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Coordination and correlation in Markov rational belief equilibria*

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Summary. This paper studies the effect of correlation in the rational beliefs of agents on the volatility of asset prices. We use the technique of generating variables to study stable and non-stationary processes needed to characterize rational beliefs. We then examine how the stochastic interaction among such variables affects the behavior of a wide class of Rational Belief Equilibria (RBE). The paper demonstrates how to construct a consistent price state space and then shows the existence of RBE for any economy for which such price state space is constructed. Next, the results are used to study the volatility of asset prices via numerical simulation of a two agents model. If beliefs of agents are uniformly dispersed and independent, we would expect heterogeneity of beliefs to have a limited impact on the fluctuations of asset prices. On the other hand, our results show that correlation across agents can have a complex and dramatic effect on the volatility of prices and thus can be the dominant factor in the fluctuation of asset prices. The mechanism generating this effect works through the clustering of beliefs in states of different levels of agreement. In states of agreement the conditional forecasts of the agents tend to fluctuate *together* inducing more volatile asset prices. In states of disagreement the conditional forecasts fluctuate *in diverse directions* tending to cancel each other's effect on market demand and resulting in reduced price volatility.

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1 Introduction

The theory of rational beliefs developed in Kurz (1994a) explains why rational and equally informed economic agents may have different probability beliefs about the dynamics of exogenous and endogenous variables. The theory leads

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both to a new concept of equilibrium called Rational Belief Equilibrium (in short, "RBE", see Kurz (1994b), (1996), Kurz and Wu (1996) and Nielsen (1996)) as well as to a radically different paradigm of the nature of uncertainty in economic systems which is called "Endogenous Uncertainty" (see Kurz (1974a)). This is the uncertainty which is propagated endogenously by the beliefs and actions of the agents.

It is clear that the effect of diversity of beliefs on market performance depends upon the correlation among the agents. We use the term "correlation" rather than "coordination" to stress the fact that we are not studying a problem of market failure which can be corrected by some collective and coordinated action. Aggregate market performance is very sensitive to the nature of this correlation. When actions of agents are "independent", aggregation across agents acts as a "market law of large numbers" and when correlation among agents is present, amplified aggregate behavior emerges. In the latter case a complex dynamics of price movement need not be caused by any exogenous factor; it can be an endogenous consequence of the structure of correlation among agents. In this sense the structure of correlation among agents is one component of the explanation of endogenous uncertainty. The main purpose of this paper is to study the effect of spontaneous correlation among agents with diverse beliefs on the volatility of equilibrium prices. Thus, suppose that agents observe data generated by a system $(\Omega, \mathscr{B}, \Pi, T)$. The main theorem of Kurz (1994a) characterizes a set $B(\Pi)$ which is the set of beliefs which are rational with respect to Π. We hold the view that there are two factors which generate correlation in beliefs. The first is the clustering (of similarity or opposition) within $B(\Pi)$ which can be considered to be caused by communication among the agents. The second factor is the fact that in forming conditional probabilities, all agents condition on the same observed data.

In order to explore these ideas we need to develop tools for simplifying the representation of stable but non-stationary processes. Nielsen (1994), (1996) takes an important first step in this direction. He studies a broad class of stable but non-stationary processes called SIDS processes for which one can characterize the set of all rational beliefs. Using the structure of SIDS processes he shows how to construct the price state space of an RBE. Kurz and Wu (1996) use SIDS processes to study the financial structure which is used to trade the endogenous uncertainty of future prices in an RBE. The main tool used in describing the non-stationarity of an SIDS process is a "Generating Process" and we briefly explain this concept with an example from the *Guest Editor's Introduction* to this volume, which is discussed in Section 2. In order to apply these ideas to our problem we need first to expand the scope of this tool to more general non-stationary settings and this is done in Section 3 under the heading of "Conditional Stability Theorem". Since this theorem is the main tool of the paper we provide in Section 4 an explanation of the perspective offered by this theorem. Our main results on correlation are developed in the context of a multi-agent, overlapping generations, stock market economy developed in section 6. This section also contains several numerical examples of RBE for this OLG economy. In Section 5 we show how the conditional stability theorem can be used to construct the price state space of a single agent stock market economy. In fact, one technical result of the paper is that the approach developed here can be applied both to the OLG economy as well as to the single, infinite lived, agent economy.

Our notation is the same as in Kurz (1994a). X is a Borel subset of R^N. Data realized in this economy are represented by infinite sequences $x = (x_0, x_1, \ldots) \in X^\infty$. The sequences x are the realizations of a stochastic process on the measurable space $(X^\infty, \mathscr{B}(X^\infty))$ where $\mathscr{B}(X^\infty)$ is the Borel σ-field of X^∞. The probability Π (or Π with a subscript) denotes the true probability of the random sequence and therefore the true dynamical system is $(X^\infty, \mathscr{B}(X^\infty), \Pi, T)$. The dynamical system $(X^\infty, \mathscr{B}(X^\infty), m, T)$ is the unique associated stationary system and we refer to m as "*the stationary measure*" of Π. In general, for any stable system $(X^\infty, \mathscr{B}(X^\infty), Q, T)$ we denote by m^Q the unique stationary measure associated with it. Since we discuss in this paper various random variables and processes, we use the following compact notation: *Random variables* are identified by lower case letters. i.e. x_t, y_t, z_t. Infinite sequences of random variables are denoted by lower case letters (i.e. x, y, z). *Stochastic processes* are denoted by either one of these notations: $\{(x_t, y_t), t = 0, 1, 2, \ldots\} \equiv [x, y]$.

2 On generating processes

Stable non-stationary processes are difficult to work with and complex to describe. One of our objectives is to simplify the description of the changes over time in the probability of the process while ensuring that the process is stable in the sense of Kurz [1994a][1]. To motivate we start with example (1), Section3, of the *Guest Editor's Introduction* to this volume. Thus pick a process $[y]$ of i.i.d. random variables taking values in $\{0, 1\}$ with probability of 1 being, say, 1/4 and generate an infinite sequence of observations $(y_0^*, y_1^*, y_2^*, \ldots)$ of this process. Now, define the process $[x]$ where $x_t \in X = \{0, 1\}$ to be a sequence of *independent* random variables satisfying

$$P\{x_t = 1\} = \begin{cases} \alpha & \text{if } y_t^* = 1 \\ \beta & \text{if } y_t^* = 0. \end{cases} \tag{1}$$

The probability on $(X^\infty, \mathscr{B}(X^\infty))$ implied by $[x]$ is denoted by Π_{y^*}[2] and the dynamical system by $(X^\infty, \mathscr{B}(X^\infty), \Pi_{y^*}, T)$. This system is stable and has a stationary measure m represented by the i.i.d. process $[w]$, $w_t \in X$ with $P\{w_t = 1\} = (1/4)\alpha + (3/4)\beta$. The i.i.d. process $[y]$ is called a *generating process* and $y^* = (y_0^*, y_1^*, \ldots)$ a *generating sequence.* A particular realization y^* is thought of as a sequence of "structural" parameters which generate the probability Π_{y^*}. Since y^* specifies how the probability of the 1's changes with time, it generates the non-stationarity in the process $[x]$. As explained in the *Guest Editor's Introduction*, a rational belief of an agent can be defined with a "private" generating process $[z]$ of i.i.d. random variables on $\{0, 1\}$ with probability of 1 being π. Using a generating

[1] The reader may consult the *Guest Editor's Introduction* to this volume which contains a brief review of the theory of Rational Beliefs.

[2] The notation Π_Y will be used to denote the *marginal* measure of $\Pi(Y^\infty, \mathscr{B}(Y^\infty))$. The notation Π_y is used to denote the probability which has a generic sequence y as a parameter and this will later be defined to be a *conditional* probability of Π on a joint space $((X \times Y)^\infty, \mathscr{B}(X \times Y)^\infty)$ given y.

sequence z^* with the frequency of $z_t^* = 1$ being π an agent can now define the perceived process $[x']$ by $P\{x_t' = 1\} = \begin{cases} \alpha' & \text{if } z_t^* = 1 \\ \beta' & \text{if } z_t^* = 0 \end{cases}$ *independently* over time. This sequence of random variables defines a stable measure P and is a rational belief if $\pi\alpha' + (1-\pi)\beta' = (1/4)\alpha + (3/4)\beta$.

We referred to $[z]$ as a "private" generating process in order to express the fact that this process may be different from the true process described above. It also stresses the idea that each agent generates the sequence but he is the only one who "observes" it. In more complex contexts this is what a private research department does in an organization. We do not view $[z^*]$ as an objective signal or data which gives objective information to an observer in the sense of a true insight into the unknown structure of some process[3]. The sequence z^* should be interpreted as a *quantitative formulation of the belief of the agent about the impact of structural change (i.e. non-stationarity) on the observed variables in the economy, the x_t's.*

One may note that in the example above we can construct a rational belief by the densities defined by any *finite* sequence of z_t^*'s from, say, date 1 to date T followed by the i.i.d. process with probability $(1/4)\alpha + (3/4)\beta$ after date T. Moreover, we could define a *joint* process $[x', z]$ as a perceived process of data and of parameters and in later sections we refer to the probability *of this joint process itself* as a rational belief if the implied distribution on the observables satisfy the rationality axioms. Under such a modeling strategy the agent takes *as known* at each date t the past observed data $(x_0, \ldots, x_t)$ as well as the past assigned parameters $(z_0, \ldots, z_t)$. This view of rational beliefs turns out to have major advantages that will be discussed in Section 4.

We conclude by stressing that a generating process is a tool which we introduce to enable both a systematic selection of the time dependency of the joint distributions of the x_t's as well as *a characterization of the statistical stability properties of the process.* In general, a generating process is a dynamical system $(Y^\infty, \mathscr{B}(Y^\infty), \mu, T)$ where μ is a *generating measure.* These terms will be used in the discussion of the Conditional Stability Theorem to which we now turn.

3 A conditional stability theorem for a joint system

We study now a system $((X \times Y)^\infty, \mathscr{B}(X \times Y)^\infty, \Pi, T)$ (the "joint" system) and throughout this section we use the simplified notation of $\Omega = (X \times Y)^\infty$ and $\mathscr{B} = \mathscr{B}((X \times Y)^\infty)$. In any application we think of y as a sequence of parameters so that the data x are realization of the true, possibly non-stationary, dynamical system $(X^\infty, \mathscr{B}(X^\infty), \Pi_y, T)$. Π_y is the conditional probability of Π given y. Our *primitive assumption* is then that the joint system is either stable or stationary and we derive all analytical conclusions from this postulate. This structure implies that the statistical properties of the parameters y are interrelated with the statistical properties of the

[3] The question of the exact sense in which z_t is "observed" and its implication to the question of rationality is discussed in Kurz and Schneider (1996, Appendix (A.III)).

data x but both are derived as a consequence of the stability properties of the joint system. In contrast with the true system $(X^\infty, \mathscr{B}(X^\infty), \Pi_y, T)$ which generates the data, we think of the joint system either as a *hypothetical system* which is used in order to assess the stability properties of $(X^\infty, \mathscr{B}(X^\infty), \Pi_y, T)$ or as a *model* which a rational agent may use in formulating his belief.

Relative to the joint system we use in this paper various terms which we define here, risking some repetition of well-known textbook terms. Given $Y \subseteq R^M$ denote by $(Y^\infty, \mathscr{B}(Y^\infty), \mu, T)$ the generating system as discussed in the previous section. Now consider the map

$$\Pi_y(\cdot): Y^\infty \times \mathscr{B}(X^\infty) \to [0, 1]. \tag{2}$$

Standard properties of conditional probabilities imply that for each $A \in \mathscr{B}(X^\infty)$, $\Pi_y(A)$ is a measurable function of y and for each y, $\Pi_y(\cdot)$ is a probability on $(X^\infty, \mathscr{B}(X^\infty))$. For $A \in \mathscr{B}(X^\infty)$ and $B \in \mathscr{B}(Y^\infty)$ we have that

$$\Pi(A \times B) = \int_B \Pi_y(A)\mu(dy). \tag{3}$$

In addition to the generating measure μ we define the marginal measure γ on $(X^\infty, \mathscr{B}(X^\infty))$ thus

$$\gamma(A) = \Pi(A \times Y^\infty) \quad \text{for all } A \in \mathscr{B}(X^\infty) \tag{4a}$$

$$\mu(B) = \Pi(X^\infty \times B) \quad \text{for all } B \in \mathscr{B}(Y^\infty). \tag{4b}$$

Now suppose that $(\Omega, \mathscr{B}, \Pi, T)$ is a stable dynamical system with a stationary measure m. Then we define the two marginal measures of m as follows:

$$m_X(A) = m(A \times Y^\infty) \quad \text{for all } A \in \mathscr{B}(X^\infty) \tag{5a}$$

$$m_Y(B) = m(X^\infty \times B) \quad \text{for all } B \in \mathscr{B}(Y^\infty). \tag{5b}$$

With this notation we also have $\Pi_X = \gamma$ and $\Pi_Y = \mu$. We now have the following result.

