Endogenous uncertainty in a general equilibrium model with price contingent contracts*

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Summary. This paper views uncertainty and economic fluctuations as being primarily endogenous and internally propagated phenomena. The most important Endogenous Uncertainty examined in this paper is price uncertainty which arises when agents do not have structural knowledge and are compelled to make decisions on the basis of their beliefs. We assume that agents adopt Rational Beliefs as in Kurz [1994a]. The trading of endogenous uncertainty is accomplished by using Price Contingent Contracts (PCC) rather than the Arrow-Debreu state contingent contracts. The paper provides a full construction of the "price state space" which requires the expansion of the exogenous state space to include the "state of beliefs." This construction is central to the analysis of equilibrium with endogenous uncertainty and the paper provides an existence theorem for a Rational Belief Equilibrium with PCC. It shows how the PCC completes the markets for trading endogenous uncertainty and lead to an allocation which is Pareto optimal. This paper also demonstrates that endogenous uncertainty is generically present in this new equilibrium.

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1 On the nature of uncertainty

The standard component of the theory of an individual decision problem in an uncertain environment is the specification of the "world" which is what the individual is uncertain about. The "state of the world" is then a complete description of the world such that once it is revealed to the individual, no uncertainty remains.[1] The "state" may be an entirely subjective object: it may not be observable by others or comparable to the states of other agents. There is no requirement that a decision

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[1] See Savage [1954], page 9.
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maker be able to communicate his state to other agents or that his state can even be comprehended by others. We stress these details since the theoretic framework of the individual decision problem became the basis for the treatment of uncertainty in general equilibrium analysis.

The drastic conceptual leap that was taken by Arrow [1953], Debreu [1959] and Arrow and Debreu [1954] was the assumption that the state space was common to all agents. In contrast to the individual problem where the state of the world is merely a subjective description of individual uncertainty, in general equilibrium theory it describes commodities, identifies markets and is a basis for writing contracts and specifying property rights. In an Arrow-Debreu economy all markets for the exogenously specified state contingent claims require, for their viability, an empirically coherent description of all states. Moreover, in the formulation of an equilibrium of a sequence economy where securities replace markets for state contingent claims, Arrow [1953] and Radner [1972] elevate the concept of the state to even more crucial role. In the formulation of the equilibrium they adopt the rational expectations hypothesis (which has also been called the conditional perfect foresight hypothesis) where agents are assumed to know at each date the map between future realized states and future equilibrium commodity and asset prices. Thus, in such an equilibrium with securities the realized state must resolve all individual uncertainty *including the uncertainty of future prices.*

It is widely recognized that this exogenous and objective concept of the state which is common to all agents, has empirical content only for insurance markets.[2] An exogenous state which is observable and common to all agents cannot resolve most uncertainties. On the other hand, a state which expresses all individual uncertainties consists mostly of unobservable and incomparable components. Arrow [1953] himself explicitly recognized this when explaining that markets for exogenously specified state-contingent commodity claims do not exist and therefore we must consider securities as the main vehicles for reallocating social uncertainty. However, for the exogenous state to be a useful tool for the pricing of securities agents need to know the maps from states at future dates to prices in the future and it is entirely unrealistic to assume that agents can find out what this sequence of maps is. It is then clear that the construction of an exogenous state, common to all agents, serves as a convenient mathematical device which enables an Arrow-Debreu theory to formally incorporate the complex phenomenon of uncertainty by merely relabeling commodities.

As a useful mathematical device, the exogenous state space together with the rational expectations hypothesis in the sense of Arrow [1953] have enabled extremely important developments in the fields of general equilibrium theory and finance. With the view of extending this theory further we note that the arguments against this approach are both empirical as well as theoretical.

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[2] The Arrow-Debreu state space has been useful only for insurance markets. However, in this case Malinvaud [1972] [1973] showed that since these markets handle only individual, idiosyncratic risks, the diversification achieved through such markets imply that these individual risks are of little relevance to general equilibrium considerations. The main allocation problem which general equilibrium theory must address is then the allocation of social risks; these risks are represented mostly by the fluctuations of economic variables over time.

On the empirical side we know that markets for state contingent contracts rarely exist and the exogenous "state" is hardly describable. On the other hand, an extensive array of price contingent contracts (to be clled "PCC" in this paper) are traded by agents for hedging risks. Such contracts are obviously traded daily in large volume on major financial markets across the world. Moreover, such instruments play an essential role in the ordinary conduct of business. A few examples will illustrate the point. Real estate developers and natural resource companies use options and other PCC as hedging devices in their planning of major projects. Purchasers of large scale plant and equipment such as aircrafts and gas pipelines use PCC as a normal device for handling the risk of fluctuating demand. Even owners of sport clubs use options in any labor contract to assure themselves the continuity of service without price risk. The use of PCC to trade price uncertainty is the consequence of the fact that agents do not know the Arrow [1953] maps from exogenous states to future prices. In such a framework concepts of "complete" and "incomplete" markets is devoid of empirical content since there is no empirical way of determining if a market is complete or not. In fact, the presumption should be that markets are always "complete" since under the assumptions of Arrow [1953] agents know the map from states to prices and consequently can always add enough "derivative" securities to complete the markets. This observation is due to Ross [1976]. Also, when the market structure is complete "derivative" securities are "redundant" (see Hakansson [1978]).

All the above are critically central issues to a general equilibrium theory with securities. They are, however, of minor importance relative to our major theoretical argument against the Arrow [1953] and Arrow-Debreu [1954] state space and rational expectations with securities.

The central issue at stake is *the nature of uncertainty in economic systems*. The Arrow [1953] and Arrow-Debreu [1954] formalism which was adopted by all subsequent developments, views all uncertainty in the economy as being generated by forces which are external to the economic system. In a temporal context uncertainty is mostly represented by the potential fluctuations of economic variables. Hence, according to this theory the fluctuations of asset prices, GNP or foreign exchange rates are all ultimately explainable by exogenous factors such as the weather, earthquakes, etc. Equally objectionable is the view that nothing that agents do or think has any impact on the fluctuations of economic variables.

It is our firm view that most of the uncertainty in an advanced industrial society arises from internally propagated fluctuations which are generated by the actions and beliefs of the agents about the nature of the market and by their uncertainty about the actions of other agents. This component of uncertainty was introduced by Kurz [1974a] who also introduced the term *Endogenous Uncertainty* to describe it. Such uncertainty obviously cannot arise in an Arrow [1953] context with rational expectations. Our model recognizes the fact that agents do not know the structural relationships between exogenous states and optimal actions of other agents or between these states and prices. In such circumstances agents are uncertain about future prices and Kurz [1974a] proposed that agents trade this uncertainty using PCC such as options. In such an economy PCC are not "redundant" in any sense: *they are the primary vehicles to trade price uncertainty*. These ideas were further

developed by Svensson [1081], Henrotte [1996] and Kurz [1993]. Pivotal to this approach is the existence of diversity of beliefs among the trading agents.

The central outlook of the endogenous uncertainty approach is to view uncertainty and economic fluctuations as being primarily endogenous and internally propagated phenomena. It is therefore clear that the development of this approach must be based on two elements. The first is an integration of a new theory of expectations and beliefs which is compatible with the diversity of beliefs among agents. The second is a comprehensive study of financial institutions such as PCC which enable the trading of and reallocation of endogenous uncertainty. Our approach clearly recognizes the importance of exogenous variables for equilibrium analysis. However, we define the price state space to include the states of beliefs as well as other endogenous variables such as prices or profits of firms. Such an economy contains the Arrow-Debreu economy as a special case when there is no endogenous uncertainty. In this paper we want to highlight the importance of the "state of beliefs" and therefore will exclude from the price state space other endogenous variables. We postpone to later papers the analysis of more complex price state spaces.

This paper aims to develop an equilibrium theory with endogenous uncertainty in a one commodity overlapping-generations (OLG) context and explore the role of price-contingent contracts in the allocation of risk. Our model has three components. First, agents are assumed to have Rational Beliefs (see Kurz [1994a] [1994b] [1955] [1996]) which include rational expectation as a special case. The second component of our model is the explicit introduction of a "price state space" and of price contingent contracts. In contrast with the Arrow [1953] and Radner [1972] framework where knowledge of the exogenous state carries with it the complex information needed to determine prices, our price state space is either a set of integers $\{1, 2, \ldots, M\}$ in the case of finite prices or the unit interval in the case of continuum of prices. Thus, prices are *themselves* the state variables. On the other hand, PCC are contracts that specify a delivery of commodities or securities at a future date and such deliveries are contingent on the prices which will be realized at that future date.

The third component of our model is the sequential structure where markets reopen at each date. Since agents are assumed to hold Rational Beliefs the equilibrium concept employed here is a Rational Belief Equilibrium introduced by Kurz [1994b] [1996]. In these two papers agents are infinitely lived but are not allowed to trade PCC. The novelty of the present paper is that here agents can trade PCC and the framework is an OLG model. This structure with a single consumable commodity and a single productive activity is a drastic simplification. It does enable us, however, to highlight some of the essential features of this new approach to dynamic equilibrium analysis.

2 The model and the Rational Belief Equilibrium (RBE) concept

2a The basic model

We use a standard OLG model with K young agents in each generation which we denote by $k = 1, 2, \ldots, K$. There are also K old agents in each generation but only the

young receive an endowment $\omega_t^k, t = 1, 2, \ldots$. For each k $\{\omega_t^k, t = 1, 2, \ldots\}$ is a stochastic process which will be specified below. Each young person is a replica of the old person who preceded him where the term "replica" refers to the *utilities* and *beliefs*. This is a model of "dynasties" and the simplifying assumption made here is that there is a finite number of such dynasties. In addition to the market for commodities traded in each period, two types of financial assets are included. One is the common stock of a firm and at date 1 the supply (equals to 1) of the stock is distributed among the old. This distribution initiates the financial sector and ultimately ensures integenerational efficiency. The second asset class is a PCC which enables an agent to contract for the delivery of a unit of the common stock at future dates contingent upon the *prices* which prevail at these future dates rather than upon some abstract "states" which are realized.

