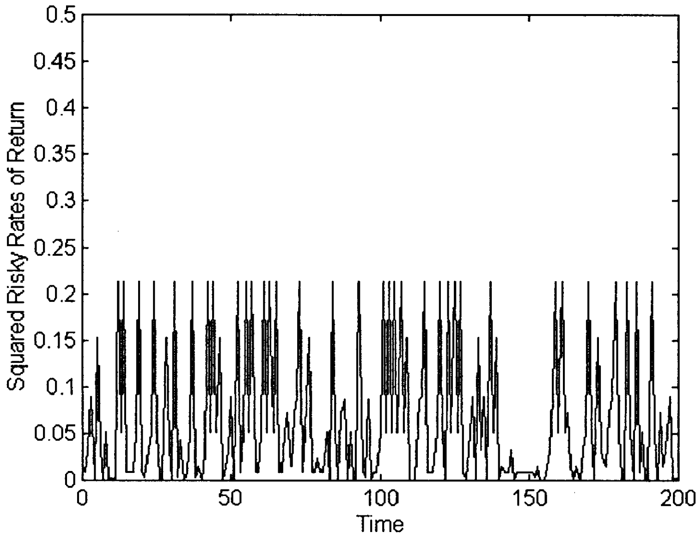

Figure 3. RBE simulation

DIS). Now suppose that at some date the state of belief is OO. From OO the economy can move to all states of beliefs. If it moves to PP it remains in the regime of agreement and if from PP it moves back to OO the market completes a cycle within the regime of agreement. If, however, it moves from PP to DIS, a regime of disagreement is started with sharply different price volatility characteristics. Note the sharp spikes in Figure 2. The highest price occurs only in the regime of agreement when the state of belief is PP while the lowest “crash” price occurs in the recession when $d_t = d^L$ and beliefs are in DIS. As the states of belief change over time, returns move among different volatility regimes. Indeed, stochastic volatility of returns is a Markov process with varying degrees of persistence since the state of belief is a Markov process with varying degrees of persistence. Hence the GARCH property of asset return is caused by the dynamic properties of the regimes of belief.

Table 5. The autocorrelation function of the residuals of the squared return regression

lag	1	2	3	4	5	6	7	8	9	10
	0.026	0.044	0.016	0.007	−0.003	−0.005	0.0007	0.0003	.001	.004
	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)

To formally examine the GARCH property of returns we simulated 100,000 observations of R_t^2 in the RBE and estimated the regression $R_t^2 = \xi_0 + \xi_1 d_t + \varepsilon_t$. We report in Table 5 the first 10 terms of the autocorrelation function of the residual of R_t^2. The first three terms are large and the majority of terms are positive but decline rapidly, a result which is compatible with the empirical record (see Brock and LeBaron, 1996). We have explored several models that may best describe the behavior of the data over time. Following the Akaike Information Criterion, we found that the following E-GARCH(1, 1) model fits the data best:

$$R_t^2 = \underset{(0.0003)}{-0.3192} + \underset{(0.0002)}{0.3541} d_t + \varepsilon_t, \qquad \varepsilon_t \sim N(0, h_t)$$

where

$$\log(h_t) = \underset{(0.0216)}{-5.8139} - \underset{(0.0040)}{0.2873} \log(h_{t-1}) - \underset{(0.0064)}{1.6924} \left| \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}} \right| + \underset{(0.0038)}{0.4938} \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}}.$$

(iii) Regressions of long returns. A large literature has explored the predictability of asset returns. (see, for example, Fama and French, 1988; Poterba and Summers, 1988; Campbell and Shiller, 1988). Despite some disagreement on the interpretation of the empirical record, the stylized facts appear to be as follows: (i) short returns of one day or one month are too volatile and exhibit no statistically significant predictability; (ii) long returns exhibit mean reversion but the effect declines with the returns' length; (iii) the price\dividend ratio is the best variables to predict long returns with correlation coefficients of ten year returns regression around 0.300.

The unit of time in our model is one year and we cannot test the results regarding short returns. To compare the model predictions with the record we generated a random sample of 30,000 observations and estimated the regression models of Fama and French (1988) and of Campbell and Shiller (1988) for duration from 1 year to 20 years. This space is too limited for a detailed report[10] of the results. We can report that they exhibit (i) mean reversion of returns

[10] These detailed results are available in the pre-printed version of this paper which is posted on the web page of the first author at www.stanford.edu\ ∼mordecai\

which declines with the length of returns, (ii) the price\dividend ratio is the best predictor of long returns and the correlation coefficient of the 10 year regression model is .383. These results are consistent with the empirical evidence but differ from some of the details in the papers reported.