Theorem 1. *Let $(\Omega, \mathscr{B}, \Pi, T)$ be a stable dynamical system then*

(a) the systems $(X^\infty, \mathscr{B}(X^\infty), \gamma, T)$ and $(Y^\infty, \mathscr{B}(Y^\infty), \mu, T)$ are also stable with m^γ and m^μ as the associated stationary measures.
(b) $m^\gamma = m_X$ and $m^\mu = m_Y$.
(c) If $(\Omega, \mathscr{B}, \Pi, T)$ is stationary then the two marginal systems are also stationary with $\Pi = m$, $\mu = m_Y = \Pi_Y$ and $\gamma = m_X = \Pi_X$.

For each $y \in Y^\infty$ we denote by $(X^\infty, \mathscr{B}(X^\infty), \Pi_y, T)$ the dynamical system defined by the conditional probability Π_y. Our primary interests are the stability properties of this conditional dynamical system for different sequences $y \in Y^\infty$.

In theorem 2 which is central to this paper we postulate that the joint system is stable and ergodic. Unfortunately, conditional probabilities do not, in general, inherit the ergodic properties of the unconditional probability measures from which they are derived. The Lemma below will examine this question with respect to a special class of conditional probabilities which is of interest to us. In order to do this denote by $\mathbb{C}$ the set of cylinders in $\mathscr{B}$ and for $C \in \mathbb{C}$ let

$$K_C = \left\{ (x,y) \in \Omega\colon \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} 1_C(T^k(x,y)) \quad \text{exists} \right\}. \tag{6}$$

Now define the following sub σ-field of $\mathscr{B}$

$$\mathscr{L} = \sigma\{S : S = X^\infty \times B, B \in \mathscr{B}(Y^\infty)\}. \tag{7}$$

Lemma 1. *Let $(\Omega, \mathscr{B}, \Pi, T)$ be a stable dynamical system and consider the conditional probability $\Pi(\cdot|\mathscr{L})(x,y)$ on $\mathscr{B}$. Then, for any $C \in \mathbb{C}$*

$$\Pi(K_C|\mathscr{L})(x,y) = 1 \text{ for } \Pi \text{ almost all } (x,y). \tag{8}$$

Proof. Since $\Omega = (X \times Y)^\infty$ is a complete and separable metric space we take $\Pi(\cdot|\mathscr{L})(x,y)$ as a regular conditional probability and since $(X^\infty \times Y^\infty) \in \mathscr{L}$ it follows from the definition of conditional probability (see Ash (1972), page 249) that we have

$$\int_{(X \times Y)^\infty} \Pi(K_C|\mathscr{L})(x,y)\Pi(d(x,y)) = \Pi(K_C \cap (X \times Y)^\infty) = \Pi(K_C).$$

From (6) and the assumption of joint stability we have that $\Pi(K_C) = 1$. Since the measurable function $\Pi(K_C|\mathscr{L})(x,y)$ satisfies $0 \le \Pi(K_C|\mathscr{L})(x,y) \le 1$ for all $(x,y) \in \Omega$ it follows from Theorem 1.6.6 of Ash (1972) (page 47) that $\Pi(K_C|\mathscr{L})(x,y) = 1$ for Π-almost all (x,y). □

Remark. The thrust of the Lemma is that given a stable probability on $\mathscr{B}((X \times Y)^\infty)$, the conditional probability on $\mathscr{B}(X^\infty)$ will inherit the stability properties of the unconditional probability if the conditioning is done on a *fixed* σ-field of $\mathscr{B}(Y^\infty)$. In Theorem 2 we shall be concerned with sets of the form $(A \times Y^\infty)$ where $A \in \mathscr{B}(X^\infty)$. In that case we shall write

$$\Pi_y(A) \equiv \Pi(A \times Y^\infty|\mathscr{L})(x,y) \quad \text{independent of } x \text{ due to (7).}$$

Theorem 2. (*Conditional Stability Theorem*) *Let $(\Omega, \mathscr{B}, \Pi, T)$ be a stable and ergodic dynamical system and let Y be countable. Then*

(a) $(X^\infty, \mathscr{B}(X^\infty), \Pi_y, T)$ is stable and ergodic for Π a.a. y.

(b) The stationary measure of Π_y is independent of y and if we denote it by m^Π then it satisfies the condition $m^\Pi = m_X$.

(c) If $(\Omega, \mathscr{B}, \Pi, T)$ is stationary then the stationary measure of Π_y is the X^∞ marginal measure of Π. That is,

$$m^\Pi = \gamma = m_X = \Pi_X. \tag{9}$$

Proof. See Appendix.

4 Interpreting rational beliefs with the perspective of the conditional stability theorem

As stressed in Section 2 the function of the generating sequence is to identify the parameter structure of the economy at each date. For example, in the case of an

SIDS process the value y_t selects a one period probability on $(X, \mathscr{B}(X))$ from a finite or countable set of such probabilities. If agents think that the observables form a Markov process, the generating sequence specifies at each date which among a finite or countable set of transition matrices is to be in effect at t.

For processes such as SIDS, y_t at each date t is independent of x_{t-1}. The crucial feature of the Conditional Stability Theorem is that it permits a fully general interdependence between sequences x and y. More specifically, within the joint system where observable variables and parameters are random variables, past values of economic variables and parameters jointly *cause* the determination of their future values. Hence, if an agent uses a joint process as a model of his belief then at date t he observes $(x_0, x_1, \ldots, x_t)$ and he "knows" $(y_0, y_1, \ldots, y_t)$ which his model generated. His "research department" can generate y_{t+1} but it is not feasible to generate the *entire* infinite sequence of y's from $t+1$ on without knowing the future values of the observables x's. Suppose then that the agent specifies his belief based only on *finite* sequences of y's and suppose also that the joint system with probability Q is stationary. Each finite vector $y^t = (y_0, y_1, \ldots, y_t)$ defines a stable system $(X^\infty, \mathscr{B}(X^\infty), Q_{y^t}, T)$ with a stationary measure m^{y^t}. The Conditional Stability theorem tells us that $m^{y^t} = Q_X$ for all t. Note that Q_{y^t} and Q_{y^τ}, $t \neq \tau$, are different measures on $(X^\infty, \mathscr{B}(X^\infty))$, hence by conditioning on the varying sequence y^t the agent adopts *an entire family of stable beliefs* $\{Q_{y^t}, t = 1, 2, \ldots\}$ *each of which has* Q_X *as its stationary measure.* But then if the agent uses past data of the x's to determine future values of the y's then it is useful to define Q to be a rational belief of the agent on $((X \times Y)^\infty, \mathscr{B}(X \times Y)^\infty)$ together with the specification that the probability measures on $(X^\infty, \mathscr{B}(X^\infty))$ which he uses at each date t is Q_{y^t}. This definition applies to any stable probability Q defined on $((X \times Y)^\infty, \mathscr{B}(X \times Y)^\infty)$ and Π defined on $((X \times Z)^\infty, \mathscr{B}(X \times Z)^\infty)$ where Y and Z are countable spaces. The Conditional Stability Theorem then motivates us to define Q to be a rational belief with respect to Π if $m_X^Q = m_X^\Pi$. Our approach also raises some questions with regard to the statistical and economic interpretation of the generating variables. This issue is addressed in detail by Kurz and Schneider (1996, Appendix (A.III)).

A rational belief as a probability Q on $((X \times Y)^\infty, \mathscr{B}(X \times Y)^\infty)$ offers significant analytical simplifications. To see this suppose that $\{(x_t, y_t), t = 1, 2, \ldots\}$ is jointly a Markov process. Then, an agent who solves a dynamic programming problem treats the pair (x_t, y_t) symmetrically: he observes x_t in the market and y_t in his own private research department. For the state space $X \times Y$ and a realization $(x_t, y_t) \in X \times Y$ the optimization of the agent is entirely routine. In contrast consider the case when y is an infinite sequence and the belief is represented by Q_y. At any date the optimizing agent faces an infinite sequence $(y_t, y_{t+1}, y_{t+2}, \ldots)$ which defines his "state" at t. This means that no matter what the dimensions of X and Y are, the state space for his optimization is of infinite dimension raising the level of complexity of the problem at hand.

In the rest of this paper we represent beliefs of agents as stationary probabilities on a joint space of observables and structural parameters. We use Markovian systems on finite state spaces and hence a belief is represented by a finite dimensional transition matrix on $(X \times Y) \times (X \times Y)$.

5 Asset prices in a single agent economy: an application of the conditional stability theorem

One of the conclusions of this paper is that the mathematical structure of RBE of the multi-agent OLG stock market economy of section 6 is essentially the same as the structure of RBE of economies with a single, infinitely lived, agent. The asset pricing theory of such an RBE is an analogue to the Lucas (1978) Rational Expectations theory. Since that model has been extensively used, the study of RBE for a single agent economy has an added interest on its own. We thus proceed in two steps. First we introduce the single agent model and its assumptions with the aim of explaining how the definition of an RBE is simplified by the Conditional Stability Theorem. We then specialize to the case of finite state spaces and study the construction of a consistent state space for endogenous uncertainty and the existence of RBE. This construction is then extended in a natural way to the OLG set-up of Section 6 below.

5a The model and the definition of an RBE

To describe the single agent stock market economy let $[d]$ be an exogenous dividend process where $d_t \in D \subseteq \mathbb{R}^N$ are N vectors of output of a single perishable consumption good whose price is denoted by p_t^c. The state space D is a Borel subset of $\mathbb{R}^N$. The dividend process represents N activities each producing a random stream of output. These are organized as firms and we denote by $p_t \in \mathbb{R}_+^N$ the price vector of the firms' ownership shares which are traded in competitive markets. The supply of ownership shares is $\vec{1}$. The price process $\{(p_t^c, p_t) \in P, t = 1, 2, \dots\}$ is the main object of our study and the Borel subset $P \subseteq \mathbb{R}_+^{N+1}$ of feasible prices will be of central importance in the analysis below.

Proceeding as in the Conditional Stability Theorem, let Y be a *countable* subset of $\mathbb{R}$; it is the domain of the generating sequence. As in Section 3, the primitive of the agent's belief is a joint process of generating variables and observables and we assume that this joint process is Markov. That is, we define the agent's belief Q on $((D \times P \times Y)^\infty, \mathscr{B}((D \times P \times Y)^\infty))$ and assume:

Assumption 5.1. *Under the belief Q the process $\{d_t, p_t^c, p_t, y_t), t = 1, 2, \dots\}$ is jointly a Markov process and the system $((D \times P \times Y)^\infty, \mathscr{B}((D \times P \times Y)^\infty), Q, T)$ is stationary and ergodic.*

We can now formulate the standard portfolio problem of the agent given his belief Q:

$$\max_{(x,\theta)} E_Q \sum_{t=1}^{\infty} \delta^t u(x_t)$$

subject to (10)

$$p_t^c x_t + \theta_t \cdot p_t \le \theta_{t-1} \cdot (p_t + p_t^c d_t) \quad t = 1, 2, \dots$$
$$0 \le \theta_t \le \bar{\theta}, \quad \bar{\theta} \ge \vec{1}, \quad x_t \ge 0 \text{ all } t.$$

Here $\theta_t \in \mathbb{R}^N$ is the stock holdings at t and x_t is the consumption at t. Our assumption on the utility function is standard:

Assumption 5.2. $u(\cdot)$ *is a C^1, strictly increasing and strictly concave utility function.*

Given that Q is jointly stationary on $((D \times P \times Y)^\infty, \mathscr{B}((D \times P \times Y)^\infty)$ the standard theorems of dynamic programming apply when we assume that at each date the agent knows (d_t, p_t^c, p_t, y_t). This is in the sense that he *observes* the prices and dividends in the market while he *knows* the parameter y_t which is *generated internally*. We then derive a pair $(\varphi_x, \varphi_\theta)$ of demand functions

$$x_t = \varphi_x(d_t, p_t^c, p_t, y_t, \theta_{t-1}) \tag{11a}$$

$$\theta_t = \varphi_\theta(d_t, p_t^c, p_t, y_t, \theta_{t-1}). \tag{11b}$$

An equilibrium of the stock market economy requires that $\theta_t = \vec{1}$ for all t and a solution of these $N+1$ equations implies that there is a map $\Phi^* : \mathbb{R}^{(N+1)} \to \mathbb{R}^{(N+1)}$ such that

$$\begin{pmatrix} p_t^c \\ p_t \end{pmatrix} = \Phi^*(d_t, y_t) \quad t = 1, 2, \ldots \tag{12}$$

Equation (12) shows in the clearest way what the RBE theory maintains: that equilibrium prices depend both upon the exogenous state d_t as well as the "state of beliefs" represented by y_t!