The infinitely lived firm is assumed to be extremely simple: it generates exogenously a deterministic sequence $\{R_t, t = 1, 2, \ldots\}$ of dividend payments. This production uses no resources and in most of the development we shall assume that $R_t = R > 0$. The PCC developed in Svensson [1981], Henrotte [1996] and Kurz [1993] enables agents to contract for the future delivery of commodities and securities in a setting with multiple commodities and securities. However, it is immediate that conditions of "no arbitrage" imply that the only factor that matters is the ability of agents to transfer *purchasing power* across time rather than any specific commodities or securities. We shall therefore specify that the PCC below permit an agent to contract for the future delivery of the shares of the firm. Naturally, the price of such insurance could be so prohibitive that an agent may elect not to be fully insured. We stress that the only uncertainty faced by the agents is the uncertainty about the price of the stock and under our assumptions the market is "complete" *when this term means that there is a feasible way of trading all price uncertainty in the second period.*

In our model endowment is random but this uncertainty raises some interesting questions. Since this uncertainty exists *before* the agents are born and since at "birth" they are told what ω_t^k is, this ex-post variability of the endowment is a useful tool of analysis. We could reallocate this uncertainty by assuming that each agent is born with a PCC and consequently the uncertainty would be shifted to the old agents. We see no advantage in such a device since both observability and incentives make it impossible to directly trade the uncertainty of individual endowments and our concern in this paper is the market mechanism for allocation of risk.

The notation we employ in this paper is as follows:

x_t^{1k} – the consumption of k when young at t;
x_{t+1}^{2k} – the consumption of k when old at $t + 1$. This indicates that k was born at date t;
θ_t^k – stock purchase of young agent k at t;
θ_0^k – endowment of the stock to an old agent k at date 1 where $\theta_0^k > 0$ for all k;
ω_t^k – endowment of k when young at t. This means that $k = 1, 2, \ldots, K$ is among the young born at t. Writing ω_t^{1t} is unnecessary since only the young receive an endowment;
p_t – the price of the common stock at date t;

p_t^c – the price of the consumption good at date t;

$f_t^k(p_{t+1})$ – the conditional density of k when young representing his belief at date t regarding the distribution of p_{t+1} at date $t+1$. The subscript t denotes the time dependency and suppresses the conditioning on I_t – the history up to t.

$\beta_t(p_{t+1})$ – the price at date t of one unit of a PCC for delivery of 1 unit of the common stock at date $t+1$ contingent on p_{t+1}.

$z_t^k(p_{t+1})$ – the amount of PCC purchased at date t by agent k, each for delivery of one unit of the stock contingent on p_{t+1}. The owner of such a PCC receives at date $t+1$, contingent upon p_{t+1}, both the stock *as well as* the dividends on the stock at date $t+1$. This assumption is merely a convention which we follow for analytical simplicity.

We assume that the young are being informed of the realization of *their own endowments* and consequently the optimization problem of agent k when young is as follows:

$$\underset{(x^{1k},\theta^k,z^k)}{\text{Max}} \left\{ \int_0^\infty u^k(x_t^{1k}, x_{t+1}^{2k}(p_{t+1})) f_t^k(p_{t+1}) dp_{t+1} \right\} \tag{1}$$

subject to

$$p_t^c x_t^{1k} + p_t \theta_t^k + \int_0^\infty \beta_t(p_{t+1}) z_t^k(p_{t+1}) dp_{t+1} = p_t^c \omega_t^k \tag{2a}$$

$$p_{t+1}^c x_{t+1}^{2k}(p_{t+1}) = \theta_t^k(p_{t+1} + p_{t+1}^c R_{t+1}) + z_t^k(p_{t+1})(p_{t+1} + p_{t+1}^c R_{t+1}). \tag{2b}$$

The budget equations (2a)–(2b) are homogenous of degree zero in prices and because of price normalization, the uncertainty about p_{t+1} is all the uncertainty an agent faces. For the problem above to make sense, the functional expressing the value of the amount z_t^k of the PCC purchased by k must be well defined and this leads to complex technical difficulties (see Svensson [1981], Henrotte [1996] and Kurz [1993]). In all of our analysis below we work with the case in which both $\beta_t(\cdot)$ and p_t *take finite number of values*. In fact, an important conclusion of this paper is that by introducing the conditions of rationality of beliefs, the price state space could be made to consist of only a finite number of elements thus dispensing with most of the technical problems mentioned above. Also, in the case when $\{R_t, t = 1, 2, \ldots\}$ is a random process, issues of incentives make it impossible to trade at date t the uncertainty of R_{t+1}. That is, since in the real economy management decisions induce the stochastic process $\{R_t, t = 1, 2, \ldots\}$ such incentives would be affected by the ability of managers to bet on the outcome of their own decisions. The ability to trade common stocks has some element of these same incentive problems, but to a much lesser extent and even then substantial public regulations have been instituted to prevent the distortions of such incentive effects. As a modeling strategy we assume that all PCC can be traded but do not permit the trading of contracts which are contingent on the profits of the firm. Our previously specified assumption of $R_t = R$ is then justified by the desire for simplification. It also ensures that *each generation* will have a complete set of markets to trade uncertainty.

Questions of arbitrage free pricing are important. To see what restrictions they impose note that if an agent purchases 1 unit of the common stock at the price p_t then

he receives the amount of $(p_{t+1} + p_{t+1}^c R)$ units of consumption in period $t + 1$. On the other hand, suppose that he buys the constant PCC contract – as a *function* of future prices – $z_t^k(p_{t+1}) = 1$. He then receives 1 unit of the common stock for sure. Since by our convention he also receives the dividend next period, this contract generates at $(t + 1)$ the same value as the ownership of the common stock. The cost of this composite PCC is $\int_0^\infty \beta_t(p_{t+1}) dp_{t+1}$. These considerations lead us to introduce the following: *A price system* (p, p^c, β) *is said to be arbitrage free* if for all t

$$p_t = \int_0^\infty \beta_t(p_{t+1}) dp_{t+1}. \tag{3}$$

In the development below we require that the price vectors (p_t, p_t^c, β_t) satisfy (3).

2b Rational belief equilibria of the OLG model with price-contingent contracts

Our development here follows Kurz [1994a] and [1994b]. Rational Belief Equilibrium (RBE) requires market clearance. Thus, we say that *markets clear at all dates* if, for all histories

$$\sum_{k=1}^K \theta_t^k = 1 \quad t = 1, 2, \ldots \tag{4a}$$

$$\sum_{k=1}^K z_t^k(\cdot) = 0 \quad t = 1, 2, \ldots \tag{4b}$$

If follows from (2a)–(2b) and (4a)–(4b) that when markets clear then

$$p_t^c x_t^1 + p_t = p_t^c \omega_t \quad t = 1, 2, \ldots \tag{5a}$$

$$p_t^c x_t^2 = p_t + p_t^c R \quad t = 1, 2, \ldots \tag{5b}$$

where x_t^1, x_t^2 and ω_t are the aggregates defined by

$$x_t^i = \sum_{k=1}^K x_t^{ik} \quad i = 1, 2 \tag{6a}–(6b)$$

$$\omega_t = \sum_{k=1}^K \omega_t^k. \tag{6c}$$

The non-arbitrage condition (3) and a natural normalization are both used in the selection of an appropriate price space S_v for $v_t = (p_t, p_t^c)$. We need such a space to be a compact subset of a complete and separable metric space and then $(S_v^\infty, \mathscr{F}(S_v^\infty))$ is a measurable space where $\mathscr{F}(\cdot)$ denotes the Borel σ-field of the space in question.

To define a *Rational Belief Equilibrium* let $(x_t^{1k}, x_t^{2k}, \theta_t^k, z_t^k, t = 1, 2, \ldots)$ be a sequence of optimal decision functions which are maps from histories to actions for $k = 1, 2, \ldots, K$. These functions induce a market clearing process of prices $\{(p_t, p_t^c), t = 1, 2, \ldots\}$ over the space $(S_v^\infty, \mathscr{F}(S_v^\infty), \Pi)$ and an associated sequence of functions $(\beta_t, t = 1, 2, \ldots)$. One of the objectives of this paper is to show how the rationality of belief conditions enable us to work with a space with finite number of elements. Note the crucial observation that the market clearing probability Π was induced by the beliefs $(Q^1, Q^2, \ldots, Q^k)$ of the K dynasties (represented earlier by the

conditional densities $(f^1, f^2, \ldots, f^k)$). In a RBE we have the dual property that Π is induced by $(Q^1, Q^2, \ldots, Q^K)$ and each one of the Q^k is a Rational Belief relative to Π (as in Kurz [1994a] [1994b]). This motivates our basic concept.

Definition 1. *A Rational Belief Equilibrium with Price Contingent Contracts is a sequence of decision functions* $\{(x_t^{1k}, x_t^{2k}, \theta_t^k, z_t^k, k = 1, 2, \ldots, K) \, t = 1, 2, \ldots\}$, *a stochastic process of prices* $\{(p_t, p_t^c) \, t = 1, 2, \ldots\}$ *on* $(S_v^\infty, \mathscr{F}(S_v^\infty), \Pi)$, *a sequence of functions* $\{\beta_t(p_{t+1}), t = 1, 2, \ldots\}$ *and a set of probability beliefs* $(Q^1, Q^2, \ldots, Q^K)$ *such that*

(i) $(x_t^{1k}, x_t^{2k}, \theta_t^k, z_t^k)$ *is optimal relative to* Q^k *and* $\{p_t, p_t^c, \beta_t(p_{t+1})\}$, $t = 1, 2, \ldots$.

(ii) *The markets clear at all dates and for all histories.*

(iii) Q^k *is a Rational Belief relative to* Π *for* $k = 1, 2, \ldots, K$.

The definition of RBE does not address directly the issue of multiple equilibria. Keep in mind that we are modeling the economy as a dynamical system in which infinite random draws are associated with *definitive* sequences of realized economic allocations. This means that if at any date the economy can have multiple market clearing outcomes, then as part of the dynamics postulated there is a procedure for selecting a particular one of them which, in turn, generates the data observed in the economy. This, indirectly, addresses also the issue of sunspot equilibria. Such equilibria require a device for alternating random selections from among multiple equilibria of some underlying economy over time. If such an equilibrium is to be realized then this selection must be part of the description of the dynamical system. Moreover, a formal coordination among agents is feasible only if one of the observable exogenous variables provides the needed signal for joint action and then we must interpret the fluctuations of the economy which are due to the commonly observed sunspot variable as *exogenously* caused. In an RBE where an exogenous sunspot signal is not present, it is possible that the agents form beliefs which will vary over time and would, in a spontaneous way, be prefectly coordinated. In this narrow sense, a sunspot equilibrium can be realized as an RBE but is a most unlikely equilibrium.