(iv) The dynamics and correlation of beliefs. A unique characteristic of the RBE studied earlier is the fact there are 4 potential market states of beliefs (i.e. 4 values of (y_t^1, y_t^2)) and over time, the state of belief fluctuates. This might lead one to conclude the fluctuations of market states of belief are essential to our volatility conclusions. Indeed, in the formulation of the parameter space where we used 4 parameters, two were specifically introduced to regulate the correlation between the beliefs of agent. We also claimed in Section 3.4 (see "Fact A") that only 2 parameters are needed for explaining the moments and the correlation plays an important role only in price dynamics. We now show that fluctuations of market states of belief and correlation among individual states of belief *have little to do with the moments of Tables 3 and 4:* these are fully explained by the two parameters (α, λ) while the correlation is crucial only to understanding the *dynamics of prices.*

To explain the claim above, we follow Kurz (1998) and study an OLG economy as before except that now we introduce a *continuum of agents.* The assessment variables are identically distributed across agents and are *independent across any countable collection of agents.* Assume also that each assessment variable is the same as in the model presented earlier: $y_t^k \in \{0, 1\}$ and is i.i.d. over time with $P(y_t^k = 1) = 0.57$. It follows that in this economy there is only one market state of belief which is (0.57, 0.43). This is the fixed distribution of beliefs in the economy: at any date t, 57% of the agents are optimistic and 43% are pessimistic about high prices at $t + 1$. Now consider the time series of this economy, compute the moments discussed earlier and compare with the moments in, say, the middle box (0.57, 0.57) of Table 4. Table 6 presents the results where column 1 reports the moments in the economy with a continuum of agents. It shows the model with a single market state of belief generates about the same volatility (measured by the moments) as the model reported in Tables 3 and 4. Moreover, the moments predicted by the model with a single market state of belief are reasonably close to the empirical record. These results are explained by exactly the same factor that enables the model to generate the moments reported in Tables 3 and 4. This factor is the amplification property of the RBE due to the asymmetry between the intensities of the pessimists and optimists. That is, what enables both models to generate moments close to the empirical record is that in both models the two parameters satisfy (i) $\alpha_1 = \alpha_2 = 0.57$ and (ii) $\lambda = \mu = 1.754$. These two imply that in both models the optimists are in the majority but the RBE rationality conditions require the pessimists to have a higher intensity level. The correlation parameters have little impact on moments in Tables 3 and 4.

The natural question is then why should we consider RBE with multiple states of belief which fluctuate over time and why should we be concerned with the correlation among individual states of beliefs of agents in the economy?

Table 6. Volatility comparison of models with and without correlation of beliefs

Variable	RBE with a Single Market State of Belief	RBE with Correlation and Four Market States of Belief	The Empirical Record
σ_p	4.10	3.18	6.48
R	9.55%	8.68%	8.00%
σ_R	31.00%	23.35%	18.08%
r^F	0.43%	0.71%	1.00%
σ_{r^F}	24.30%	18.24%	5.67%
ρ	9.98%	7.97%	7.00

The reason is that the model with a single market state of belief generates results which are counter-factual with regard to the dynamics of prices. Examples of such results are: (i) it implies that the variations in prices are *perfectly* correlated with the observed exogenous shocks and hence are *completely* explainable by these exogenous changes; (ii) it is a fact that major market declines are associated with recessions, but it is also a fact that a fraction of major market declines incorrectly forecast recessions that never materialize. This fact contradicts the first implication which holds that the market declines only in recessions. A model with a single state of belief implies that all Endogenous Uncertainty is an amplification of exogenous shocks; (iii) it implies that there are no extreme market price increases and no market crashes; (iv) it fails to generate the stochastic volatility property of asset returns.

In short, there are two central reasons for our analysis of an RBE model in which the market distribution of beliefs fluctuates over time and individual states of belief are correlated. First, it is an empirical fact that existing measures of the distribution of beliefs (e.g. distribution of price and earning forecasts on Wall Street, published forecast distributions of inflation, etc.) fluctuate over time. Second, if the model is to explain the empirical record it must go beyond the simple moments discussed in this paper. It also need to exhibit price dynamics which is compatible with the characteristics of observed price dynamics. We think that the model with fluctuating state of belief and some correlation among beliefs is better suited for that goal.