We now turn to a description of the true primitives of our economy as distinct from the perception of the agent as reflected in his belief Q. Although Π_D is an *exogenously* given probability on $(D^\infty, \mathscr{B}(D^\infty))$ the true dynamic movement of the economy is influenced by the realized generating sequence y. To understand how the generating mechanism comes to be, think of the agent as constructing a model of how the structure of the economy evolves. Such a model combines stochastic specifications which the agent knows exactly and conditional dependence upon past observed values of dividend payments and asset prices as well as past realized values of the generating process. For example, the agent may select a sequence of random variables $\{\varepsilon_t, t = 1, 2, \ldots\}$ where ε_t and d_t may be correlated. He then defines the stochastic sequence

$$y_t = \begin{cases} 1 & \text{if} \quad \alpha + \beta d_{t-1} + \gamma p_{t-1} + \eta y_{t-1} + \varepsilon_t \geq \alpha + \beta \bar{d} + \gamma \bar{p} + \eta \bar{y} \\ 0 & \text{if} \quad \alpha + \beta d_{t-1} + \gamma p_{t-1} + \eta y_{t-1} + \varepsilon_t < \alpha + \beta \bar{d} + \gamma \bar{p} + \eta \bar{y} \end{cases} \tag{13}$$

which, together with the price functions (12), establishes an *actual* stochastic relation between the processes $[d, p]$ and $[y]$. This implies that the joint probability distributions of the d_t and the ε_t together with (10) and (12) define the true probability measure Π_{DY} which *we take as a primitive* in the development below. We stress that these relationships need not be the same as those perceived by the agent since he knows neither the true probability distributions of the d_t's nor the market clearing map (12) and his belief is formed *without this knowledge*. Moreover, although he selects the mechanism which generates y this does not mean that he perceives correctly the correlation between d_t and y_t. By construction he knows Q_Y, the true unconditional probability of the generating process $[y]$, and hence our construction implies that

$$Q_Y = \Pi_Y. \tag{14}$$

Assumption 5.3. *Under Π_{DY} the dynamical system $((D \times Y)^\infty, \mathscr{B}((D \times Y)^\infty), \Pi_{DY}, T)$ is stable and ergodic with a stationary measure m_{DY}.*

The probability Π_{DY} and conditions (12) define a probability Π_p on $(P^\infty, \mathscr{B}(P^\infty))$. The stability of the process $[d, y]$ together with the fact that Φ^* is a measurable function imply (by Proposition 1 of Nielsen [1996]) that $[p^c, p]$ is an ergodic and stable process with probability Π_p induced by Φ^* and by Π_{DY}. Denote the stationary measure of Π_p my m_p. Next, for any rectangle $(A \times B \times C) \in (\mathscr{B}(D^\infty) \times \mathscr{B}(P^\infty) \times \mathscr{B}(Y^\infty))$ define

$$\Pi'(A \times B \times C) = \Pi_{DY}((A \times C) \cap \Phi^{*-1}(B)) \tag{15}$$

where $\Phi^{*-1}(B) = \{(d, y) \in (D \times Y)^\infty : (p^c, p) \in B \text{ and } (p_t^c, p_t) = \Phi^*(d_t, y_t)\}$. As in (4), one extends Π' to a measure Π on the full space $((D \times P \times Y)^\infty, \mathscr{B}((D \times P \times Y)^\infty)$. We have thus specified two measures Q and Π which, in the case at hand, are defined on the same space. We then have

Lemma 2. *The dynamical system $((D \times P \times Y)^\infty, \mathscr{B}((D \times P \times Y)^\infty), \Pi, T)$ is stable and ergodic.*

We denote the stationary measure of the system $((D \times P \times Y)^\infty, \mathscr{B}(D \times P \times Y)^\infty)$, $\Pi, T)$ by m. The proof follows from (12), (15), Assumption 5.3 and Proposition 1 of Nielsen [1996]. The Conditional Stability Theorem motivated the rationality conditions which Q must then satisfy:

Q is a Rational Belief with respect to Π if $Q_{DP} = m_{DP}$.

We can finally define a Rational Belief Equilibrium:

Definition 1. *A triple $\{\Pi, Q, \varphi_\theta(d_t, p_t^c, p_t, y_t, \theta_{t-1})\}$ is a Rational Belief Equilibrium of the single agent stock market economy if*

1. $Q_{DP} = m_{DP}$,
2. $\varphi_\theta(d_t, p_t^c, p_t, y_t, \vec{1}) = \vec{1}$ $\quad \Pi$ a.e.

5b The construction of a consistent price state space and the existence of RBE

The study of RBE raises the important methodological problem of how to construct a consistent state space for prices which, in principle, may change with the RBE considered. Recall that for the purpose of studying endogenous uncertainty in an economy we characterize it in part by the "state of belief". The *space* of states of belief is a component of the agent's probability belief and consequently, from a general equilibrium perspective, the determination of the state space over which prices are defined is "endogenous" in the sense that it is part of the description of an RBE. We now show how to solve this problem for the case where the three state spaces D, Y, and P are finite. Thus, *we now assume* that $D = \{d^1, d^2, \ldots, d^N\}$ and $Y = \{y^1, y^2, \ldots, y^J\}$ and focus on the two finite state spaces

$$S_D = \{1, 2, \ldots, N\} \tag{16a}$$

$$S_Y = \{1, 2, \ldots, J\}. \tag{16b}$$

Note now that from (12) it follows that the *maximal* number M of different prices which can ever be observed in this economy is finite and equals NJ. This leads to the observation that for the problem at hand the state space of the price process can be viewed as

$$S_P = \{1, 2, \ldots, M\} \qquad \text{where } M = NJ. \tag{16c}$$

Now, any probability on a finite collection such as D or Y can be redefined as a probability on the state space on which these objects are defined. In our case these are the spaces S_D, S_Y and S_P. Without introducing new notation *we shall, therefore, consider Q and Π to be probabilities on the measurable space* $((S_D \times S_p \times S_Y)^\infty, \mathscr{B}(S_D \times S_p \times S_Y)^\infty))$. We then regard Assumption 5.1 as applicable to the dynamical system $((S_D \times S_p \times S_Y)^\infty, \mathscr{B}((S_D \times S_p \times S_Y)^\infty), Q, T)$ and Assumption 5.3 to the dynamical system $((S_D \times S_Y)^\infty, \mathscr{B}(S_D \times S_Y)^\infty), \Pi_{DY}, T)$. However, the translations of the probabilities appear to depend upon the particular map Φ^* in (12). We now claim that one can separate between the statement of the probabilities which *do not depend* upon the map Φ^* in (12) and the specification of the value of the market clearing prices which *do depend* upon Φ^*. To see this we let $\Phi: S_D \times S_Y \to S_P$ be any 1–1 and onto invertible map of the form

$$s_{(k,j)} = \Phi(k, j) \qquad (k, j) \in S_D \times S_Y. \tag{17}$$

The map Φ shows that the state space S_P can be partitioned by the exogenous variables into N blocks $(S_P^1, S_P^2, \ldots, S_P^N)$ where S_p^k is identified by $d^k \in D$ as $S_P^k = \{(k, j)$, for all $j \in S_Y\}$ and $k \in S_D$. However, it is possible that the market clearing map Φ^* associates *the same* value of the price with different members of the subset S_p^k of indices. To put it differently, the map Φ associates with (k, j) and (k, i) different price indices but the map Φ^* does not necessarily imply that $p_{k,j} \neq p_{k,i}$. If $p_{k,j} = p_{k,i}$ the probability of the common value is simply the sum of the probabilities of the indices over which the value of the price is the same. This argument shows that for any economy with a market clearing map Φ^* one may specify the probabilities Q and Π only on the indices specified by the state spaces and without any reference to the absolute value of the market clearing prices which are, in turn, specified by Φ^*. "Endogenous Uncertainty" is then the additional variability of prices which results from variability in the states of belief rather than changes in the dividend process. This amounts to requiring that some prices are different on different members of each block S_p^k.

We conclude that under assumptions 5.1, 5.2 and 5.3 the probabilities Q and Π can be formulated with the aid of an arbitrary invertible map Φ as in (17) and with no reference to Φ^*. That is, both the probabilities (Q, Π) as well as the conditions of rationality of belief can be formulated entirely on the spaces of indices. To formalize this idea we introduce:

Definition 2. *The collection* (S_D, S_P, S_Y) *is called a consistent price state space*[4] *for* (Q, Π, Φ) *if*

1. Condition (17) is satisfied;
2. $Q_{DP} = m_{DP}$.

[4] A "Consistent Price State Space" is similar to Nielsen's (1996) concept of a "Rational Belief Structure".

Considering now the single agent stock market economy we take the probability Π_{DY} as the given primitive. To relate our terminology to Definition 2 we shall then say that given a probability Π_{DY} the single agent stock market economy with belief Q *has a consistent state space for* (Q, Π) *if* (S_D, S_P, S_Y) *is a consistent price state space for* (Q, Π, Φ). Definition 2 intends to pave the way to a proof of existence of an RBE by identifying the conditions which a pair of probabilities Q and Π on abstract spaces of indices must satisfy in order to become candidates for an RBE. We stress again the fact that these conditions are specified without any reference to the equilibrium values of prices defined on S_P. With the aid of the Conditional Stability Theorem we have then demonstrated in this section an important practical principle. Suppose that the single agent economy satisfies Assumptions 5.2–5.3 on u and Π_{DY}. Then there is a family of probabilities Q which satisfy Assumption 5.1 such that *without specifying the equilibrium values of prices* we can construct a consistent price state space for (Q, Π).

Turning now to the question of existence we note that given a pair (Q, Π) for which a consistent price state space exists what is left to do is to determine the equilibrium values of prices by solving the system of equations in (12). In the stationary case when the conditional probability Q_y is independent of the y the existence theorem of Lucas (1978) applies since an RBE is a rational expectations equilibrium. Our result is broader:

Theorem 3. *Consider a single agent stock market economy which satisfies Assumptions 5.1, 5.2 and 5.3. For any pair* (Q, Π) *for which the economy has a consistent price state space there exists an RBE.*

Proof. Theorem 3 is a special case of Theorem 4 which is stated in the next section on OLG stock market economies.

Remark. The presence of endogenous uncertainty in an RBE is essential if the theorem is to be of interest. We address this issue below (see remark after Theorem 4).

Although Definition 1 of RBE does not address the issue of multiple equilibria we note that the economy is modeled as a dynamical system in which infinite random draws are associated with *definitive* sequences of economic allocations. If at any date the economy can have multiple market clearing outcomes, then as part of the dynamics postulated there is a procedure for selecting a particular one which generates the data observed. In the Markov case at hand this implies that if at two different dates the economy reaches the same state of the exogenous variables and the same state of belief then equilibrium prices should be the same at both dates.

Before concluding this section we make a technical remark. The rationality condition 2 in Definition 2 needs some clarification. Note that rational agents do not know the equilibrium map Φ^*. From observations they can deduce that the following set of states never occurs:

$$A^\circ = \{(s_p, k) \in S_P \times S_D \mid \text{there is no } j \in S_Y \text{ such that } s_p = \phi(k, j)\}. \tag{18}$$

This means that the stationary measure m assigns zero unconditional probability to the set of states in A°. Thus the partition $(S_P^1, S_P^2, \ldots, S_P^N)$ of S_P is respected by the stationary measure of a rational belief Q in the sense that given $d_t = d^k$ the stationary measure places a conditional probability of 1 on the block S_P^k. This is not a condition which Q itself must satisfy.

6 Correlation among agents in an overlapping generation rational belief equilibrium

We turn now to the problem of correlation among multiple agents which will be studied in the context of an OLG stock market economy.

6a The RBE of an OLG stock market economy

We formulate our stock market economy as a standard OLG model with K young agents in each generation which we denote by $k = 1, 2, \ldots, K$. There are also K old agents at each date but only the young receive an endowment ω_t^k, $t = 1, 2, \ldots$ of a single, perishable, consumption good. The stochastic processes $[\omega^k]$ for $k = 1, 2, \ldots, K$ of the endowment will be specified below. Each young person is a replica of the old person who preceded him where the term "replica" refers to the *utilities* and *beliefs* and hence this is a model of a finite number of "dynasties". The economy has N infinitely lived firms which produce the consumption good. The firms generate exogenously a stochastic process $\{d_t \in \mathbb{R}^N_{++}, t = 1, 2, \ldots\}$ of strictly positive outputs with no inputs. These net outputs are then paid out to the owners of the shares of the firms as dividends at the date at which the net outputs are produced.