Given the definition of an RBE the rest of the paper is organized as follows. Section 3 works out a simple example which demonstrates how the price state space of an RBE is constructed and why the rationality conditions allow us to work with a finite state space. Our use of the rationality conditions in the construction of the price state space may be contrasted with the treatment in Svensson [1981], Henrotte [1996] and Kurz [1993] who do not use any rationality conditions and end up needing to work with a space of the order of the continuum. The section also provides a definition of the important concept of Endogenous Uncertainty. Section 4 provides a proof of the existence of an RBE with PCC for the family of SIDS processes developed by Nielsen [1994]. The proof demonstrates how the PCC "complete" the market for trading endogenous uncertainty for every generation once the endowment of the young is known. Section 5 discusses the Pareto optimality properties of the equilibrium and the role of the PCC in the optimality properties.

3 Endogenous uncertainty and rational beliefs

The study of RBE in full generality entails complex technical difficulties but the properties of the state space and endogenous uncertainty become very clear even in

simple models. Consequently, this section will be devoted to explore a special case where four simplifications are made:

(i) $R_t = R > 0$ for all t,

(ii) $K = 2$,

(iii) (ω_t^1, ω_t^2) can take only two values $\omega^H = (\omega^{1H}, \omega^{2H})$ and $(\omega^{1L}, \omega^{2L})$,

(iv) $u^k(x_t^{1k}, x_{t+1}^{2k}(p_{t+1})) = u_1^k(x_t^{1k}) + u_2(x_{t+1}^{2k}(p_{t+1}))$,

(v) In order to avoid issues of price normalization we use the term "price" in this section to mean the pair (p_t, p_t^c). However, young agents form beliefs only about the relative price p_t/p_t^c and thus the term "price" in this section means "relative price p_t/p_t^c".

3a Rational beliefs and price states

We think of $\{\omega_t = (\omega_t^1, \omega_t^2), t = 1, 2, \ldots\}$ as a stochastic process defined on an exogenous state space S_ω which is the state space for exogenous variables. In the present case, this space is very simple

$$S_\omega = \{H, L\}. \tag{7}$$

The true process is assumed to be non-stationary and is constructed in the following way. Select a partition $\{D, V\}$ of the positive integers such that on any infinite set of dates $\{t, t + 1, \ldots\}$ the fraction of members of D in the set is, say, π_D. A simple mechanism to select the sets D and V is to toss, at each date t, a coin with probability of H being π_D. If H is realized you declare the date $t + 1 \in D$, if not $t + 1 \in V$. We select the value of $\pi_D = 2/5$ and this is achieved by an i.i.d. coin-tossing process with a probability of 1 being 2/5. Thus we have $\pi_D = 2/5$, $\pi_V = 3/5$. Finally, we select $\{(\omega_t^1, \omega_t^2), t = 1, 2, \ldots\}$ to be an independent sequence of random variables with two densities g_ω^1 and g_ω^2 such that

$$P(\omega_t^1, \omega_t^2) = \begin{cases} g_\omega^1(\omega_t^1, \omega_t^2) & \text{if } t + 1 \in D \\ g_\omega^2(\omega_t^1, \omega_t^2) & \text{if } t + 1 \in V. \end{cases} \tag{8}$$

Foe example, we select

$$g_\omega^1\{(\omega_t^1, \omega_t^2) = (\omega^{1H}, \omega^{2H})\} = \tfrac{3}{4}$$

$$g_\omega^2\{(\omega_t^1, \omega_t^2) = (\omega^{1H}, \omega^{2H})\} = \tfrac{1}{3}.$$

Denote by Π_ω the probability of the stochastic process $\{\omega_t, 1, 2, \ldots\}$ given $\{D, V\}$. This non-stationary process is an example of an SIDS process studied by Nielsen [1994], [1996]. It follows from Nielsen [1996] that it is stable and has a stationary measure m_ω defined by the i.i.d. process where the probability of $(\omega^{1H}, \omega^{2H})$ is $\tfrac{1}{2}$. This is so since $\tfrac{3}{4} \cdot \tfrac{2}{5} + \tfrac{1}{3} \cdot \tfrac{3}{5} = \tfrac{18 + 12}{60} = \tfrac{1}{2}$.

We stress its non-stationary character in comparison with $(S_\omega^\infty, \mathcal{F}(S_\omega^\infty), m_\omega)$ by noting that one must interpret the true process as a composition of a selection rule of dates in D or V together with the probabilities which apply in D or in V. Thus, if you were to bet at t on the outcome at $t + 1$ then the knowledge of Π_ω will lead you to

bet differently at different dates. This is not so under m_ω where the process $\{\omega_t, t = 1, 2, \ldots\}$ is viewed as an i.i.d. process and hence stationary. Neither one of the two agents know the true stochastic process of the endowments. Moreover, despite the simplicity of the process in the model, it aims to represent the process of exogenous variability due to technology, inventions, climate changes, discovery of natural resources, etc. Most of these are neither observable nor fully understood and the model simplification of all these down to a binary process with two different probabilities is purely a matter of illustration. The natural urge is to exploit the model's simplicity to extract valuable information. From our perspective this is not constructive since we need to think of the process $\{\omega_t, t = 1, 2, \ldots\}$ as representing the complexity of the exogenous environment of modern society. We thus propose to think of the *aggregate* endowment ω_t as unobservable.

Agents observe their own endowments and market prices and need to form beliefs about future prices. We assume that agent type 1 believes that his environment is stationary. The stationary measure representing his belief will be derived below.

Agent type 2 believes that his economic environment consists of two different regimes of price distributions. Such a belief structure means that there are two different states of belief (denoted by 1 and 2) at which the probabilities adopted by agent type 2 are f^1 and f^2. Combined with the two exogenous states, there are four possible prices which may be realized. Instead of thinking about the *values* of prices we focus on *the space on which prices are later defined by an equilibrium process.* We thus see that we have here four "price states": two are induced by exogenous factors and two are induced endogenously by the belief of agent 2. This means that f^1 and f^2 are defined on a state space of dimension four. Denote the two probabilities (f^1, f^2) by

$$f^1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, f^2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}. \tag{9}$$

Note that at any moment of time agent 2 believes that any of the four prices may be realized except that he does not believe that the distribution is stationary. To formulate his belief on infinite sequences of prices he selects first a rule of partitioning the integers into $\{D^1, V^1\}$ in a manner similar to the rule of selection $\{D, V\}$ so that on any infinite set $\{t, t+1, t+2, \ldots\}$ the proportions are $\pi_{D^1} = \frac{1}{3}$, $\pi_{V^1} = \frac{2}{3}$. He then selects at date t the probabilities representing his belief about prices at date $t+1$ according to the following rule:

$$\begin{cases} \text{at } t \text{ select } f_t = f^1 & \text{if } t+1 \in D^1 \\ \text{at } t \text{ select } f_t = f^2 & \text{if } t+1 \in V^1 \end{cases}. \tag{10}$$

We denote by Q the Rational Belief specified above on the space $(S_p^\infty, \mathscr{F}(S_p^\infty))$ of infinite price sequences.

In summary, the four price states of the model *may be thought of* in the following manner

$$
\begin{array}{l}
\text{Price state } 1: (\omega = \omega^H, f = f^1) \\
\text{Price state } 2: (\omega = \omega^H, f = f^2) \\
\text{Price state } 3: (\omega = \omega^L, f = f^1) \\
\text{Price state } 4: (\omega = \omega^L, f = f^2).
\end{array}
\tag{11}
$$

We stress the fact that one may only *think* about the states in this manner for two reasons. First, neither ω^H nor f_t, are observable and from the point of view of the agents there are simply four prices that may be observed. Second, the *interpretation* of the "price state" depends upon the structure of information given to the agents. For this reason we define the Price State Space by

$$
S_p = \{1, 2, 3, 4\} \tag{11'}
$$

and the stochastic process $\{p_t,\ t = 1, 2, \ldots\}$ of equilibrium prices is an infinite sequence of random variables on S_p.

The assumption of non-structural knowledge by the agents is central to our work. It means that agents know that in equilibrium there are only four possible prices without the knowledge of the structure which induces these states. We use this simple structure because it is analytically convenient. However, agents are required to disregard their individual impact on the structure. More specifically, agent type 2, whose belief induces some of the variability of prices, is assumed to be "competitive" or "small" and is therefore *specifically prohibited from knowing how his own belief structure contributes to the nature of uncertainty of future prices*. In a model where the endowment ω_t^k may take any random value in a subset of $\mathbb{R}_+^n$, where the dividend process R_t may take any value in a compact interval and where the number of agents is very large, the assumption that each agent is "competitive" is then naturally made and would hardly be questioned.