(v) The forward discount bias in foreign exchange markets. Kurz (1997b) and Black (1997) developed a model which is similar to ours except for the addition of a second country and two more short term nominal debt instruments. To define the problem that was addressed in these papers suppose that you estimate a regression of the form

$$\frac{ex_{t+1} - ex_t}{ex_t} = c + \zeta(r_t^D - r_t^F) + \varepsilon_{t+1} \tag{24}$$

where $(ex_{t+1} - ex_t)$ is the change of the exchange rate between t and $t + 1$ while $(r_t^D - r_t^F)$ is the difference between the short term nominal interest rates in the domestic and the foreign economies. Under rational expectations $(r_t^D - r_t^F)$

should provide an unbiased predictor of the $(ex_{t+1} - ex_t)$. This means that apart from a technical correction for risk aversion, the parameter ζ should be close to 1. In 75 empirical studies ζ was estimated to be significantly less than 1 and in many studies it was estimated to be negative (see Froot, 1990; Engel, 1996 for an extensive survey). The failure of this parameter to exhibit estimated values close to 1 is known as the "Forward Discount Bias" in foreign exchange markets. Applying the RBE theory to this problem, Kurz (1997b) and Black (1997) estimated ζ to be 0.152. However, their specifications were different from ours and violated the condition of anonymity which we have imposed. We thus discuss first the reformulation of the model for this narrower parameterization.

We think of the first agent as the "domestic U.S." and the second agent as a "foreign economy" and hence need to allow the introduction of two nominal interest rates, two different monetary policies and a different stochastic structure. We thus assume that there is only one stock market in the home currency and the stochastic process of dividends is as in (10a). As in our model above we also assume that the endowment\dividend ratio of the domestic agent is a constant ω and the domestic economy has a real bill which is traded by both agents. But then, how should we model the second country? What is the meaning of an exogenous shock in the foreign country? With such difficulties we (along with Kurz, 1997b; Black, 1997) model a *hypothetical* foreign economy which is characterized as follows:

(1) the endowment\dividend ratio ω^* of the foreign agent is a random variable with two states $(\omega^{*H}, \omega^{*L})$ which is i.i.d. with the probability of $\omega^* = \omega^{*H}$ being 0.8;

(2) the shocks to endowment are small, say of 2%–3% hence in the REE $\omega^{*H} = 24.6$ and $\omega^{*L} = 23.4$ and in the RBE $\omega^{*H} = 26.6$ and $\omega^{*L} = 25.4$. Monetary policy in the home economy is responsive to the dividend shocks and monetary policy in the foreign country is responsive to the endowment shock in the foreign economy. The main reason for the endowment shock in the foreign economy is to allow the determination of the exchange rate in any REE;

(3) an RBE requires a selection of a Γ^* matrix to generate the stationary measure of the equilibrium dynamics. A matrix that satisfies the requirements specified is

$$\Gamma^* = \begin{bmatrix} 0.8\phi A & 0.8(1-\phi)A & 0.2\phi A & 0.2(1-\phi)A \\ 0.8(1-\phi)B & 0.8\phi B & 0.2(1-\phi)B & 0.2\phi B \\ 0.8\phi C & 0.8(1-\phi)C & 0.2\phi C & 0.2(1-\phi)C \\ 0.8(1-\phi)D & 0.8\phi D & 0.2(1-\phi)D & 0.2\phi D \end{bmatrix}, \quad (25)$$

where A, B, C, and D are matrices of the form (16). As in the domestic model, the parameter neighborhood is specified by $\alpha_1 = \alpha_2 = 0.57$ and $\lambda_s = \lambda = \mu = \mu_s = 1.7542$.

(4) in our basic domestic model we set $A = B$ and $a = b = (0.50, 0.14, 0.14, 0.14)$ which we shall continue to assume. Given that the probability of $\omega^* = \omega^{*H}$ is 0.8, it follows from the structure of the matrix Γ^* that 80% of the time,

the international economy will look very much like our domestic economy when the second agent has endowment of ω^{*H}. But now, how should we select C and D? What about the other 20% of the time when the lower part of Γ^* is realized? To consider this point note that the *arbitrary* stochastic structure introduced by the i.i.d. process of $\{\omega_t^*, t = 1, 2, ...\}$ introduces into Γ^* a new and arbitrary element which may have nothing to do with the way the international economy *actually* works. This change must have some effect on the dynamics of the states of beliefs. The effect that we found was entirely minimal and is represented by the simple specification $c = a = b = (0.50, 0.14, 0.14, 0.14)$ but $d = (0.57, 0.14, 0.57, 0.14)$. Hence we can view the international model as a proper extension of our earlier model.