The stock market economy has $N + 1$ markets. First, there is a market for the single consumption good which is traded in each period with an aggregate supply which equals total endowment plus total production. In addition, there are N markets for ownership shares of the firms and at date 1 the supply (which equals 1 for each common stock) of the securities is distributed among the old. This distribution initiates the financial sector and ultimately ensures intergenerational efficiency by allowing intergenerational redistribution of the endowment.

There is some similarity between the assumptions made here and those made by Kurz and Wu (1996). However, Kurz and Wu (1996) focus on the structure of Price Contingent Contracts used to allocate price uncertainty and in that case the randomness of the endowments and dividends is not essential: they assume that aggregate endowment is not observable and the dividends are not random. Our focus here is on the mechanism of correlation among agents and for our purpose it is sufficient to focus on the randomness of dividends and asset prices. For this reason we assume that *the endowment is not random and for each k it is a known constant* ω^k *over time*. This assumption is standard in OLG production models in which a young agent is assumed to have a certain endowment of one unit of labor which, in our model, is transformed into a unit of the consumption good. The notation which we employ is as follows:

x_t^{1k} – the consumption of k when young at t;
x_{t+1}^{2k} – the consumption of k when old at $t + 1$. This indicates that k was born at date t;

θ_t^k – vector of stock purchases of young agent k at t;
θ_0^k – endowment vector of stock to an old agent k at date 1 where $\theta_0^k > 0$ for all k;
ω^k – endowment of k when young at t. This means that $k = 1, 2, \ldots, K$ is among the young born at t. Writing ω^{1k} is unnecessary since only the young receive an endowment;
p_t – the price vector of common stocks at date t;
p_t^c – the price of the consumption good at date t;
d_t – the vector of positive dividends paid by the n firms at date t $d_t \in D \subseteq \mathbb{R}_+^N$;
P – the price space. When normalized we can set $P = \Delta = \{x \in \mathbb{R}_+^{N+1} : \sum_{j=1}^{N+1} x_j = 1\}$;
Y^k – the state space for the generating sequences of agent k with generic element y_t^k;
Q^k – the probability belief of k which is a measure on $((D \times P \times Y^k)^\infty, \mathscr{B}(D \times P \times Y^k)^\infty)$; without changing notation we shall later think of Q^k as a probability on the subsets of the index space $(S_D \times S_P \times S_{Y^k})^\infty$;
$u^k(\cdot, \cdot)$ – the utility function of agent k.

We now make the following assumptions:

Assumption 6.0. *Dividends are strictly positive in all states and short sales are not permitted so that for all t and k* $\theta_t^k \geq 0$.

Assumption 6.1. *For all k, under the belief* Q^k *the process* $\{d_t, p_t^c, p_t, y_t^k), t = 1, 2, \ldots\}$ *is jointly a Markov process and the dynamical system* $((D \times P \times Y^k)^\infty, \mathscr{B}((D \times P \times Y^k)^\infty), Q, T)$ *is stationary and ergodic. The non-stationarity induced by each private generating sequence* y^k *is a selection, at each date, of a Markov transition function (a matrix if the set of prices is countable) which is determined by the value taken by* y_t^k.

Assumption 6.2. $u^k(\cdot, \cdot)$ *is a* C^1, *strictly increasing and strictly concave utility function, all k.*

Given Assumptions 6.1 and 6.2 we can define the optimization problem of k when young:

$$\underset{(x^{1k}, \theta^k, x^{2k})}{\text{Max}} \quad E_{Q^k}\{u^k(x_t^{1k}, x_{t+1}^{2k}) | (p_t^c, p_t, d_t, y_t^k)\} \tag{19a}$$

subject to

$$p_t^c x_t^{1k} + p_t \cdot \theta_t^k \leq p_t^c \omega^k \tag{19b}$$

$$p_{t+1}^c x_{t+1}^{2k} \leq \theta_t^k \cdot (p_{t+1} + p_{t+1}^c d_{t+1}). \tag{19c}$$

The budget constraints (19b)–(19c) are homogenous of degree zero in prices and due to price normalization, the uncertainty about (p_{t+1}, d_{t+1}) is all the uncertainty an agent faces.

An RBE requires market clearance at all dates. Thus, we say that *markets clear at all dates* if, for almost all histories

$$\sum_{k=1}^{K} \theta_t^k = \vec{1} \qquad t = 1, 2, \ldots \tag{20}$$

It follows from (19b)–(19c) and (20) that when markets clear then

$$p_t^c x_t^1 + p_t \cdot \vec{1} = p_t^c \omega_t \qquad t = 1,2,\ldots \tag{21a}$$

$$p_t^c x_t^2 = (p_t + p_t^c d_t) \cdot \vec{1} \qquad t = 1,2,\ldots \tag{21b}$$

where x_t^1, x_t^2 and ω_t are the aggregates defined by $x_t^i = \sum_{k=1}^K x_t^{ik}$ $i = 1,2$ and $\omega_t = \sum_{k=1}^K \omega_t^k$. It follows from Assumptions (6.1)–(6.2) that the demand functions of the young have the form

$$\begin{aligned} \theta_t^k &= \varphi_\theta^k(p_t^c, p_t, d_t, y_t^k) \\ x_t^{1k} &= \varphi_x^{1k}(p_t^c, p_t, d_t, y_t^k). \end{aligned} \tag{22}$$

Market clearing requires $\sum_{k=1}^K \varphi_\theta^k(p_t^c, p_t, d_t, y_t^k) = \vec{1}$ which implies that we can solve for (p_t^c, p_t) in the general form

$$\begin{pmatrix} p_t^c \\ p_t \end{pmatrix} = \Phi^*(d_t, y_t) \tag{23}$$

where $y_t = (y_t^1, y_t^2, \ldots, y_t^K) \in Y \equiv Y^1 \times Y^2 \times \cdots \times Y^K$. Clearly, (23) and (12) are the same and hence, following the development in the single agent economy we can then define the measure Π_{DY} on the measurable space $((D \times Y)^\infty, \mathscr{B}((D \times Y)^\infty)$ as a primitive of the theory. By the same argument note that by the construction of the K generating processes we must have equality of the marginal measures, i.e.

$$(\Pi_{DY})_{Y^k} = Q_{Y^k}^k. \tag{24}$$

In comparison with (14), (24) applies *to each k separately*. Our corresponding assumption is

Assumption 6.3. *Under Π_{DY} the dynamical system $((D \times Y)^\infty, \mathscr{B}((D \times Y)^\infty), \Pi_{DY}, T)$ is stable and ergodic with a stationary measure m_{DY}.*

As in the single agent economy we show that $[p^c, p]$ is an ergodic and stable process with probability Π_P induced by Φ^* and by Π_{DY}. The stationary measure of Π_P is denoted by m_P. Proceeding as in (15) the probability Π_{DY} together with the map Φ^* in (23) induce a stable and ergodic equilibrium system $((D \times P \times Y)^\infty, \mathscr{B}((D \times P \times Y)^\infty), \Pi, T)$ with a stationary measure m and a marginal measure m_{DP}. This enables us to restate (24) in the form $\Pi_{Y^k} = Q_{Y^k}^k$ as in (14). With this noted the Conditional Stability Theorem suggests the following:

Definition 3. *$\{\Pi, \{Q^k, \varphi_\theta^k(p_t^c, p_t, d_t, y_t^k), k = 1,2,\ldots,K\}\}$ is a Rational Belief Equilibrium of the heterogenous agent stock market OLG economy if*

1. $m_{DP} = Q_{DP}^k$ *for* $k = 1,2,\ldots,K$
2. $\sum_{k=1}^K \varphi_\theta^k(p_t^c, p_t, d_t, y_t^k) = \vec{1}$ $\quad \Pi$ a.e.

The joint system on $(D \times Y)^\infty$ with probability Π_{DY} is the central dynamic mechanism of our RBE. The process of dividends is exogenous but the probability Π_{DY} is determined jointly by the dividend process and by the collective generating sequence $(y_t^1, y_t^2, \ldots, y_t^K)$ for $t = 1,2,\ldots$ of the K agents. The correlation among these variables arises from two sources: communication among the agents and the fact that agents form their beliefs conditional on past data. To put it differently y_t^j and y_t^i may be positively or negatively correlated partly because agents i and j communicate with each other and thus influence each other's models. But in selecting their models the random process of selecting y_t^j is conditioned on the past values of the y^js as well as past observed variables such as dividends and prices; the Markov condition in Assumption 6.3 does not prohibit the dependence of y_t^j upon past realized values of the observed variables.

Proceeding as in the single agent economy *we now assume that D and Y^k for all k are finite* and conclude that in any RBE only a finite number of prices, M, is observed. Thus define

$S_D = \{1, 2, \ldots, J\}$ with a generic element s_d

$S_{Y^j} = \{1, 2, \ldots, N_j\}$ with a generic element s_{y^j}

$S_Y = \{s_y = (s_{y^1}, s_{y^2}, \ldots, s_{y^K}) \in S_{Y^1} \times S_{Y^2} \times \cdots \times S_{Y^K}\}$

$S_P = \{1, 2, \ldots, M\}$ with a generic element s_P where $M = (J \cdot N_1 \cdot N_2 \cdots N_K)$.

Given these state spaces select any 1–1 invertible map $\Phi: S_D \times S_Y \to S_p$ which we write as

$$s_p = \Phi(s_d, s_y) \tag{25}$$

Without changing our notation we now view the Q^k as probabilities on the spaces $((S_D \times S_P \times S_{Y^k})^\infty, \mathscr{B}((S_D \times S_P \times S_{Y^k})^\infty))$ and Π on $((S_D \times S_P \times S_Y)^\infty, \mathscr{B}((S_D \times S_P \times S_Y)^\infty))$.

Definition 4. *$(S_D, S_P, S_{Y^1}, \ldots, S_{Y^K})$ is a consistent price state space for $(\Pi, Q^1, \ldots, Q^K, \Phi)$ if*

1. *Condition (25) is satisfied;*
2. $m_{DP} = Q^k_{DP}$ *for* $k = 1, 2, \ldots, K$.

We shall use simpler terminology in applying Definition 4 to the multi-agent OLG stock market economy when Π_{DY} and Q^k for $k = 1, 2, \ldots, K$ are being considered. We then say that *the economy has a consistent price state space for $(\Pi, Q^1, Q^2, \ldots, Q^K)$ if $(S_D, S_P, S_{Y^1}, S_{Y^2}, \ldots, S_{Y^K})$ is a consistent price state space for $(\Pi, Q^1, Q^2, \ldots, Q^K, \Phi)$.*

6b The existence of an RBE for every consistent price state space

The general question of existence of an RBE is resolved by Gottardi (1990) if all agents ignore their generating sequences. In that case an RBE is a stationary rational expectation equilibrium. Our result is much broader and extends Theorem 3.

Theorem 4. *Consider a multi-agent OLG stock market economy which satisfies Assumptions 6.0, 6.1, 6.2 and 6.3. For any pair (Q, Π) for which the economy has a consistent price state space there exists an RBE.*

Proof. See Appendix (A.II).

Remark. As suggested earlier, the presence of endogenous uncertainty in an RBE is of central importance if Theorems 3 and 4 are to be of interest. We shall not give a formal proof which can be constructed using a transversality argument. Instead, the numerical simulations below will be used to illustrate the observation of sensitivity of equilibrium prices to the belief parameters. This result is true as long as the utility functions are "forward looking" in the sense that changes in beliefs change individual portfolio demands.

The explanation why endogenous uncertainty is generically present will be provided with the results of the simulation in Section (6.c.2).

6c *Correlation and price volatility: simulations of a two agent economy*

In Section (6.b) we showed that the probability Π_{DY} determines the dynamics of the price states in any RBE. Moreover, Π_{DY} measures the implicit correlation among the generating variables of the agents and how this correlation is affected by the common data which they observe. Here we formulate a relatively simple model which is numerically solved with the aim of showing the impact of the correlation among the agents on aggregate price dynamics. We stress the *implicit* nature of this correlation since agents do not incorporate it directly in their belief.