We now turn to the specification of the restrictions of rationality on the beliefs of a type 2 agent. This agent selects $f_t = f^1$ with frequency $\pi_{D^1} = \frac{1}{3}$ when $t + 1 \in D^1$ and $f_t = f^2$ with frequency $\pi_{V^1} = \frac{2}{3}$ when $t + 1 \in V^1$. This selection is done *independently* of the realization in $\{\omega^H, \omega^L\}$ which are selected with probabilities of $\frac{3}{4}$ when $t \in D$ and $\frac{1}{3}$ when $t \in V$. This independence induces two sets of conditions on the stationary vector $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ of price probabilities. From the point of view of type 2 agents, we must have $\pi_{D^1} f^1 + \pi_{V^1} f^2 = \mu$ and consequently

$$
\frac{1}{3} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}.
$$

On the other hand, consider each of the μ_i. Given the independent selection of D and D^1 it follows that on the events $DD^1 = \{t \in D,\ t + 1 \in D^1\}$, $DV^1 = \{t \in D,\ t + 1 \in V^1\}$, $VD^1 = \{t \in V,\ t + 1 \in D^1\}$ and $VV^1 = \{t \in V,\ t + 1 \in V^1\}$ we have the frequencies

$$
\pi_{DD^1} = \tfrac{2}{15},\ \pi_{DV^1} = \tfrac{4}{15},\ \pi_{VD^1} = \tfrac{3}{15},\ \pi_{VV^1} = \tfrac{6}{15}.
$$

To calculate the μ_i consider, for example, μ_1. It is generated when $\omega_t = \omega^H$ and $f_t = f^1$. The event $f_t = f^1$ is realized only on dates in DD^1 and VD^1. Given that $t \in D$ and $t + 1 \in D^1$ the probability of $\omega_t = \omega^H$ is $\frac{3}{4}$ and when $(t \in V, \ t + 1 \in D^1)$ this probability is $\frac{1}{3}$. This leads to $\mu_1 = \frac{1}{6}$ and the calculations of the rest of then follows:

$$\mu_1 = \tfrac{3}{4}\pi_{DD^1} + \tfrac{1}{3}\pi_{VD^1} = \tfrac{1}{6} \qquad \mu_2 = \tfrac{3}{4}\pi_{DV^1} + \tfrac{1}{3}\pi_{VV^1} = \tfrac{1}{3}$$

$$\mu_3 = \tfrac{1}{4}\pi_{DD^1} + \tfrac{2}{3}\pi_{VD^1} = \tfrac{1}{6} \qquad \mu_4 = \tfrac{1}{4}\pi_{DV^1} + \tfrac{2}{3}\pi_{VV^1} = \tfrac{1}{3}.$$

These calculations imply that the rationality restrictions on the beliefs (f^1, f^2) of agent type 2 are

$$\tfrac{1}{3}a_1 + \tfrac{2}{3}b_1 = \tfrac{1}{6} \tag{12a}$$

$$\tfrac{1}{3}a_2 + \tfrac{2}{3}b_2 = \tfrac{1}{3} \tag{12b}$$

$$\tfrac{1}{3}a_3 + \tfrac{2}{3}b_3 = \tfrac{1}{6} \tag{12c}$$

$$\tfrac{1}{3}a_4 + \tfrac{2}{3}b_4 = \tfrac{1}{3}. \tag{12d}$$

Added to (12a)–(12d) are the natural restrictions on probabilities which are

$$a_1 + a_2 + a_3 + a_4 = 1 \tag{12e}$$

$$b_1 + b_2 + b_3 + b_4 = 1 \tag{12f}$$

We thus have 5 independent equations with 8 unknowns implying 3 degrees of freedom which indicate the size or the dimension of indeterminacy leading to multiple RBE.

Although we assumed that agents cannot observe the aggregate endowment relaxing this assumption would have made no difference to the calculations of the stationary measure. The reason is that an inspection of (11) reveals that when $t + 1 \in D^1$ the agent believes (at t) that the event $\{p_1 \text{ or } p_2\}$ occurs with probability $(a_1 + a_2)$ and this is also the probability of the event $\{\omega_t = \omega^H\}$. When $t + 1 \in V^1$ he believes (at t) that the event $\{p_1 \text{ or } p_2\}$ occurs with probability $(b_1 + b_2)$. Since $\pi_{D^1} = \frac{1}{3}$ and $\pi_{V^1} = \frac{2}{3}$ this knowledge requires that

$$\tfrac{1}{3}(a_1 + a_2) + \tfrac{2}{3}(b_1 + b_2) = \tfrac{1}{2}. \tag{13}$$

The $\frac{1}{2}$ on the right hand side of (13) comes about from the stationary measure m_ω of the endowment process. However, (13) adds no restriction since it is implied by (12a)–(12f).

An example of a Rational Belief of agent type 2 is therefore

$$f^1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, \qquad f^2 = \begin{bmatrix} \frac{1}{8} \\ \frac{3}{8} \\ \frac{1}{8} \\ \frac{3}{8} \end{bmatrix}$$

where $\pi_{D^1} f^1 + \pi_{V^1} f^2 = \mu$. Now denote by S_p^∞ the space of infinite sequences s_p of members of S_p; by $\mathscr{F}(S_p^\infty)$ the Borel σ-field of subsets of S_p^∞ and by $(S_p^\infty, \mathscr{F}(S_p^\infty), \Pi)$ the equilibrium probability space. Π is the *true* probability of price state sequences $s_p \in S_p^\infty$ induced by Π_ω in (8) and by the selection rules specified in (10). The stationary

measure on $(S_p^\infty, \mathscr{F}(S_p^\infty))$ is denoted by m and is defined by the i.i.d. process with density μ at each date. We present in the table below an example of the three probability measures of our example. Π is the true probability; $Q^1 = m$ is the belief of agent type 1 and $Q^2 = Q$ is the belief of agent type 2.

Probabilities of p_{t+1} at t

$(t, t+1) \in$	DD^1			DV^1			VD^1			VV^1		
	Π	$Q^1 = m$	Q	Π	$Q^1 = m$	Q	Π	$Q^1 = m$	Q	Π	$Q^1 = m$	Q
Price state 1	3/4	1/6	1/4	0	1/6	1/8	1/3	1/6	1/4	0	1/6	1/8
Price state 2	0	1/3	1/4	3/4	1/3	3/8	0	1/3	1/4	1/3	1/3	3/8
Price state 3	1/4	1/6	1/4	0	1/6	1/8	2/3	1/6	1/4	0	1/6	1/8
Price state 4	0	1/3	1/4	1/4	1/3	3/8	0	1/3	1/4	2/3	1/3	3/8

We now turn to the definition of the crucial concept of "Endogenous Uncertainty." As is clear from the description of the state spaces S_ω and S_p we have that

$$S_p = \{1, 2, 3, 4\} \text{ and } S_\omega = \{H, L\}. \tag{14}$$

However, thinking of H and L as subsets of the price state space it follows from (11) that

$$H = \{1, 2\} \text{ and } L = \{3, 4\}.$$

It is typical that *the collection of all sets in the exogenous state space is a non-trivial partition of the price state space*. The term "non-trivial" means that at least one member of S_ω contains more than one member of S_p.

Now suppose that agents had full structural knowledge. They would then know that at dates in D the density of ω_t is g_ω^1 but $\omega_t = \omega^L$ or $\omega_t = \omega^H$. In V the density is g_ω^2 but $\omega_t = \omega^L$ or $\omega_t = \omega^H$. Hence the four price states under full structural knowledge are

$$\begin{aligned}
\text{Price state 1:} \quad & (\omega = \omega^H, g_\omega = g_\omega^1) \\
\text{Price state 2:} \quad & (\omega = \omega^H, g_\omega = g_\omega^2) \\
\text{Price state 3:} \quad & (\omega = \omega^L, g_\omega = g_\omega^1) \\
\text{Price state 4:} \quad & (\omega = \omega^L, g_\omega = g_\omega^2)
\end{aligned}$$

In contrast, price variations in an RBE occur only as a result of variations in the realized value of exogenous variables and variations in the "state of beliefs" of the agents. With this in mind we define $s(\omega, f)$ to be the price states in an RBE where ω is the exogenous state and f is the vector of one period probability beliefs of the agents. However, it is important to see that the mere fact that prices may take more values than is warranted by the number of different values taken by the exogenous variables *is only a necessary condition for the presence of Endogenous Uncertainty*. This is

so since in the example here it may be true that $p_1 = p_2$ and $p_3 = p_4$. In this case the *variations in the beliefs of agents type 2 have no impact on actual variability of equilibrium prices.* In this case the four price states are reduced to two and there is no endogenous uncertainty: all the stochastic variability of prices is entirely caused by stochastic variations in the exogenous variables.

To formally define the concept of Endogenous Uncertainty we need to take into account the variability of g_ω, the true probabilities of the exogenous variables. Note first that these probabilities are known in a rational expectations equilibrium but not in an RBE. As a result, prices in a rational expectations equilibrium vary when g_ω varies but this is not the case in an RBE. To enable a distinction between these two equilibrium concepts we define the pair (g_ω, s) as the *extended state* of the economy. Although the first component does not influence prices in an RBE the construction is useful in clarifying the following concept:

Definition 2. *Endogenous Uncertainty is said to be present in an RBE if*

(a) *the exogenous state space is a non-trivial partition of the price state space;*
(b) *there exist two extended states (g_ω^1, s^1) and (g_ω^2, s^2) satisfying $s^1 = (\omega, f^1)$ and $s^2 = (\omega, f^2)$, $g_\omega^1 = g_\omega^2$ and $f^1 \neq f^2$ such that $p(s^1) \neq p(s^2)$.*

Endogenous Uncertainty thus arises when the beliefs of agents influence equilibrium prices when the true probability of the exogenous variables remains the same. This allows for the possibility that an RBE is a rational expectations equilibrium and the definition stipulates that Endogenous Uncertainty *cannot be present* in a rational expectations equilibrium. That is, suppose that at two different dates the exogenous variables have the same realized ω but different true probabilities of future values of the exogenous variables. Under rational expectations agents *know* that the true probabilities are different and hence prices will be different but this variability, according to our definition, does not constitute endogenous uncertainty.

3b When the utility function is logarithmic endogenous uncertainty is not present

The first order conditions require that the decision functions $(x_t^{1k}, x_{t+1}^{2k}, \theta_t^k, z_t^k)$ depend upon all variables $(p_t, p_{t+1}, \beta_t(\cdot), \omega_t^k)$. In writing x_t^{1k} we suppress this dependence but in writing $x_{t+1}^{2k}(j)$ we explicitly recognize the dependence of x_{t+1}^{2k} on the realized price state j at $t+1$. Supposing that $u^k(x, y) = \log x + \log y$, let $p_t^*(j) = \dfrac{p_t(j)}{p_t^c(j)}$, we have

$$\frac{1}{x_t^{1k}} = \lambda_t^k p_t^c \tag{15a}$$

$$\sum_{j=1}^{4} \frac{1}{x_{t+1}^{2k}(j)} f_t^k(p_{t+1}^*(j)) \left(\frac{p_{t+1}(j) + p_{t+1}^c(j)R}{p_{t+1}^c(j)} \right) = \lambda_t^k p_t \tag{15b}$$

$$\frac{1}{x_{t+1}^{2k}(j)} f_t^k(p_{t+1}^*(j))(p_{t+1}(j) + p_{t+1}^c(j)R) = \lambda_t^k \beta_t(p_{t+1}^*(j))p_{t+1}^c(j) \quad j = 1, 2, 3, 4. \tag{15c}$$

Equations (15a)–(15b) imply that

$$x_t^1 = \tfrac{1}{2}\omega_t \qquad x_t^2 = \tfrac{1}{2}\omega_t + R \qquad p_t = \tfrac{1}{2}\omega_t. \tag{16}$$