Summary of specification: $\phi = 0.43, \beta_1 = \beta_2 = 0.90, \gamma_1 = \gamma_2 = 3.25, \alpha_1 = \alpha_2 = .57, \lambda_s = \lambda = \mu = \mu_s = 1.7542, a = b = c = (0.50, 0.14, 0.14, 0.14), d = (0.57, 0.14, 0.57, 0.14)$. In the REE ($\omega = 24, \omega^{*H} = 24.6, \omega^{*L} = 23.4$); in the RBE ($\omega = 26, \omega^{*H} = 26.6, \omega^{*L} = 25.4$).
Table 7 presents the simulation results for the REE and the RBE of the specified international model. In this table "ex" denotes the "exchange rate" and σ_{ex} is the standard deviation of the exchange rate. Note first that the results for the REE are essentially the same as the results in Table 2 and the parameter ζ is computed to be 0.95, as is expected. From the point of view of comparing the RBE with the REE the only new result is the much larger variance of the foreign exchange rate in the RBE relative to the REE. Since the foreign economy is hypothetical we do not suggest any particular value for ex and σ_{ex}. Turning finally to the RBE, we observe that the results here are essentially the same as in Tables 3 or 4 but the new result is the simulated equilibrium value of $\zeta = 0.47$ which is significantly less than 1. We thus conclude that the Forward Discount Bias is another anomaly which is explained *by the same RBE model*. Sharper results for ζ could probably be obtained by formulating a more realistic foreign sector.

Table 7. Results for the reformulated international model

Variable	REE	RBE	Empirical Record
p	23.31	23.94	23
σ_p	0.37	2.70	6.48
R	6.21%	7.80%	8.00%
σ_R	4.72%	19.34%	18.08%
r^F	5.64%	1.52%	1.00%
σ_{r^F}	1.89%	16.37%	5.67%
ρ	0.57%	6.28%	7.00%
ex	0.68	0.67	—-
σ_{ex}	1.29%	9.93%	—-
ζ	0.95	0.47	diverse < 1

Why does the RBE predict a value for ζ which is lower than 1? If $\zeta < 1$ then in an REE agents can make an *expectational* arbitrage: they can borrow today in one currency, invest in the other and *expect* that the net return on their investment *next period* is larger than the depreciation of the currency. In a world of securities (rather than an Arrow-Debreu world of contingent claims) this is not an arbitrage in the textbook sense since the trades do not take place at the same time. But, in a stationary world of an REE all agents hold the same self-fulfilling expectations and the *expectational* arbitrage becomes a *real* arbitrage. Hence, $\zeta \approx 1$ must hold in equilibrium.

In an RBE a condition of differential nominal interest rates across countries offers an investment opportunity but now such investment is subjected to endogenous uncertainty. This results in a true, equilibrium, process of the exchange rate which exhibits excess fluctuations in part due to variability in the states of belief of the agents. As a result, a differential of nominal interest rates between the two countries is not an unbiased estimate of the rate of depreciation of the exchange rate one period later. If an RBE is to explain the data, why should we expect that under Rational Beliefs $\zeta < 1$? In an RBE agents know that the true distribution of future exchange rates is not known and therefore the mean value of the depreciation in the exchange rate is subject to endogenous uncertainty. But since ζ is part of the return on foreign currency investments, risk-averse foreign currency investors would demand a risk premium for endogenous uncertainty and, on average, the difference $(1 - \zeta)$ is the *proportional premium on nominal interest differential* demanded by currency investors for carrying foreign currency positions. For a positive premium we need to have $\zeta < 1$.

4 A final comment

We developed in this paper a simple model for the study of the volatility characteristics of financial markets. With the aid of this model we explained diverse, empirically observed, features of market volatility in the U.S. However, using the simulation method as a theoretical device we also offered a theoretical foundation for the success of the model. This does not prove that our theory is correct and much additional research, both empirical as well as theoretical, is needed in order to build on the ideas developed here. However, we think that the results so far are very promising and future research should test empirically the effect of expectations on market volatility and develop more realistic, infinite horizon, models to further explore the hypotheses advanced here.

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