6c.1 The simulation model

(a) *The economy*. The simulation model is a relatively simple one with two agents denoted $k = 1, 2$ who have the same utility function over consumptions (x^1, x^2) of the form

$$u(x^1, x^2) = \frac{1}{1-\gamma}(x^1)^{1-\gamma} + \frac{\beta}{1-\gamma}(x^2)^{1-\gamma} \qquad \gamma > 0. \tag{26}$$

The economy has only one consumed good, $N = 1$ with $D = \{d^H, d^L\}$, $Y^k = \{1, 0\}$ for $k = 1, 2$. The individual generating processes are i.i.d. with the probabilities of 1 being α_1 and α_2. Normalizing the prices select $p_t^c = 1$ for all t. It follows that $M = 8$ and $(P_1, P_2, \ldots, P_8)$ is the vector of 8 possible prices of the common stock of the single firm. The map Φ in (25) is then

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \Phi \begin{bmatrix} d_1 = d^H, & y_1^1 = 1, & y_1^2 = 1 \\ d_2 = d^H, & y_2^1 = 1, & y_2^2 = 0 \\ d_3 = d^H, & y_3^1 = 0, & y_3^2 = 1 \\ d_4 = d^H, & y_4^1 = 0, & y_4^2 = 0 \\ d_5 = d^L, & y_5^1 = 1, & y_5^2 = 1 \\ d_6 = d^L, & y_6^1 = 1, & y_6^2 = 0 \\ d_7 = d^L, & y_7^1 = 0, & y_7^2 = 1 \\ d_8 = d^L, & y_8^1 = 0, & y_8^2 = 0 \end{bmatrix} \tag{27}$$

We refer to d^H as the "high dividends" and to d^L as the "low dividends".

The two agents believe that the variables in the economy constitute a Markov process. In specifying the beliefs Q^k for $k = 1, 2$ Assumption 6.1 stipulates the state space of agent k to be $D \times P \times Y^k$. For simplicity we assume that each agent has two

Markov transition matrices on the observables (d_t, p_t) which he selects depending upon the value of y_t^k. More specifically, when $y_t^k = 1$ one matrix is used while when $y_t^k = 0$ the second is employed. The y_t^k are assumed serially independent. Note that since the agent not know the map (27) his transition matrices are of dimension 16×16. However, given the map (27) it is clear that 8 of the states in the agent's models will never be observed (e.g.: p_2 with d^L). The rationality conditions obviously ensure that in each of the two matrices which represent the models of the agents, the stationary probability of these states is 0 and since the system is ergodic these are transient states and have no effect on the long run statistics of the economy. Since we are concerned here only with the long run characteristics of the economy we simplify the exposition considerably by ignoring the transient states and consider only the 8 non-transient states specified in the map (27). We later specify the two Markov transition matrices of the agents on $P \times P$ with a full understanding that the first four states are the "high dividend" states and the last four are the "low dividend" states.

Formulating the budget constraints of the two agents in terms of the price states (as in (A.9a)–(A.9b), Appendix) we have for $k = 1, 2$ and $s, j = 1, 2, \ldots, 8$

$$x_s^{1k} = \omega^k - \theta_s^k P_s \tag{28a}$$

$$x_{sj}^{2k} = \theta_s^k (P_j + d_j). \tag{28b}$$

Denote by $Q^k(j \mid s, y^k)$ agent k's conditional probability of price state j given price state s and the value of y^k but without k's knowledge of the map (27). Optimization with respect to θ_s^k, implies that the first order conditions for $k = 1, 2$ can be stated in the well known form

$$\left[\beta \sum_{j=1}^{8} \left(\frac{P_j + d_j}{P_s}\right)^{1-\gamma} Q^k(j \mid s, y^k)\right]^{-1/\gamma} = \frac{\frac{\omega^k}{P_s} - \theta_s^k}{\theta_s^k} \qquad s = 1, 2, \ldots, 8. \tag{29}$$

Once we specify (Q^k, ω^k) for $k = 1, 2$ and the vector $(d_1, d_2, \ldots, d_8)$ of dividend values we can compute θ_s^k for $k = 1, 2$, as a function of the 8 prices. In equilibrium

$$\theta_s^1 + \theta_s^2 = 1 \quad \text{for all } s. \tag{30}$$

(29)–(30) implies a system of 8 equations in prices.

We now select the probability Π_{DY} so that its marginal measure Π_D is compatible with the Markovian assumptions commonly made in the literature (e.g. Mehra and Prescott [1985]). This specifies the stationary measure of the dividend process with a transition matrix of the form

$$\begin{bmatrix} \phi, 1-\phi \\ 1-\phi, \phi \end{bmatrix} \tag{31}$$

We also want the joint process $[y^1, y^2]$ of the generating variables to be simple so that we can state the rationality conditions with ease. We then require that

$$\text{the marginal measures } \Pi_{Y^k} \text{ specify } y_t^k \text{ to be i.i.d. with } P\{y_t^k = 1\} = \alpha_k. \tag{32}$$

The following transition matrix Γ defines a stationary probability Π_{DY}[5] satisfying these conditions:

$$\Gamma = \begin{bmatrix} \phi A, (1-\phi)A \\ (1-\phi)A, \phi A \end{bmatrix} \tag{33}$$

where A is a 4×4 matrix, characterized by the 6 parameters (α_1, α_2, a) and $a = (a_1, a_2, a_3, a_4)$:

$$A = \begin{bmatrix} a_1, \alpha_1 - a_1, \alpha_2 - a_1, 1 + a_1 - \alpha_1 - \alpha_2 \\ a_2, \alpha_1 - a_2, \alpha_2 - a_2, 1 + a_2 - \alpha_1 - \alpha_2 \\ a_3, \alpha_1 - a_3, \alpha_2 - a_3, 1 + a_3 - \alpha_1 - \alpha_2 \\ a_4, \alpha_1 - a_4, \alpha_2 - a_4, 1 + a_4 - \alpha_1 - \alpha_2 \end{bmatrix}. \tag{34}$$

It follows from (34) that under Π_{DY}, $P\{y_t^k = 1\} = \alpha_k$ for $k = 1, 2$; this is compatible with our individual specifications in (32). Also, although each process $[y^k]$ for $k = 1, 2$ is very simple, the joint process may be complex; it allows joint correlation among the y_t^k over time, and this is our next topic.

(b) *Correlation and implicit consensus.* Our interest in correlation arises from the fact that correlated actions of agents have far reaching effects on the dynamics of the economy. However, there are two basic ways in which correlated actions are realized in the model at hand. The first is purely statistical: when the generating variables (y_t^1, y_t^2) are correlated the generating states $((1,1), (0,1), (1,0), (0,0))$ are not equally likely. However, the lack of statistical independence of these variables is not meaningful without the second form of "implicit consensus" which we define as *the degree of commonality of the two Markov matrices selected by the agents given the realized values of their private generating variables.* In short, we think of "correlation" as a statistical relationship between generating variables. On the other hand "implicit consensus" is related to the way the agents *interpret* the private generating variables and in our model can be defined in terms of a specific collection of matrices which the agents may select. This will be explicitly formulated below.

Turning first to statistical correlation we aim to identify parameters of the model whose change leads to increased correlation between the generating variables. However, observe that statements about "the effect of increased degree of correlation on price volatility" depend upon the baseline for comparison. The reason for this is that when we change a parameter which changes correlation *we also change equilibrium prices.* Hence changing any parameter has two effects on price volatility: the first is a pure correlation effect which *changes the stationary distribution of a given set of equilibrium prices* and the second is the effect on equilibrium prices. In order to simplify the evaluation of these two effects we take the pure i.i.d. case as the basic reference. This case is defined by the parameters $\phi = .5$, $\alpha_1 = \alpha_2 = .5$ and $a_i = .25$ for $i = 1, 2, 3, 4$. It is easy to see that in this case the stationary distribution $(\pi_1, \pi_2, \ldots, \pi_8)$

[5] Thus Π_{DY} is not necessarily stationary but if non-stationarity in the dividend process is represented with a generating variable then this variable is required, in the simple case of this paper, to be independent of the individual generating variables.

implied in (33) is $\pi_i = .125$ for all i. If we assume, in addition, that the agents adopt the stationary measure as their belief, then equilibrium prices are constant. Hence the variance $\sigma_p^2 = 0$ in this rational expectations equilibrium provides a natural reference point to the endogenous effect of correlation on price volatility in an RBE.

Under $\phi = \alpha_1 = \alpha_2 = .5$ we calculate the stationary distribution $(\pi_1, \pi_2, \ldots, \pi_8)$ implied by (33). Let $\xi = a_1 + a_4 - a_2 - a_3$ then the solution is

$$\pi_1 = \pi_4 = \pi_5 = \pi_8 = \frac{1}{8}\,\frac{2(a_2 + a_3)}{1 - \xi} \tag{35}$$

$$\pi_2 = \pi_3 = \pi_6 = \pi_7 = .25 - \pi_1. \tag{36}$$

In this case, each of the sums $(a_1 + a_4)$ and $(a_2 + a_3)$ is a scalar measure of correlation: an increase in each of them increases the probabilities $(\pi_1, \pi_4, \pi_5, \pi_8)$ of the states of agreement when $y_s^1 = y_s^2$. These probabilities can move between 0 and .25. In the calculations below we evaluate some models in which the condition $\phi = \alpha_1 = \alpha_2 = .5$ does not hold. We turn now to the second issue of implicit consensus but in order to discuss it we must first review the structure of Rational Beliefs of the two agents.

(c) *Rational Beliefs.* The probability beliefs of the agents are constructed with the aid of two pairs of matrices, (F_1, F_2) for agent 1 and (G_1, G_2) for agent 2. Given the definition of the generating sequences it follows from Nielsen (1994, page 40) that Rationality of Beliefs requires

$$\alpha_1 F_1 + (1 - \alpha_1) F_2 = \Gamma, \qquad \alpha_2 G_1 + (1 - \alpha_2) G_2 = \Gamma. \tag{37}$$

Given this condition, the following conditional probabilities (where F_1^{sj} is the (s, j) element of F_1)

$$Q_t^1(j \mid s, y_t^1) = \begin{cases} F_1^{sj} & \text{if } y_t^1 = 1 \\ F_2^{sj} & \text{if } y_t^1 = 0 \end{cases} \qquad Q_t^2(j \mid s, y_t^2) = \begin{cases} G_1^{sh} & \text{if } y_t^2 = 1 \\ G_2^{sj} & \text{if } y_t^2 = 0 \end{cases} \tag{38}$$

define the beliefs Q^k for $k = 1, 2$. We next select the four matrices (F_1, F_2, G_1, G_2) by using two sets of 8 parameters $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_8)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_8)$ which will be motivated later. To do that we introduce the notation for the row vectors of A:

$$A^j = (a_j, \alpha_1 - a_j, \alpha_2 - a_j, a_{j4}) \qquad a_{j4} = 1 + a_j - (\alpha_1 + \alpha_2).$$

With this notation we define the 2 matrix functions of $z = (z_1, z_2, \ldots, z_8)$ as follows:

$$A_1(z) = \begin{bmatrix} z_1 A^1 \\ z_2 A^2 \\ z_3 A^3 \\ z_4 A^4 \end{bmatrix}, \quad A_2(z) = \begin{bmatrix} (1 - \phi z_1) A^1 \\ (1 - \phi z_2) A^2 \\ (1 - \phi z_3) A^3 \\ (1 - \phi z_4) A^4 \end{bmatrix}. \tag{39}$$

We then finally define

$$F_1 = \begin{bmatrix} \phi A_1(\lambda), & A_2(\lambda) \\ (1 - \phi) A_1(\lambda), & A_2(\lambda) \end{bmatrix} \qquad G_1 = \begin{bmatrix} \phi A_1(\mu), & A_2(\mu) \\ (1 - \phi) A_1(\mu), & A_2(\mu) \end{bmatrix} \tag{40}$$

and (F_2, G_2) determined by (38). The motivation for this construction of the matrices F_1 and G_1 is that the parameters λ_s and μ_s are multiplied by the rows of the matrix A and hence are interpreted as proportional adjustments of the conditional probabilities of the four states (1, 2, 3, 4) and (5, 6, 7, 8) relative to the stationary measure represented by Γ. Since the first four are the "high dividend" states and the last four are the "low dividend" states, $\lambda_s > 1$ means that when using F_1 agent 1 perceives the conditional probabilities of the high dividend state given s to be larger by a factor λ_s than the corresponding probabilities in Γ. $\mu_s > 1$ has a similar interpretation for agent 2. This means that the 16 parameters (λ, μ) measure the relative degree of "optimism" or "pessimism" of an agent in a given states about the probabilities of the states of high or low dividends. We can now relate the above definitions to the concept of agreement or "implicit consensus". The central case which we consider in the simulations below is the one where $\lambda_s > 1$ and $\mu_s > 1$ for all s. It follows from the definitions (39)–(40) that in this case both agents become relatively more optimistic about the high dividend states when $y_t^k = 1$ for $k = 1, 2$ and both become relatively more pessimistic when $y_t^k = 0$ for $k = 1, 2$. Given the way in which the agents interpret y_t^k we say that *the states* (0, 0) *and* (1, 1) *are states of agreement or implicit consensus and when* $\lambda = \mu$, this common value measures the intensity of such *consensus when they agree or, simply, the degree of variability in the beliefs of the agents.* Consequently, a natural case which we shall consider is $\lambda_s = \mu_s = z$ for all s and then we use the number z as a scalar measure of intensity. We shall examine below how increasing this measure changes long term price volatility.