In then follows from (16) that no Endogenous Uncertainty can be present in this RBE *since the demand of the young is independent of their price expectations.* Equations (15a)–(15c) also imply that the consumption of the old in price state j satisfies

$$p_{t+1}^c(j)x_{t+1}^{2k}(p_{t+1}(j)) = \frac{1}{2}\omega_t^k \frac{f_t^k(p_{t+1}^*(j))}{\beta_t(p_{t+1}^*(j))}(p_{t+1}(j) + p_{t+1}^c(j)R). \tag{17}$$

Equation (17) shows that although there is no Endogenous Uncertainty in the model, the beliefs of the agents influence the allocation. Adding (17) over k reveals the equilibrium function β,

$$\beta_t(p_{t+1}^*(j)) = \sum_{k=1}^{2} \frac{\omega_t^k}{2R+\omega_t}\left(\frac{p_{t+1}(j) + p_{t+1}^c(j)R}{p_{t+1}^c(j)}\right)f_t^k(p_{t+1}^*(j)). \tag{18}$$

It is seen in (19) that the price of a PCC is exactly what one can call *the market belief*: it is the weighted average of the probabilities of the agents when their relative endowments provide the weights. We can then conclude as follows:

Proposition 1. *When $u^k(x,y) = \log x + \log y$, all k, there is no Endogenous Uncertainty in any RBE.*

3c Endogenous uncertainty is generally present in a RBE

The logarithmic utility function is a borderline case when future perceived investment opportunities have no effect on current consumption and consequently price expectations have no effect on current consumption. For any other utility function the diversity of beliefs in an RBE gives rise to Endogenous Uncertainty. To illustrate, consider an alternative case where for all k

$$u^k(x,y) = Ax - \tfrac{1}{2}x^2 + By - \tfrac{1}{2}y^2 \qquad (x \leq A, y \leq B). \tag{19}$$

The first order conditions of the individual optimization are

$$A - x_t^{1k} = \lambda_t^k p_t^c \tag{20a}$$

$$\sum_{j=1}^{4}(B - x_{t+1}^{2k}(j))f_t^k(p_{t+1}^*(j))\left(\frac{p_{t+1}(j) + p_{t+1}^c(j)R}{p_{t+1}^c(j)}\right) = \lambda_t^k p_t \tag{20b}$$

$$(B - x_{t+1}^{2k}(j))f_t^k(p_{t+1}^*(j))(p_{t+1}(j) + p_{t+1}^c(j)R) = \lambda_t^k \beta_t(p_{t+1}^*(j))p_{t+1}^c(j). \tag{20c}$$

In this case the expectations of the future are of central importance. Note that due to the properties of SIDS processes, conditional and unconditional expectations are the same. Hence, condition (20c) together with the no-arbitrage condition (3) imply that if Q^k, $k = 1, 2$ are the beliefs of the two types of agents then

$$2A - x_t^1 = \sum_{k=1}^{2} E_{Q^k}\left[(B - x_{t+1}^{2k})\left(\frac{p_{t+1} + p_{t+1}^c R}{p_{t+1}^c}\right)\right]\frac{p_t^c}{p_t}. \tag{21}$$

Since by (5a) $p_t^c x_t^1 = p_t^c \omega_t - p_t$ it follows that

$$\frac{p_t}{p_t^c} = \omega_t - 2A + \frac{p_t^c}{p_t} \sum_{k=1}^{2} E_{Q^k} \left[(B - x_{t+1}^{2k}) \left(\frac{p_{t+1} + p_{t+1}^c R}{p_{t+1}^c} \right) \right]. \tag{22}$$

To see how endogenous uncertainty is reflected in (22) note that p_t is different when $\omega_t = \omega^H$ or $\omega_t = \omega^L$. However, it is also different depending upon whether $t + 1 \in D^1$ or $t + 1 \in V^1$ since the price expectations of agent type 2 are different in these two cases.

The price of a PCC still reflects the "market belief" but here the weights are different. This follows from the fact that (20a) and (20c) imply that

$$\beta_t(p_{t+1}^*(j)) = \sum_{k=1}^{2} \frac{B - x_{t+1}^{2k}(j)}{A - x_t^{1k}} f_t^k(p_{t+1}^*(j)) \left(\frac{p_{t+1}(j) + p_{t+1}^c(j)R}{p_{t+1}^c(j)} \right). \tag{23}$$

This example shows that resource allocations of an agent with non-logarithmic utility function is sensitive to his beliefs and in equilibrium it will, generally, translate into endogenous uncertainty.

4 Rational Belief Equilibrium with PCC and endogenous uncertainty

This section demonstrates the existence of an RBE for the economy of Sections 2 and 3. An existence theorem requires not only a proof of the existence of market clearing prices but also a demonstration that agents hold rational beliefs and that equilibrium quantities and prices constitute a stable dynamical system. Anticipating the stability requirement we carry out the analysis in two stages. In the first stage we construct the price state space by selecting the probability of the endowment process and the probability beliefs of the agents to be jointly SIDS measures since it follows from Nielsen [1996] that this induces stable equilibrium system. Such an SIDS system generalizes the example of Section 3. In the second stage we prove the existence of equilibrium prices and quantities for the specified systems.[3]

4a The structure of uncertainty and beliefs

We return to the model of Section 2. The process $\{\omega_t = (\omega_t^1, \ldots, \omega_t^K), t = 1, 2, \ldots\}$ is a stochastic process on the exogenous state space S_ω where $(\omega_t^1, \omega_t^2, \ldots, \omega_t^K)$ is the endowment vector of the K young agents. This defines a dynamical system $(S_\omega^\infty, \mathcal{F}(S_\omega^\infty), \Pi_\omega, T)$. The assumption which we make here and which will then lead to the postulated SIDS system is:

Assumption 1. *The process* $\{\omega_t, t = 1, 2, \ldots\}$, $\omega_t \in \mathbb{R}^K$, *takes only a finite number of values in the set* $F_\omega = \{\omega^1, \omega^2, \ldots, \omega^{N_0}\}$ *of* N_0 *elements. Moreover,* Π_ω *is an SIDS probability measure under which* $\{\omega_t, t = 1, 2, \ldots\}$ *is an independent sequence of random variables.*

---

[3] This approach was proposed by Nielsen [1994], [1996] who proves existence in economies where PCC are not traded

Assumption 1 implies that we can define the exogenous state space S_ω by the coordinates. Hence

$$S_\omega = \{1, 2, \ldots, N_0\}. \tag{24}$$

We next introduce the beliefs of the agents which are probabilities Q^k, $k = 1, 2, \ldots, K$ on a measurable space $(S_p^\infty, \mathscr{F}(S_p^\infty))$ where S_p – the price state space – is a Borel subset yet to be defined. To do that we introduce

Assumption 2. *The beliefs Q^k, $k = 1, 2, \ldots, K$ specify the process $\{(p_t^c, p_t), t = 1, 2, \ldots\}$ to be a sequence of independent random variables with date t probability on $(S_p, \mathscr{F}(S_p))$ denoted by f_t^k. This probability is selected from a finite set of such probabilities $F^k = \{f_1^k, f_2^k, \ldots, f_{N_k}^k\}$ with N_k members.*

The important implication of Assumptions 1 and 2 is that we can derive from conditions (4a)–(4b) an equilibrium map which takes the form

$$(p_t^c, p_t, \beta_t(\cdot)) = \Phi^*(\omega_t, f_t^1, f_t^2, \ldots, f_t^K). \tag{25}$$

Inspection of (25) reveals that under Assumptions 1 and 2 the maximal number of prices that can be observed is $M = N_0 \times N_1 \times N_2 \times \cdots \times N_K$. This leads to the conclusion that the price state space, which is the domain of the price process, can be defined by

$$S_p = \{1, 2, \ldots, M\}. \tag{26}$$

Given the finite state spaces of the endowment process and the beliefs of the agents we want to specify the selection process $\{(\omega_t, f_t^1, f_t^2, \ldots, f_t^K), t = 1, 2, \ldots\}$ to be jointly stable. Formally, note that in view of the equilibrium map (25) it follows that we can think of the exogenous state space S_ω as a partition of the price state space S_p as explained in (14) and this implies that we can think of the probabilities Π_ω and Q^k as measures on the same space. With this in mind we specify (see Nielsen [1996] Proposition 6):

Assumption 3. *The probability measures $(\Pi_\omega, Q^1, Q^2, \ldots, Q^K)$ are jointly SIDS.*

We remark that Assumptions 1 and 3 are essentially assumptions about the stochastic process $\{(\omega_t, f_t^1, f_t^2, \ldots, f_t^K), t = 1, 2, \ldots\}$ which is called a "generating process" and is the driving mechanism of an SIDS process (see Nielsen [1996] Section 4-5).

The joint SIDS property of the endowment and the beliefs of the agents is central to our existence argument since a proof of the existence of an RBE requires a demonstration that all market clearing variables constitute a stable process and the beliefs of the agents are rational with respect to the probability of the equilibrium process. An SIDS system is "self referential" or "closed" in the precise sense that if Assumptions 1, 2, 3 are satisfied then equilibrium prices are also SIDS and for each exogenous process there exist SIDS beliefs which are rational with respect to the resulting equilibrium. Moreover, for each dynamical system $(S_\omega^\infty, \mathscr{F}(S_\omega^\infty), \Pi_\omega, T)$ the equilibrium dynamics of prices and the rationality conditions on the beliefs $(Q^1, Q^2, \ldots, Q^K)$ are all specified in terms of the dynamical system $(S_p^\infty, \mathscr{F}(S_p^\infty), \Pi, T)$ of *price states* without specifying the numerical values of equilibrium prices. Also,

the stationary measure of the equilibrium stochastic process of prices is independent of the particular sequence $(\omega_t, f_t^1, f_t^2, \ldots, f_t^K)$ which is realized; it depends only on the *generating process itself*. This self-referential property, which is a substitute for a fixed point argument, has been extended by Kurz and Schneider [1996] to subclasses of Markov processes.