6c.2 Simulation results

In this section we present four sets of calculations and in the next section discuss the interpretation of their results. In all cases we use the following parameter values: $\beta = .85$, $\gamma = 2$, $d^H = 10$, $d^L = 0$ and in case (a), (b), (c) we set $\omega^1 = 100$, $\omega^2 = 100$. In case (d) we consider the effect of income distribution by varying the ω's.

Case (a) *The presence of endogenous uncertainty.* We promised earlier to use the simulations in order to illustrate how variations in the state of belief cause variations in equilibrium prices. Thus, consider our model with the following parameters: $a_i = .25$ for all i, $\phi = .4$, $\alpha_1 = .4$, $\alpha_2 = .6$, $\lambda = (.4, .9, .5, .6, 1.3, .7, .9, 1.6)$, $\mu = (1.5, 1.4, 1.3, 1.2, 1.1, 1.0, .9, .8)$. Equilibrium prices are ($P_1 = 95.16306$, $P_2 = 95.59644$, $P_3 = 94.64301$, $P_4 = 95.12190$, $P_5 = 93.93110$, $P_6 = 94.70204$, $P_7 = 94.41132$, $P_8 = 94.47654$). The agents' beliefs are characterized by the 18 parameters $(\lambda, \mu, \alpha_1, \alpha_2)$ and we now claim that a change in any one of these parameters changes the vector of equilibrium prices.

Case (b) *Increased intensity in consensus states.* Consider now the following scalar measure of intensity mentioned above: $\lambda_s = \mu_s = z$ for all s, $a_i = .25$ for all i and then vary the number z. The results are as follows:

z	1	1.2	1.4	1.6	1.8
σ_p^2	0	.0576	.2304	.5186	.9227

Recall that the underlying economy is the pure i.i.d. case and $\lambda_s = \mu_s = 1$ is a rational expectations equilibrium in which prices are constant (hence $\sigma_p^2 = 0$). Also, note that the variations in (λ, μ) have no effect on the stationary distribution of prices which is $\pi_i = .125$ for $i = 1, \ldots, 8$. The effect of increased intensity on volatility is then *entirely a result of the change in equilibrium prices.*

Case (c) *Increased correlation with fixed level of $\lambda = \mu$.* We now fix $\lambda_s = \mu_s = 1.8$ for all s while setting $\phi = \alpha_1 = \alpha_2 = .5$. Relative to the i.i.d. economy with $a_i = .25$ for all i we increase simultaneously (a_1, a_4) or (a_2, a_3). As shown above this increases the correlation between y_t^1 and y_t^2 and the stationary probabilities π_1, π_4, π_5 and π_8. The results are then:

a	$a_1 = a_4 = .25$ $a_2 = a_3 = .25$	$a_1 = a_4 = .35$ $a_2 = a_3 = .25$	$a_1 = a_4 = .49$ $a_2 = a_3 = .25$
σ_p^2	.9227	1.1536	1.7750

a	$a_1 = a_4 = .25$ $a_2 = a_3 = .25$	$a_1 = a_4 = .25$ $a_2 = a_3 = .35$	$a_1 = a_4 = .25$ $a_2 = a_3 = .49$
σ_p^2	.9227	1.0765	1.2220

These results complement the volatility calculations of Case (b). The value $\sigma_p^2 = .9227$ is obtained in Case (b) only by increasing z while maintaining the independence of y_t^1 and y_t^2. By introducing the correlation between the generating variables the variance of prices is nearly doubled to 1.7750 in the table above. There is one subtle difference between Cases (b) and (c). In Case (b) we change only equilibrium prices while keeping the stationary distribution of the price states fixed. In Case (c) we would have liked to change only the stationary distribution *while keeping prices fixed* but this is not possible: changing $a = (a_1, a_2, a_3, a_4)$ changes equilibrium prices and this change could reduce volatility. It is interesting to note that in the case at hand the correlation effect dominates the potential price change effect on volatility. We return to these considerations in section (6.c.3) when we discuss these results.

Case (d) *Effect of income distribution.* Changes in income distribution change the demand for risky assets. To see the effect on volatility consider the case where $\phi = \alpha_1 = \alpha_2 = .5$, $a_i = .25$ and $\lambda_s = \mu_s = 1.4$ for all i and all s. Relative to this baseline consider the effect of changing the distribution of (ω^1, ω^2):

(ω^1, ω^2)	(100, 100)	(80, 120)	(60, 140)	(40,160)	(20, 180)
σ_p^2	.2304	.2396	.2672	.3133	.3778

The results do not show a dramatic effect but the changes are significant enough to be noted.

6c.3 Theoretical implications of the simulation results

The simulation results in Cases (b) and (c) show that endogenous uncertainty can be the dominating cause of price volatility in asset markets. However, the mechanism which generates price volatility needs to be explored. We stressed that the method of changing the common value of (λ, μ) has the property of having no effect on the stationary distribution of prices. Explaining the causes for increased price volatility in this case is central to this paper and is formulated in terms of the *Regime Process* $\{R_t, t = 1, 2, \dots\}$. This process is defined by

$$R_t = \begin{cases} 1 & \text{if } y_t^1 = y_t^2 \\ 0 & \text{if } y_t^1 \neq y_t^2. \end{cases} \tag{41}$$

A regime process arises in a natural way in this model of endogenous uncertainty since it provides a quantitative representation of the interaction among the generating variables of the agents. The regime process attains its significance from the correlation among the generating variables and from the interpretation of the states of the process in relation to the states of implicit consensus. In general, the regime process has a complex distribution but under the assumptions $a_1 = a_4$, $a_2 = a_3$ and $\phi = .5$, $\{R_t, t = 1, 2, \dots\}$ is a stationary Markov process with transition matrix and stationary distribution

$$\Gamma_R = \begin{bmatrix} 1 - \alpha_1 - \alpha_2 + 2a_1, & \alpha_1 + \alpha_2 - 2a_1 \\ 1 - \alpha_1 - \alpha_2 + 2a_2, & \alpha_1 + \alpha_2 - 2a_2 \end{bmatrix} \qquad \pi_R = \begin{bmatrix} \dfrac{1 - \alpha_1 - \alpha_2 + 2a_2}{1 - 2(a_1 - a_2)} \\ \dfrac{\alpha_1 + \alpha_2 - 2a_1}{1 - 2(a_1 - a_2)} \end{bmatrix}.$$

In Case (b) we assume that $\alpha_1 = \alpha_2 = .5$ and $a_1 = a_2 = .25$ and in this case all the entries in the matrix Γ_R are .5 and hence the R_t are i.i.d. with a frequency of the consensus states $((0,0),(1,1))$ being the same as that of the disagreement states $((1,0),(0,1))$. However, the economic implications of the $R_t = 1$ regime are drastically different from the implications of the $R_t = 0$ regime. To see this note that when an agent is relatively more optimistic in one state, then rationality of beliefs requires him to be relatively more pessimistic in the complementary states. But fluctuations between optimism and pessimism cause fluctuations in his demand for assets. If one agent is optimistic and the other pessimistic they tend to cancel each other out and hence the regime of disagreement ($R_t = 0$) is a low volatility regime while when $R_t = 1$ there is agreement and consequently this is the high volatility regime. In fact, in Case (b) $P_1 = P_5$, $P_2 = P_6$, $P_3 = P_7$, $P_4 = P_8$, and in all cases $P_2 = P_3$. P_1 is the low price and P_4 is the high price of the high volatility regime. As the common value of $\lambda_s = \mu_s$ increases the low price P_1 decreases and the high price P_4 rises further. The intermediate price $P_2 = P_3$ (which is the single price of the low volatility regime) remains in the same range. As the common value of $\lambda_s = \mu_s = z$ increases, the variance of prices in the volatile regime increases while the variance of prices in the low volatility regime remains zero.

To clarify how the regime structure affects the time series of prices in the economy consider the case $\lambda_s = \mu_s = 1.8$, $\phi = .5$, $a_1 = a_4 = .47$, $a_2 = a_3 = .02$. This case yields higher level of persistence of the regime process relative to the i.i.d. case.

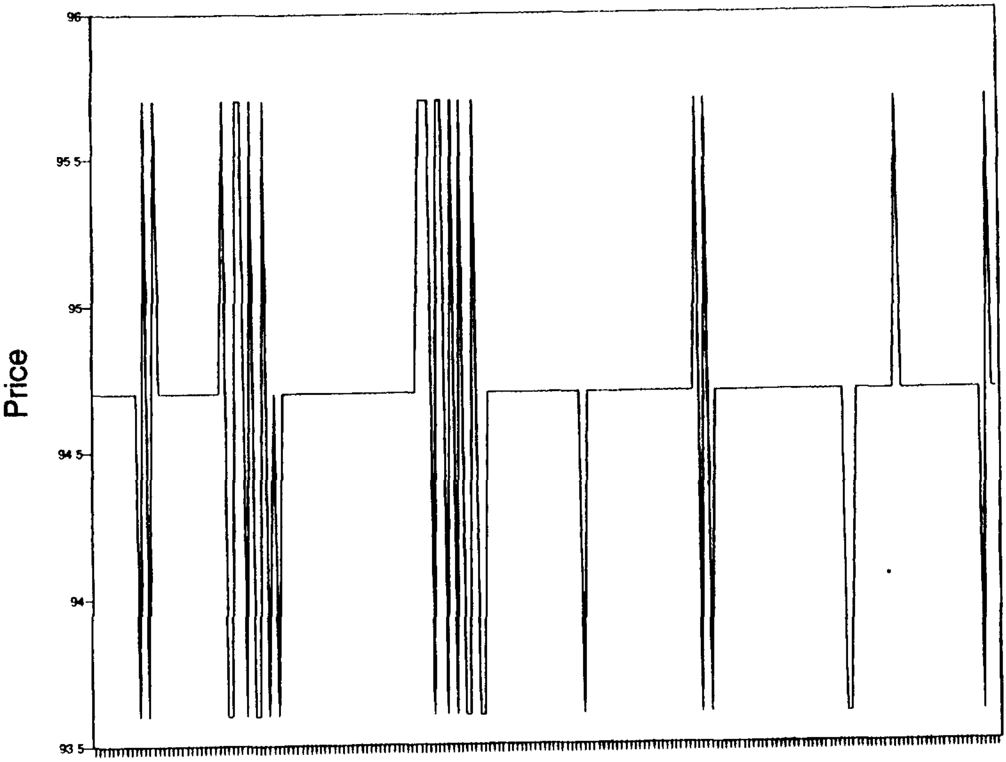

Diagram 1

As in Case (b) there are two prices in the volatile regime and a single price in the low volatility regime. A typical time series realization of 200 observations of equilibrium prices is plotted in diagram 1.

Turning to the effect of correlation we note that there we selected a fixed set of (λ, μ). As we increase $(a_1 = a_4)$ or $(a_2 = a_3)$ we increase the correlation between y_t^1 and y_t^2 and this changes the transition matrix Γ_R and the distribution π_R. In fact, in Case (c) $\alpha_1 = \alpha_2 = .5$ and hence the transition matrix is reduced to $\Gamma_R = \begin{bmatrix} 2a_1, & 1-2a_1 \\ 2a_2, & 1-2a_2 \end{bmatrix}$ and as $a_1 \to .5$ the relative frequency of the volatile regime increases towards 1. Thus, increased correlation increases long term volatility in this case mostly by increasing the *frequency* of the volatile regime. We use the term "mostly" in order to remind the reader that in this case the changes in the parameters *do change equilibrium prices* and these changes can reduce equilibrium volatility. Since the *correlation effect* dominates the *price change effect* the net result is increased volatility.