To complete the development we need to ensure that we have a consistent price state space in the sense that for each endowment process there is a map (25), an SIDS equilibrium price process and SIDS beliefs Q^k for all k such that the beliefs are rational with respect to the equilibrium dynamics of prices and the beliefs induce the equilibrium SIDS of prices. We then have the result:

Lemma 1. (Nielsen [1996]) *For any endowment dynamics $(S_\omega^\infty, \mathscr{F}(S_\omega^\infty), \Pi_\omega, T)$ satisfying Assumption 1 there exists a class of generating processes $\{(\omega_t, f_t^1, f_t^2, \ldots, f_t^K),\ t = 1, 2, \ldots\}$ such that under the map (25) a consistent price state space is induced in the sense that Assumptions 2 and 3 are satisfied and*

1. the implied equilibrium dynamics of prices $(S_p^\infty, \mathscr{F}(S_p^\infty), \Pi, T)$ is a non-stationary SIDS,

2. the implied SIDS beliefs $(Q^1, \ldots, Q^K)$ of the agents are rational with respect to Π.

Lemma 1 completes the first stage of the existence argument. What is left to show is the existence of market clearing prices which are compatible with the stochastic structure postulated. Before moving on to this problem we need to clarify the indexing of the M price states. A member of this collection is identified by a selection

$$(\omega_{i_0}, f_{i_1}^1, f_{i_2}^2, \ldots, f_{i_K}^K), \ i_k \in \{1, 2, \ldots, N_k\} \quad k = 0, 1, 2, \ldots, K. \tag{27}$$

These members of the state space are identified by the permutations $(i_0, i_1, i_2, \ldots, i_K)$ of selections from the sets $\{1, 2, \ldots, N_k\}\ k = 0, 1, 2, \ldots, K$. It is convenient to order these permutations and map them one-for-one into the set of the M integers $\{1, 2, \ldots, M\}$. We thus specify by

$$(i_0(i), i_1(i), i_2(i), \ldots, i_k(i)) \qquad i = 1, 2, \ldots, M \tag{28}$$

a rule to map each permutation into an integer i in $\{1, \ldots, M\}$. We then replace (27) with

$$(\omega_{i_0(i)}, f_{i_1(i)}^1, f_{i_2(i)}^2, \ldots, f_{i_K(i)}^K) \quad \text{for } i = 1, 2, \ldots, M. \tag{29}$$

The map (29) establishes the correspondence between each price state and the configuration of endowment and beliefs which defines that state.

It is evident that each price vector (p_i^c, p_i, β_i) in state i has $M + 2$ coordinates. Alternatively, we can think of the price vector at t as consisting of a *pair* $((p_t^c, p_t), \beta_t)$ where $(p_t^c, p_t) \in \mathbb{R}_+^2$ and $\beta_t(\cdot)$ is a *function* specifying, for each (p_{t+1}^c, p_{t+1}), the cost at t of a claim on one share at $t + 1$ contingent on the specified prices realized at $t + 1$. Since there are M possible values which (p_{t+1}^c, p_{t+1}) can take, β_t can take M different values. But since there are M different possible prices (and price states) there must also be M different functions $(\beta_1, \beta_2, \ldots, \beta_M)$ that may be realized as an equilibrium schedule at any date. Since in the model at hand a young person is uncertain, at each date t, only about the prices (p_{t+1}^c, p_{t+1}) at date $t + 1$, the dimension of each probability belief vector f_j^k is M.

4b Existence of an RBE with a complete set of PCC

4b.1 Demand correspondences and the interpretation of price states

Although the equilibrium values of each one of the price vectors (p_i^c, p_i, β_{ij}) has not been established as yet, the implication of Lemma 1 is that we must think of a "price state" as exactly such a vector. In other words, in an RBE an agent does not think of some abstract and unobservable "state" and then considers equilibrium prices to be measurable functions on this state space. Instead, *he thinks of vectors of prices as states over which he places his probability beliefs.* Equally important, the PCC contracts used to trade uncertainty are not contingent upon an exogenously specified states but rather, on specific and observable price vectors (p_i^c, p_i) that may be realized next period. In the present paper, where we assume *a complete set of PCC at each date*, the two-stage procedure of our proof has the implication that standard techniques of Arrow-Debreu theory can be used to prove the existence of an RBE despite the new financial structure which we postulate.

We return to the problem (1)–(2) of agent k assuming $R_t = R$ in order to restate it in terms of the price states (p_i^c, p_i), $i = 1, 2, \ldots, M$ and the associated prices of the PCC. Starting with this price system, it is now an $M \times M$ matrix $[\beta_{ij}]$ and the *arbitrage free* condition (3) is now written in the following form

$$p_i = \sum_{j=1}^{M} \beta_{ij} \qquad i = 1, 2, \ldots, M. \tag{30}$$

The budget equations (2a)–(2b) for $i = 1, \ldots, M$ are now written in the form

$$p_i^c x_i^{1k} + p_i \theta_i^k + \sum_{j=1}^{M} \beta_{ij} z_{ij}^k = p_i^c \omega_i^k, \tag{31a}$$

$$p_j^c x_{ij}^{2k} = (\theta_i^k + z_{ij}^k)(p_j + p_j^c R), \quad j = 1, \ldots, M. \tag{31b}$$

We observe now that the assumed complete PCC structure permits hedging, at date t, of all period $t + 1$ risks which are compatible with a single budget constraint. This makes it feasible to select riskless consumption streams. To show that all pairs of periods t and $t + 1$ consumption are feasible if they satisfy a single income constraint when young, substitute (30) into (31a) and set $\hat{\beta}_{ij} = \beta_{ij}\left(\dfrac{p_j^c}{p_j + p_j^c R}\right)$ to obtain one intertemporal budget constraint for the two vectors of consumption x_i^{1k} and $x_i^{2k} = (x_{i1}^{2k}, \ldots, x_{iM}^{2k})$ for each $i = 1, \ldots, M$

$$p_i^c x_i^{1k} + \sum_{j=1}^{M} \hat{\beta}_{ij} x_{ij}^{2k} = p_i^c \omega_i^k. \tag{32}$$

Hence we have in (32) a single budget constraint relative to which it is feasible to choose at each state i a riskless (i.e. constant) consumption stream.

The budget equations (31a)-(31b) do not apply to the old members at date 1 who trade the endowment $\theta^k > 0$, allocated initially to their "dynasty", in a competitive market. If the economy is in state j then the budget equation of such an old

agent k is

$$p_j^c y_j^k = \theta^k (p_j + p_j^c R). \tag{33}$$

In (33) we use the symbol y_j^k to identify the consumption of the old at date 1 and state j. Equation (33) reveals that although the young at any date can think of (32) as their effective constraint this is not the case for the old at date 1. This is a direct consequence of the OLG structure of the model which we need to keep in mind in establishing the existence of an RBE. We now introduce the following assumption.

Assumption 4. *For each k, the utility function u^k is continuous, quasiconcave and strictly monotonic.*

The utility function in (1) can then be written for a given belief $f^k = (f_{ij}^k)$ as

$$U^k(x_i^{1k}, x_{ij}^{2k}) = \sum_{j=1}^{M} u^k(x_i^{1k}, x_{ij}^{2k}) f_{ij}^k, \qquad i = 1, \dots, M. \tag{1'}$$

It is clear from (31a)–(31b) that if $(x_i^{1k*}, \theta_i^{k*}, z_i^{k*})$ is an optimal allocation of a young agent in state i where we use the notation $z_i = (z_{i1}, \dots, z_{iM}) \in \mathbb{R}^M$, then any other portfolio (θ_i, z_i) which satisfy the condition $\theta_i^{k*} + z_{ij}^{k*} = \theta_i + z_{ij}$ is also optimal. This indeterminancy is typical for financial models and we handle it by simply keeping the stock ownership fixed at the *initial* level

$$\theta_i^{k*} = \theta_0^k > 0 \qquad \text{for } i = 1, 2, \dots, M, \quad k = 1, 2, \dots, K. \tag{34}$$

Condition (34) means that only the vectors z_i are needed to be chosen and this determines the portfolio. We use the notation $q = (q_1, \dots, q_M) \in \mathbb{R}^{M(M+1)}$ where $q_i = (p_i^c, \beta_{i1}, \dots, \beta_{iM}) \in \mathbb{R}^{M+1}$, and given the arbitrage free condition (30) these vectors specify the price system of the economy. We also use the notation $\beta = [\beta_{ij}]$ for the matrix of PCC prices and $\beta_i = (\beta_{i1}, \dots, \beta_{iM})$ for the PCC price vectors in state i. The arbitrage free condition is then $p_i = \beta_i \cdot \vec{1}$ *for all* i where $\vec{1} = (1, \dots, 1)$. The vectors $(x_i^{1k}, z_i^k) \in \mathbb{R}^{M+1}$ denotes the vector of choices of young traders in state i while $y_i^k \in \mathbb{R}^M$ denotes the consumption vector of date 1 old in state i.

The arbitrage free condition and the homogeneity property in (31a)–(31b) give us one degree of freedom to normalize each one of the price vector q_i for $i = 1, \dots, M$. We then employ the standard simplex used in Arrow-Debreu type proofs

$$\Delta = \left\{ q_i \in \mathbb{R}_+^{M+1} \mid p_i^c + \sum_{j=1}^{M} \beta_{ij} = 1 \right\} \text{ and hence } q \in \Xi = \Delta \times \Delta \times \cdots \times \Delta \, (M \text{ times}). \tag{35}$$

The budget correspondence of young traders in state i is then written, for $k = 1, 2, \dots, K$ as

$$B_i^k(q, \omega_i^k)$$
$$= \left\{ (x_i^{1k}, z_i^k) \in \mathbb{R}_+ \times \mathbb{R}^M \mid p_i^c x_i^{1k} + \sum_{j=1}^{M} \beta_{ij} (\theta^k + z_{ij}^k) \le p_i^c \omega_i^k; \, x_{ij}^{2k} \ge 0 \text{ in (36b)} \right\} \quad q \in \Xi. \tag{35a}$$

It is important to note that the budget correspondence (35a) depends upon *the entire set of prices q* since in (31b) the agent needs to ensure non-negative consumption in all states. That is, at each price state i the agent plans his old age consumption at all possible price states j and hence it depends upon the entire price vector $p = (p_1, p_2, \ldots, p_M)$. By the arbitrage free conditions (30) the vector p is a linear function of all the elements in β.