The regime process also helps clarify the nature of the asymmetry in the response to changes in $a_1 = a_4$ vs. $a_2 = a_3$ seen in Case (c). It is clear that the stationary probability of the high volatility regime increases when either $a_1 = a_4$ or $a_2 = a_3$ is increased. However, when $\alpha_1 = \alpha_2 = .5$ then as $a_1 \to .5$ the stationary probability of

$R = 1$ tends to 1 for any $a_2 > 0$ while as $a_2 \to .5$ the stationary probability of $R = 1$ tends to $\frac{1}{2(1-a_1)}$ which is generally less than 1. The explanation for this asymmetry is that as $a_1 \to .5$ the consensus states (1, 4, 5, 8) become absorbing states and the disagreement states (2, 3, 6, 7) become transient states for any $a_2 > 0$. On the other hand $a_2 \to .5$ does not ensure that states (1, 4, 5, 8) become absorbing unless $a_1 = .5$. Since in Case (c) the baseline parameter values are $a_i = .25$ all i, it is seen that increases in $a_1 = a_4$ have a stronger effect on price volatility than increases in $a_2 = a_3$.

A more complex regime structure arises when dividends provide useful information and that is the case if $\phi \neq .5$. Consider the case $\phi = .6$, $a_1 = a_4 = .47$, $a_2 = a_3 = .02$ and $\lambda_j = \mu_j = 1.6$ and all j. In that case there are four regimes: $(R'_1 = (y_t^1 = y_t^2, d^H), R'_2 = (y_t^1 = y_t^2, d^L), R'_3 = (y_t^1 \neq y_t^2, d^H), R'_4 = (y_t^1 \neq y_t^2, d^L))$. Prices are now driven by the interaction of two independent Markov factors: the distribution of beliefs as parametrized by R'_t and by dividends. The result is a Markov process in which the regimes of agreement (R'_1, R'_2) are the high variance regimes and the regimes of disagreement (R'_3, R'_4) are the low variance regimes. The level of

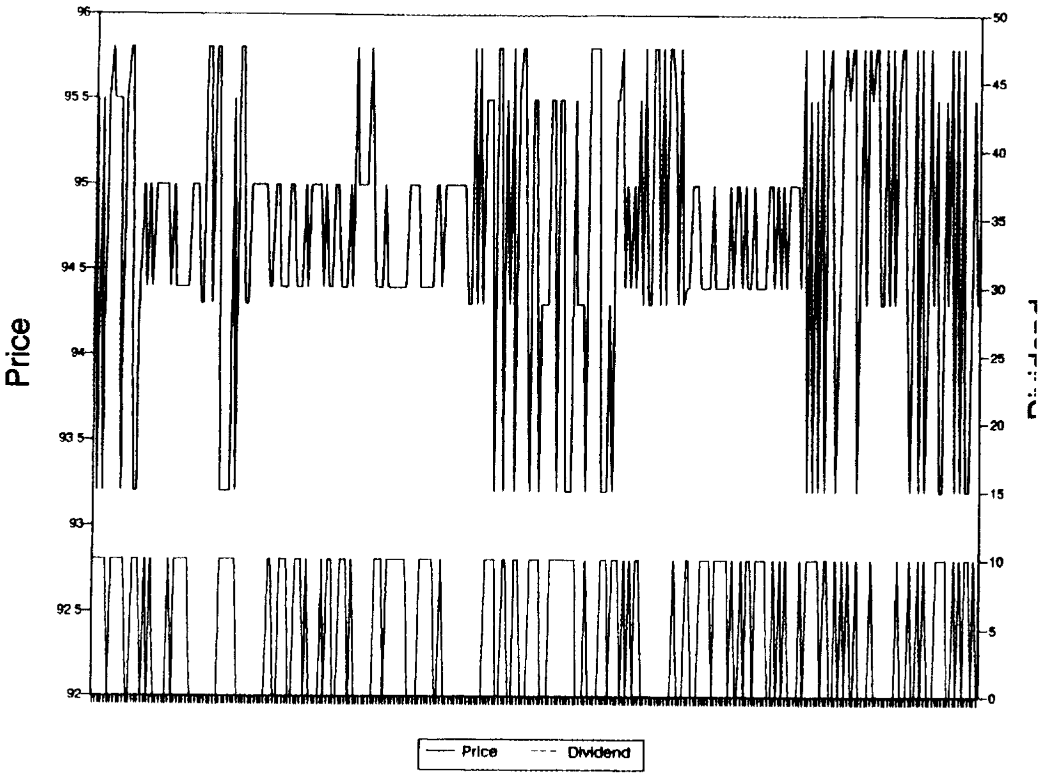

Diagram 2

prices and their variances change with dividends: in the low variance regimes each regime is characterized by a single price. The value of the stock is lower when the dividend is higher. In the high variance regime of agreement prices are *negatively correlated* with current dividends while the *variance of prices is positively correlated* with current dividend. A typical time series of 300 equilibrium prices is plotted in diagram 2.

It is obvious that other patterns can be constructed which will result in more complex volatility regimes. In particular, interesting complexity may result from the interaction between the dividend process and the process R'. However, the relatively simple cases presented here explain the nature of the non-stationary mechanism which the generating sequences play in the economy. They constitute the mechanism which drives *the endogenous propagation of diverse price volatility regimes* while the probability structure of these generating processes are the quantitative tools for describing the time variability of the volatility regimes. The variability over time of the variance of asset prices is a well documented empirical phenomenon and has given rise to the GARCH family of statistical models which describe such price movements (see, for example, Hamilton (1994) Chapter 21). The theory presented here provides a micro-economic explanation for these empirical observations. Equally important, the available empirical evidence provides a strong support for the theory advanced here.

7 Summary and conclusions

The methodological and the substantive results of this paper are interrelated. We have attempted to show that the method of generating variables, expressing the private signals which agents receive or perceive, is a useful tool for the study of stable and non-stationary processes. With the aid of such variables a rational belief is viewed in this paper as a stable joint system of generating and observable variables where the rationality conditions place restrictions on the marginal measures. We also show that the use of generating variables enables us to study the structure of a large family of RBE and prove the existence of an RBE for any economy which has a consistent price state space. This implies the existence of different RBE for an economy with the same exogenous physical characteristics.

The use of generating variables provides a powerful tool for the study of the effect of correlation among agents on price volatility. The results of this paper are central to the study of RBE in that they show that heterogeneity of beliefs has aggregate implications only to the extent that these beliefs are correlated in some way. If beliefs of agents are uniformly dispersed on the set $B(\Pi)$ and if the mechanisms for individual selection of probability distributions at each date are "independent" across agents then we should expect heterogeneity of beliefs to have little or no aggregate economic impact. On the other hand, our simulation results show that the effect of correlation across agents on the volatility of asset prices can be very dramatic and can be the dominant factor in the fluctuations of such prices. The mechanism which generates this effect works through the clustering of beliefs in states of different level of agreement. In states of consensus the conditional forecasts of the agents fluctuate, to some degree, *together* and for this reason asset prices tend to be more volatile. In states of disagreement the conditional forecasts fluctuate *in diverse directions* tending to cancel each other's effect on aggregate demand and resulting in reduced price volatility.

Appendix

(A.I) Proof of Theorem 2

Let $\mathbb{C}_X$ be the cylinders in $\mathscr{B}(X^\infty)$. To evaluate stability we need to consider cylinders of the form $(C \times Y^\infty)$ where $C \in \mathbb{C}_X$. Observe that for such cylinders one considers convergence only on the cylinders in X^∞ since $\frac{1}{N}\sum_{k=0}^{N-1} 1_{C \times Y^\infty}(T^k(x,y)) = \frac{1}{N}\sum_{k=0}^{N-1} 1_C(T^k x)$. Hence, define

$$K_C^x = \left\{ x \in X^\infty : \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} 1_C(T^k x) \quad \text{exists} \right\}.$$

We then have $K_{C \times Y^\infty} = K_C^x \times Y^\infty$. Lemma 1 implies that for each cylinder $(C \times Y^\infty)$ there exists a set $B_C \in \mathscr{B}(Y^\infty)$ such that $\Pi(K_C^x \times B_C) = 1$ and for any $y \in B_C$, $\Pi(K_C^x \times B_C | \mathscr{L})(x, y) = 1$. Stability for Π almost all y requires a set $B \in \mathscr{B}(Y^\infty)$ such that $\Pi(K_C^x \times B | \mathscr{L})(x, y) = 1$ independent of C. By an argument similar to the one used in Proposition 2 of Nielsen (1996) it follows that such a set exists if Y is countable. This proves that $(X^\infty, \mathscr{B}(X^\infty), \Pi_y, T)$ is stable for Π almost all y with a stationary measure which we denote by m^{Π_y}. This establishes the stability part of (a). Now let $G \in \mathbb{C}$ be defined by $G = A \times Y^\infty$ where $A \in \mathbb{C}_X$. Stability (proved in (a)) requires

$$\lim_{N\to\infty} \frac{1}{N} \sum 1_A(T^k x) = m^{\Pi_y}(A)(x) \quad \Pi_y \text{ a.e.} \tag{A1}$$

However, $1_A(T^k x) = 1_G(T^k(x, y))$ hence

$$\frac{1}{N} \sum_{k=0}^{N-1} 1_A(T^k x) = \frac{1}{N} \sum_{k=0}^{N-1} 1_G(T^k(x, y)). \tag{A2}$$

Now take limits on both sides of (A2). Since the joint system is stable and ergodic, the limit of the right hand side of (A2) exists and is independent of (x, y) Π a.e. Hence, modify (A1) to have

$$m^{\Pi_y}(A)(x) = m^{\Pi}(A) \quad \textit{independent of } (x, y)\ \Pi \textit{ a.e.}$$

The limit of the right hand side of (A2) is $m(A \times Y^\infty) = m_y(A)$. Hence, on $\mathbb{C}_X$ the two measures m^Π and m_X agree and this implies that $m^\Pi(S) = m_X(S)$ for all S in $\mathscr{B}(X^\infty)$. This proves (b).

To prove that $(X^\infty, \mathscr{B}(X^\infty), \Pi_y, T)$ is ergodic for Π a.a. y we recall again that $\mathbb{C}_X$ is a generating field of $\mathscr{B}(X^\infty)$. Hence, it follows from the proof of Lemma 6.7.4 of Gray (1987) that it is sufficient to prove that for any cylinder $A \in \mathbb{C}_X$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \Pi_y(T^{-k} A \cap A) = m^\Pi(A)\Pi_y(A). \tag{A3}$$

To show (A3) note that for Π a.a. y we have

$$\Pi_y(T^{-k}A\cap A)=\int_A 1_{T^{-k}A}(x)\Pi_y(dx)=\int_A 1_A(T^kx)\Pi_y(dx). \tag{A4}$$

Hence

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\Pi_y(T^{-k}A\cap A)=\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\int_A 1_A(T^kx)\Pi_y(dx). \tag{A5}$$

By the bounded convergence theorem we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\Pi_y(T^{-k}A\cap A)=\int_A\left(\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}1_A(T^kx)\right)\Pi_y(dx). \tag{A6}$$

But by (A1), (A2) the limit on the right hand side of (A6) is $m^{\Pi}(A)$ and is independent of x Π a.e. and hence Π_y a.e. as well. Hence we can conclude that for Π a.a. y

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\Pi_y(T^{-k}A\cap A)=m^{\Pi}(A)\Pi_y(A) \tag{A7}$$

and this proves (A3). Conclusion (c) is immediate. □

(A.II) Proof of Theorem 4

To prove the existence of an RBE for any (Q,Π) which has a consistent price state space we need to prove that (23) in the text has a solution. Our argument is brief at times since it is similar to the existence proof of Kurz and Wu (1996). We translate the agent's maximization to the price state space by writing the utility in (19a) for a belief $Q^k(j\,|\,i,y^k)$ as

$$U^k(x_i^{1k},x_{ij}^{2k})=\sum_{j=1}^{M}u^k(x_i^{1k},x_{ij}^{2k})Q^k(j\,|\,i,y^k),\quad i=1,\ldots,M. \tag{19a'}$$

We use the notation $q=(q_1,\ldots,q_M)\in\mathbb{R}^{M(N+1)}$ where $q_i=(p_i^c,p_{i1},\ldots,p_{iN})\in\mathbb{R}^{N+1}$ and the vector $(x_i^{1k},\theta_i^k)\in\mathbb{R}^{N+1}$ denotes the choices of young traders in state i. The homogeneity of (19b)–(19c) with respect to prices leads us to select the standard simplex

$$\Delta=\left\{q_i\in\mathbb{R}_+^{N+1}\,|\,p_i^c+\sum_{f=1}^{N}p_{if}=1\right\}\text{ and hence } q\in\Xi=\Delta\times\Delta\times\cdots\times\Delta\ (M\text{ times}). \tag{A8}$$

The budget correspondence of young traders in state i is then written, for $k=1,2,\ldots,K$ as

$$B_i^k(q,\omega^k)=\left\{(x_i^{1k},\theta_i^k)\in\mathbb{R}_+\times\mathbb{R}^N\,\middle|\,\begin{matrix}p_i^cx_i^{1k}+\theta_i^k\cdot p_i\le p_i^c\omega^k\\ 0\le x_{ij}^{2k}=\theta_i^k\cdot(p_j+p_j^cd_j)\end{matrix}\right\}\ q\in\Xi. \tag{A9a}$$

The correspondence (A9a) is sufficient to allow a trader to select his consumption and portfolio when young so as to maximize (19a): the consumption when old (i.e. the vectors x^{2k}) are determined by the choices when young. We do introduce the optimization of the period 1 old which is rather trivial, but useful. Their budget set is

$$B_j^{2k}(q_j, \theta_0^k) = \{y_j^k \in \mathbb{R}_+ \mid p_j^c y_j^k \leq \theta_0^k \cdot (p_j + p_j^c d_j), \quad \theta_0^k \geq 0\}, \quad q_j \in \Delta. \tag{A9b}$$

The following is standard:

Lemma 3. *The budget set correspondences of the young and the period 1 old are non-empty and for each q they are convex and compact valued, and continuous on the interior of* Ξ.

We now encounter the usual problem where demand correspondences are not defined on the boundary of the simplex. We denote by $\mathscr{E}$ our real economy and introduce a sequence of economies $\mathscr{E}^n$ where for each n the economy $\mathscr{E}^n$ is bounded in a cube nW. W is a compact cube centered on the zero vector and all the original budget sets are then intersected with nW to create new budget sets which are compact subsets of nW even when some prices equal 0. These budget correspondences are non-empty, convex and compact valued, and continuous *at all price vectors in* Ξ. A construction of the economies $\mathscr{E}^n$ requires complex additional notation. Since this is a standard procedure we avoid this added notation (for details see, for example, Kurz (1974b, sections 6-7)). Thus, when we say below that "a variable takes the value $+n$ in $\mathscr{E}^n$" we mean that it is on the boundary of the restricted budget set of the agent in $\mathscr{E}^n$.

Our notation for the demand correspondences of the young for $k = 1, \ldots, K$ and $i = 1, \ldots, M$ is $x_i^{1k} \in \varphi_{i0}^k$, $\theta_{if}^k \in \varphi_{if}^k$, $f = 1, \ldots, N$, $\varphi_i^k = (\varphi_{i0}^k, \varphi_{i1}^k, \ldots, \varphi_{iN}^k)$, $\varphi^k(q) = (\varphi_1^k, \ldots, \varphi_M^k)$. For period 1 old agents we use the notation $y_i^k \in \varphi_i^{2k}(q_i, \theta_0^k)$, $\varphi^{2k}(q) = (\varphi_1^{2k}, \ldots, \varphi_M^{2k})$. Thus, define the demand correspondences

$$\varphi_i^k(q, \omega^k) = \{(x_i^{1k}, \theta_i^k) \in \mathbb{R}_+ \times \mathbb{R}^N \mid (x_i^{1k}, \theta_i^k) \text{ maximizes (19a') on } B_i^k(q, \omega^k)\} \quad q \in \operatorname{int} \Xi. \tag{A10a}$$

$$\varphi_i^{2k}(q_i, \theta_0^k) = \{y_i^k \in \mathbb{R}_+ \mid y_i^k \text{ maximizes (19a') on } B_i^{2k}(q_i, \theta_0^k)\} \quad q_i \in \operatorname{int} \Delta. \tag{A10b}$$

It then follows from the theorem of the maximum and from Lemma 3 that

Lemma 4. *The demand correspondences* $(\varphi^k(q), \varphi^{2k}(q))$ for $k = 1, 2, \ldots, K$ *are non empty, convex and compact valued, and upper hemicontinuous on* int Ξ. *In each of the uniformly bounded economies* $\mathscr{E}^n$, *the vector of demand correspondences* $(\varphi^k(q), \varphi^{2k}(q))$ *is non-empty, convex and compact valued, and upper hemicontinuous on the entire price space* Ξ.

The market clearing conditions stipulate that the aggregate consumption of the young and the old at date t has to add to the total supply. That is, let $\omega_i = \sum_{k=1}^K \omega_i^k$ then we require that

$$\sum_{k=1}^K x_i^{1k} + \sum_{k=1}^K x_{ji}^{2k} = \omega_i + d_i \cdot 1 \quad \text{for } i, j = 1, \ldots, M. \tag{A11}$$

Now let $x_i^1 = \sum_{k=1}^K x_i^{1k}$, $y_i = \sum_{k=1}^K y_i^k$. We construct an RBE in which all dates are symmetric hence date 1 material balance requires that $X_i^1 + y_i = \omega_i + d_i \cdot 1$. Comparing with (A11) we can conclude that in equilibrium $\sum_{k=1}^{K} x_{ji}^{2k} = y_i$ holds at all dates. We summarize this observation by

Lemma 5. *For all q, i and j,* $\sum_{k=1}^{K} p_i^c x_{ji}^{2k} = \sum_{k=1}^{K} p_i^c y_i^k$. *Also, in equilibrium* $\sum_{k=1}^{K} x_{ji}^{2k} = \sum_{k=1}^{K} y_i^k \equiv y_i$.

By Lemma 5 we can reformulate the equilibrium condition (A11) to require

$$\sum_{k=1}^{K} x_i^{1k} + \sum_{k=1}^{K} y_i^k = \omega_i + d_i \cdot 1, \qquad \text{for all } i = 1, 2, \ldots, M. \tag{A12}$$

Next, the financial markets must clear and since all stocks have unit supply we require that

$$\sum_{k=1}^{K} \theta_i^k = 1 \qquad \text{for all } i = 1, 2, \ldots, M. \tag{A13}$$

(A12)–(A13) is a system of $M(N+1)$ market clearing conditions. Now use the notation introduced earlier to define the excess demand correspondences for $i = 1, \ldots, M$ by

$$\begin{aligned} \zeta_{i0}(q) &= \sum_{k=1}^{K} \varphi_{i0}^k(q) + \sum_{k=1}^{K} \varphi_i^{2k}(q_i) - (\omega_i + d_i \cdot 1) \\ \zeta_{ij}(q) &= \sum_{k=1}^{K} \varphi_{ij}^k(q) - 1, \qquad j = 1, \ldots, N. \end{aligned} \tag{A14}$$

We shall prove that any q satisfying $0 \in \zeta_i(q) = (\zeta_{i0}(q), \zeta_{i1}(q), \ldots, \zeta_{iN}(q))$ for $i = 1, \ldots, M$ is an equilibrium price system. A standard calculation shows that Walras' Law applies:

Lemma 6. *Under Assumption 4,* $q_i \cdot \zeta_i(q) = 0$ *for* $i = 1, 2, \ldots, M$, *that is,*

$$p_i^c \zeta_{i0}(q) + \sum_{f=1}^{M} p_{if} \zeta_{if}(q) = 0 \tag{A15}$$

Recall that we use the notation $(x_i^1 \in \mathbb{R}, x_{ij}^2 \in \mathbb{R}, y_i \in \mathbb{R})$ for the aggregates over k. We now use the notation $(x, \theta, y) \in \mathbb{R}^{KM(N+2)}$ for the *entire array*

$$(x, \theta, y) = ((x_i^{1k}, \theta_i^k, y_i^k), \quad k = 1, 2, \ldots, K, \quad i = 1, 2, \ldots, M). \tag{A16}$$

Define the maps μ_i by

$$\mu_i(x, \theta, y) = \{q_i \in \Delta \mid p_i^c(x_i^1 + y_i - \omega_i - d_i \cdot 1) + p_i \cdot (\theta - 1) \text{ is maximized over } \Delta\}$$

and $\mu(x, \theta, y) = \times_{i=1}^{M} \mu_i(x, \theta, y)$. Finally, define the map Ψ by

$$\Psi((x, \theta, y), q) = \zeta(q) \times \mu(x, \theta, y).$$

It follows from Lemma 4 that in $\mathscr{E}^n$, Ψ is a non-empty, convex and compact valued, and upper hemicontinuous correspondence from $nW \times \Xi$ into itself. It follows from the Kakutani fixed point theorem that it has a fixed point $(x^*, \theta^*, y^*, q^*)$ in $nW \times \Xi$ (Note: as our practice, we do not designate the variables in $\mathscr{E}^n$ by n).

Hence we conclude that

$$q^* \in \mu(x^*, \theta^*, y^*) \tag{A17a}$$

$$(x^*, \theta^*, y^*) \in \zeta(q^*). \tag{A17b}$$

Condition (A17a) states that for any $(p_i^c, p_i) \in \Delta$

$$p_i^{c*}(x_i^{1*} + y_i^* - \omega_i - d_i \cdot 1) + p_i^* \cdot (\theta_i^* - 1) \geq p_i^c(x_i^{1*} + y_i^* - \omega_i - d_i \cdot 1) + p_i \cdot (\theta_i^* - 1). \tag{A18}$$

Condition (A17b) states that (x^*, θ^*, y^*) are individually optimal in $\mathscr{E}^n$ relative to q^* and hence satisfy the budget constraints (A9a)–(A9b). But then by Lemma 6 we have that for all i

$$p_i^c(x_i^{1*} + y_i^* - \omega_i - d_i \cdot 1) + p_i \cdot (\theta_i^* - 1) \leq 0 \qquad \text{for all } (p_i^c, p_i) \in \Delta. \tag{A19}$$

(A19) implies that

$$x_i^{1*} + y_i^* - \omega_i - d_i \cdot 1 \leq 0 \quad \text{for } i \tag{A20a}$$

$$\theta_i^* - 1 \leq 0 \qquad \text{all } i. \tag{A20b}$$

Lemma 7. *For large n, the fixed point $(x^*, \theta^*, y^*, q^*)$ in $\mathscr{E}^n$ satisfies $(p^{c^*}, p) \gg 0$ for all (i, j).*

Proof. Suppose that $p_i^{c*} = 0$ for some i. $q_i^* \in \Delta$ implies $p_i^{c*} + p_i^* \cdot 1 = 1$ and hence $p_i^* \cdot 1 = 1$. But then (A9b) and Assumption 4 imply that in $\mathscr{E}$, y_i^* is unbounded and in $\mathscr{E}^n$ is equal to $+n$. Hence for large n such that the cube is larger than $\text{Max}_i(\omega_i + d_i \cdot 1)$, (A20a) is violated. This proves $p_i^{c*} > 0$ for all i. A symmetric argument shows that $p_{if}^* > 0$ since dividends are strictly positive in all states. □

The argument up to now has then demonstrated that for large enough n there exists an RBE in $\mathscr{E}^n$ with $(p^{c*}, p^*) \gg 0$. We complete the existence proof by noting that there exists a convergent subsequence of the equilibria in $\mathscr{E}^n$ which is an equilibrium in $\mathscr{E}$ with positive prices. To see this note that by the material balance which is satisfied in $\mathscr{E}^n$, all real equilibrium quantities are in a compact set since

$$0 \leq x_i^{1k} \leq \text{Max}_i(\omega_i + d_i \cdot 1), \quad 0 \leq y_i^k \leq \text{Max}_i(\omega_i + d_i \cdot 1),$$
$$0 \leq x_{ij}^{2k} \leq \text{Max}_i(\omega_i + d_i \cdot 1) \quad \text{for all } i, j \text{ and } k.$$

Also, $q \in \Xi$. Hence a convergent subsequence exists. To prove that the limit prices are positive note that since $p_j^{c*} y_j^* = p_j^* \cdot 1 + p_j^{c*} d_j \cdot 1$ and $p_j^{c*} + p_j^{c*} \cdot 1 = 1$ hold for all n it follows from the strict positivity of the dividends that $p_j^{c*} > 0$ in the limit for all j. The positivity of the limit p_i^* follows from the same argument since when all $p_j^{c*} > 0$, the convergence of prices to zero will generate unbounded demand. Finally, the sequence of θ^k is in a compact set by Assumption 6.0 and hence converges. □

Remark. We note that the technical Assumption 6.0 is used in Lemma 3 and in the last step of the proof. Under the assumption of non-negative dividends some assets may have zero value in states in which the beliefs of all the agents place positive conditional probabilities only on subsequent states in which dividends are zero. This conclusion would complicate the argument and would require some condition on the distribution of beliefs to ensure that prices of some assets remain positive in each state. Subject to such a additional technical condition the assumption of strictly positive dividends is not essential for the validity of Theorem 4. The assumption of no short sales is standard for existence proofs in finance.

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