The budget correspondence of old traders $k = 1, 2, \ldots, K$ at date 1 in state i is then

$$B_i^{yk}(q_i, \theta^k) = \{y_i^k \in \mathbb{R}_+ \mid p_i^c y_i^k \leq \theta^k(p_i + p_i^c R), \quad p_i = \beta_i \cdot \vec{1}\}, \quad q_i \in \Delta. \tag{35b}$$

The following is then standard:

Lemma 2. *The budget set correspondences of the young and the old are non-empty and for each q they are convex and compact valued, and continuous on the interior of Ξ.*

In the following proof we encounter the usual problem where demand correspondences are not defined on the boundary when some prices equal to 0. We denote by $\mathcal{E}$ the real economy with which we work. We now introduce a sequence of economies $\mathcal{E}^n$ where for each n the economy is bounded in a cube nW. The set W is a compact cube centered on the zero vector and all the original budget sets are then intersected with nW to create new budget sets which are then compact subsets of nW even when some prices equal 0. these budget correspondences are non-empty, convex and compact valued, and continuous *at all price vectors in Ξ.* A construction of the economies $\mathcal{E}^n$ requires complex additional notation. Since this is a standard procedure we shall avoid such added notation (for details on this procedure see Kurz [1974b, sections 6-7]). Thus, when we say below that "a variable takes the value $+n$ in $\mathcal{E}^n$" we mean that it is on the boundary of the restricted budget set of the agent in $\mathcal{E}^n$.

Turning to demand correspondences, for $k = 1, \ldots, K$ and $i = 1, \ldots, M$ the notation used for the young is $x_i^{1k} \in \varphi_{i0}^k$, $z_{ij}^k \in \varphi_{ij}^k$, $j = 1, \ldots, M$, $\varphi_i^k = (\varphi_{i0}^k, \varphi_{i1}^k, \ldots, \varphi_{iM}^k)$, $\varphi^k(q) = (\varphi_1^k, \ldots, \varphi_M^k)$. For old agents we use the notation $y_i^k \in \varphi_i^{yk}(q_i, \theta^k)$, $\varphi^{yk} = (\varphi_1^{2k}, \ldots, \varphi_M^{2k})$. Now, define the demand correspondences

$$\varphi_i^k(q, \omega_i^k) = \{(x_i^{1k}, z_i^k) \in \mathbb{R}_+ \times \mathbb{R}^M \mid (x_i^{1k}, z_i^k) \text{ maximizes } (1') \text{ on } B_i^k(q, \omega_i^k)\} \quad q \in \text{int } \Xi. \tag{36a}$$

$$\varphi_i^{yk}(q_i, \theta^k) = \{y_i^y \in \mathbb{R}_+ \mid y_i^k \text{ is maximal on } B_i^{yk}(q_i, \theta^k)\} \quad q_i \in \text{int } \Delta. \tag{36b}$$

It then follows from the theorem of the maximum and from Lemma 2 that

Lemma 3. *The demand correspondences $(\varphi^k(q), \varphi^{yk}(q))$ for $k = 1, 2, \ldots, K$ are non empty, convex and compact valued, and upper hemicontinuous on int Ξ. In each of the uniformly bounded economies $\mathcal{E}^n$, the vector of demand correspondences $(\varphi^k(q), \varphi^{yk}(q))$ is non-empty, convex and compact valued, and upper hemicontinuous on the entire price space Ξ.*

4b.2 Existence proof

In the OLG economy at hand the market clearing conditions (5a)-(5b) stipulate that no matter what the state at date $t - 1$ is, in an RBE the aggregate consumption of the

young and the aggregate consumption of the old at date t has to add to the total supply. That is,

$$\sum_{k=1}^{K} x_t^{1k} + \sum_{k=1}^{K} x_{ji}^{2k} = \omega_i + R \qquad \text{for } i, j = 1, \ldots, M. \tag{37}$$

Lemma 4. *For all states i and j, in equilibrium $\sum_{k=1}^{K} x_{ji}^{2k} = \sum_{k=1}^{K} y_i^k \equiv y_i$.*

Proof. At date 1 the requirement of material balance specifies that $x_i^1 + y_i^2 = \omega_i + R$. However, the demand of the young at any date depends only upon the state at that date and hence this condition holds for all dates. Comparing with (37) we can conclude that in equilibrium $\sum_{k=1}^{k} x_{ji}^{2k} = y_i$ holds at all dates. $\quad\square$

By Lemma 4 we rewrite (37) to require

$$\sum_{k=1}^{K} x_i^{1k} + \sum_{k=1}^{K} y_i^k = \omega_i + R, \qquad \text{for all } i = 1, 2, \ldots, M. \tag{38}$$

Next, the financial markets must clear and since all PCC are in zero net supply we require that

$$\sum_{k=1}^{K} z_{ij}^k = 0 \qquad \text{for all } i, j = 1, 2, \ldots, M. \tag{39}$$

Equations (38) and (39) is a system of $M(M + 1)$ market clearing conditions. With these requirements in mind we now use the notation introduced earlier to define the excess demand correspondences for $i = 1, \ldots, M$ by

$$\zeta_{i0}(q) = \sum_{k=1}^{K} \varphi_{i0}^k(q) + \sum_{k=1}^{K} \varphi_i^{yk}(q_i) - (\omega_i + R) \tag{40}$$

$$\zeta_{ij}(q) = \sum_{k=1}^{K} \varphi_{ij}^k(q), \qquad j = 1, \ldots, M.$$

We shall demonstrate that for any q satisfy $0 \in \zeta_i(q) = (\zeta_{i0}(q), \zeta_{i1}(q), \ldots, \zeta_{iM}(q))$ for $i = 1, \ldots, M$ is an equilibrium price system. First we show that Walras' Law applies:

Lemma 5. *Under Assumption 4, $q_i \cdot \zeta_i(q) = 0$ for $i = 1, 2, \ldots, M$, that is,*

$$p_i^c \zeta_{i0}(q) + \sum_{j=1}^{M} \beta_{ij} \zeta_{ij}(q) = 0 \tag{41}$$

Proof. Under Assumption 4, the budget constraints in (35a)–(35b) holds with equality. By summing (31a) over k we have that

$$p_i^c \left(\sum_{k=1}^{K} x_i^{1k} - \omega_i \right) + \sum_{j=1}^{M} \beta_{ij} \sum_{k=1}^{K} (0^k + z_{ij}^k) = 0. \tag{42}$$

In the calculations below we add and subtract bundles from correspondences. To avoid extra notation we use the symbols for the correspondences to represent such

feasible bundles:

$$p_i^c \zeta_{i0}(q) + \sum_{j=1}^{M} \beta_{ij} \zeta_{ij}(q)$$

$$= p_i^c \left(\sum_{k=1}^{K} \varphi_{i0}^k(q) - \omega_i \right) + \sum_{j=1}^{M} \beta_{ij} \sum_{k=1}^{K} \varphi_{ij}^k(q) + p_t^c \left(\sum_{k=1}^{K} \varphi_i^{yk}(q_i) - R \right)$$

$$\text{(by the definitions)}$$

$$= p_i^c \left(\sum_{k=1}^{K} \varphi_i^{yk}(q_i) - R \right) - \sum_{j=1}^{M} \beta_{ij} \quad \left(\text{by (42) and } \sum_{k=1}^{K} \theta_0^k = 1 \right)$$

$$= p_i^c \sum_{k=1}^{K} \varphi_i^{yk}(q_i) - (p_i^c R + p_i) \qquad \text{(by (30))}$$

$$= 0. \qquad \text{(by (33))} \quad \square$$

Recall that we use the notation $(x_i^1 \in \mathbb{R}, x_{ij}^2 \in \mathbb{R}, y_i \in \mathbb{R}, z_{ij} \in \mathbb{R})$ for the aggregates over k. We now use the notation $(x, y, z) \in \mathbb{R}^{KM(M+2)}$ for the *entire array*

$$(x, z, y) = ((x_i^{1k}, z_{ij}^k, y_i^k), k = 1, \ldots, K, i, j = 1, \ldots, M). \qquad (43)$$

Define the maps μ_i for $i = 1, 2, \ldots, M$ by

$$\mu_i(x, z, y) = \left\{ q_i \in \Delta \,\middle|\, p_i^c(x_i^1 + y_i - \omega_i - R) + \sum_{j=1}^{M} \beta_{ij} z_{ij} \text{ is maximized over } \Delta \right\} \qquad (44)$$

and $\mu(x, z, y) = \underset{i=1}{\overset{M}{\times}} \mu_i(x, z, y)$. Finally, define the map Φ by

$$\Phi((x, z, y), q) = \zeta(q) \times \mu(x, z, y). \qquad (45)$$

It follows from Lemma 3 and from the definition (43) that in $\mathscr{E}^n$, Φ is a non-empty, convex and compact valued, and upper hemicontinuous correspondence from $nW \times \Xi$ into itself. It then follows from the Kakutani fixed point theorem that it has a fixed point (x^*, z^*, y^*, q^*) in $\mathscr{E}^n$ (Note: as our custom, we do not designate the variables in $\mathscr{E}^n$ by n). Hence we conclude that

$$q^* \in \mu(x^*, y^*, z^*) \qquad (46a)$$

$$(x^*, z^*, y^*) \in \zeta(q^*). \qquad . \qquad (46b)$$

Condition (46a) states that for any $(p_i^c, \beta_i) \in \Delta$

$$p_i^{c*}(x_i^{1*} + y_i^* - \omega_i - R) + \sum_{j=1}^{M} \beta_{ij}^* z_{ij}^* \geq p_i^c(x_i^{1*} + y_i^* - \omega_i - R) + \sum_{j=1}^{M} \beta_{ij} z_{ij}^*. \qquad (47)$$

Condition (46b) states that (x^*, z^*, y^*) are individually optimal in $\mathscr{E}^n$ relative to q^* and hence satisfy the budget constraints (31a), (31b) and (33). But then by Lemma 5 we have that for all i

$$p_i^c(x_i^{1*} + y_i^* - \omega_i - R) + \sum_{j=1}^{M} \beta_{ij} z_{ij}^* \leq 0 \qquad \text{for all } (p_i^c, \beta_i) \in \Delta. \qquad (48)$$

(48) implies that

$$x_t^{1*} + y_i^* - \omega_i - R \leq 0 \qquad \text{all } i \qquad (49a)$$

$$z_{ij}^* \leq 0 \qquad \text{all } i, j. \qquad (49b)$$

Lemma 6. *For large n, the fixed point (x^*, z^*, y^*, q^*) in $\mathscr{E}^n$ satisfies $p^{c*} \gg 0$ and $\beta_{ij}^* > 0$ for all (i, j) such that $f_{ij} = \sum_{k=1}^K f_{ij}^k > 0$ and for these configurations $(49a)-(49b)$ hold with equalities.*

Proof. Suppose that $p_i^{c*} = 0$ for some i. $q_i^* \in \Delta$ implies $p_i^{c*} + p_i^* = 1$ and hence $p_i^* = 1$. But then (36b) and Assumption 4 imply that in $\mathscr{E}$, y_i^* is unbounded and in $\mathscr{E}^n$ is equal to $+n$. Hence for large n such that the cube is larger than $(\omega_i + R)$, (49a) is violated. This proves that $p_i^{c*} > 0$ for all i. Now suppose that $\beta_{ij}^* = 0$ for same (i, j) such that $f_{ij} > 0$. Since $p_j^{c*} > 0$ it follows from (31b) and Assumption 4 that it is optimal for all agents who believe that state j can occur after state i (hence for them $f_{ij}^k > 0$) to select large z_{ij}. This is feasible in (31a) since $\beta_{ij}^* = 0$. Hence in $\mathscr{E}$, z_{ij}^* is unbounded and in $\mathscr{E}^n$ is equal to $+n$ for such (i, j) and this violates (49b). Hence $\beta_{ij}^* > 0$ for such (i, j). For (i, j) where $f_{ij} = 0$, we have $\beta_{ij}^* = 0$ but z_{ij}^* is not relevant since insurance against a state that cannot occur has no utility. Using Walras Law (Lemma 5) we conclude that (49a) holds with equalities for all i and (49b) holds with equality for all (i, j) with $f_{ij} > 0$. $\square$

The argument up to now has then demonstrated

Lemma 7. *For large enough n there exists an RBE in $\mathscr{E}^n$ with $p^{c*} \gg 0$, $\beta_{ij}^* > 0$ for all (i, j) with $f_{ij} > 0$ and $(49a)-(49b)$ holding with equalities for these configurations.*

We complete the proof by noting that there exists a convergent subsequence of the equilibria in $\mathscr{E}^n$ which is an equilibrium in $\mathscr{E}$ with positive prices when $f_{ij} > 0$. To see this note that by Assumption 1 and the material balance all real equilibrium quantities are in a compact set since $0 \leq x_i^{1k} \leq \hat{\omega} + R$, $0 \leq y_i^k \leq \hat{\omega} + R$, $0 \leq x_{ij}^{2k} \leq \hat{\omega} + R$ for all i, j and k where $\hat{\omega} = \max_i \omega_i$. Also, $q \in \Xi$. Hence a convergent subsequence exists. The positivity of the limit prices is seen in the following way. Since $p_j^{c*} y_j^* = p_j^* + p_j^{c*} R$ and $p_j^{c*} + p_j^* = 1$ hold for all n it follows that $p_j^{c*} > 0$ in the limit economy for all j. The strict positivity of the limit p_i^* follows from the same argument as above since when all $p_i^{c*} > 0$ the price of the composite PCC exceeds the value of the dividend. Finally, since (31b) holds for all n the limit of the z_{ij}^{k*} exists for all (i, j) with $f_{ij} > 0$. z_{ij}^{k*} for (i, j) with $f_{ij} = 0$ may be taken equal to 0.

Theorem 1. *Given Assumption 1–4 then for each price state space as speicified Lemma 1, there exist a Rational Belief Equilibrium with Price-Contingent Contracts.*

Remark. In general there is a large collection of generating processes which induce consistent price state space (See Nielsen [1996] for SIDS processes and Kurz and Schneider [1996] for Markov processes). The thrust of our theorem is that RBE exists whenever the consistency conditions specified in Lemma 1 hold.

4c On the generic presence of endogenous uncertainty in an RBE

The existence of an RBE does not guarantee the presence of endogenous uncertainty. More precisely, since $p \in \mathbb{R}^M$ the vector p consists of N_0 blocks, each of dimension $N = (M/N_0) = (N_1 \times N_2 \times \cdots \times N_k)$. The first block consists of all prices associated with states in which $\omega = \omega^1$, the second with $\omega = \omega^2$ etc. (see Assumption 1). This implies that the vector $(p_1, p_2, \ldots, p_N)$ of the first block is associated with the realization $\omega = \omega^1$. Thus, without loss of generality, we say that endogenous uncertainty is present if $(p_1^c, \beta_1) \neq (p_2^c, \beta_2)$. Recall, however, that the no-arbitrage condition (30) can be expressed as $p = B \cdot \vec{1}$ where $\vec{1} = (1, 1, \ldots, 1)$ and B is the matrix with rows β_i, $i = 1, 2, \ldots, M$. Given the price normalization it follows that $p_1 \neq p_2$ if β_1 is sufficiently different from β_2. Thus, examination of the presence of endogenous uncertainty boils down to the examination of the rows of B.

Turning to the RBE suppose, for simplicity that the utility function is additive with functions at the two dates being (u_1^k, u_2^k). Then, in equilibrium agent k carries out the optimization of (1′) subject to (32). This leads to the first order conditions

$$\frac{\partial u_2^k(x_{ij}^{2k})}{\partial x_{ij}^{2k}} f_{ij}^k = \frac{\partial u_1^k(x_i^{1k})}{\partial x_i^{1k}} \left(\frac{\beta_{ij}}{p_i^c} \right) \left(\frac{p_j^c}{p_j + p_j^c R} \right) \qquad j = 1, 2, \ldots, M. \tag{50}$$

Now consider a perturbation of the set $f^k = \{ f_1^k, f_2^k, \ldots, f_{N_k}^k \}$ of probabilities which k may select. Agent k is said to be *future oriented* if his consumption demands are sensitive to changes in his beliefs f_{ij}^k. This excludes the logarithmic utility function. Under the assumption that the utility functions of the agents are continuously differentiable we can use a standard transversality argument to show that if all agents are *future oriented* then any perturbation in the set f^k for any k (subject to the rationality conditions), would change equilibrium prices $q = (p^c, \beta)$. Hence, generically speaking, the rows $(\beta_1, \beta_2, \ldots, \beta_M)$ of B can be taken as different and consequently the prices q_i are different, giving rise to endogenous uncertainty.

5 Risk Allocation and Pareto Optimality

This section explores how the PCC affects resource allocation with endogenous uncertainty in the OLG economy under consideration. We first show that even with a full array of PCC the agents in the economy do not optimally seek full insurance against endogenous price uncertainty in an RBE. An agent is said to have achieved "full insurance" against endogenous uncertainty if his consumption in the second period is independent of which price is realized. We then demonstrate that a Rational Belief Equilibrium is Pareto Optimal.

We have defined an asset market structure to be "complete" if at each t there exists a PCC for each possible price p_{t+1}.[4] We now note that even with a complete asset market, an overlapping generation economy with one consumption good does not achieve a full *ex-post* insurance against endogenous uncertainty in any RBE. This can be seen from the aggregate budget constraint (5b) which requires

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[4] When markets are incomplete, the results of Huang and Wu [1994] [1995] can be extended to study existence and efficiency of equilibrium with incomplete price-contingent contracts.

$p_t^c x_t^2 = \sum_{k=1}^{K} p_t^c x_t^{2k} = p_t + p_t^c R$. If all agents achieve full insurance in an RBE, then x_t^2 must be independent of (p_t^c, p_t). The right hand side of the equation (5b) shows that this is not possible in the aggregate. Therefore, given equilibrium PCC prices, not all of the agents in the economy will *want* to fully hedge against endogenous uncertainty even when there is a complete set of price-contingent contracts. We now turn to show that the allocation in an RBE is Pareto Optimal in a sense to be explained.

In order to define "feasibility" in our financial economy we need to observe that "allocations" in our terminology are always contingent upon the M states in the economy and therefore feasibility of an allocation can be defined with the aid of the financial instruments which achieve them. In view of the completeness of the set of PCC as demonstrated in (32), this places no restriction on the set of contingent allocations.

Definition 3. *An allocation* $(x_t^{1k}, x_t^{2k}, \theta_t^k, z_t^k)$, $t = 1, 2, \ldots, k = 1, \ldots, K$ *is feasible if*

(i) $\quad \sum_{k=1}^{K} x_t^{1k} + \sum_{k=1}^{K} x_t^{2k} = \sum_{k=1}^{K} \omega_t^{1k} + R \quad$ *for all realizations,*

(ii) $\quad \sum_{k=1}^{K} \theta_t^k = 1,$

(iii) $\quad \sum_{k=1}^{K} z_t^k = 0.$

In order to discuss the optimality of such allocations we need to adopt a definition of "Pareto optimality" for stochastic OLG economies. Various definitions have been proposed for such economies (see, for discussion, Peled [1982]) and this is a somewhat debated issue. The focus of these alternative definitions is the interaction between the inefficiency inherent in the *integenerational allocation* of OLG economies and the structure of information available to agents in a stochastic environment. Throughout this paper we have avoided the intergenerational transfer problem of OLG economies and concentrated on the study of the beliefs of agents and the institutions for the allocation of endogenous uncertainty in competitive markets. Accordingly, in selecting a definition of Optimality we regard the endowment risk of the unborn as a secondary problem which is of questionable importance in our context. We are thus in general agreement with Peled's [1982] proposed definition of "Conditional Pareto Optimality" and use the term "Pareto Optimality" in this sense. To state it in our context and stress our desire to avoid the issue of efficiency of intergenerational allocation, we introduce the following definition:

Definition 4. *A feasible allocation* $(x_t^{1k}, x_t^{2k}, \theta_t^k, z_t^k)$ $t = 1, 2, \ldots, k = 1, \ldots, K$ *is Pareto Optimal if there does not exist another feasible allocation* $(\tilde{x}_t^{1k}, \tilde{x}_t^{2k}, \tilde{\theta}_t^k, \tilde{z}_t^k)$ $t = 1, 2, \ldots, k = 1, \ldots, K$ *such that agents are informed of the realization of the state when young and for every such realization*

$$\sum_{j=1}^{M} u^k(\tilde{x}_t^{1k}, \tilde{x}_{t+1}^{2k}(j)) f_t^k(p_{t+1}^*(j)) \geq \sum_{j=1}^{M} u^k(x_t^{1k}, x_{t+1}^{2k}(j)) f_t^k(p_{t+1}^*(j)), \quad k = 1, \ldots, K,$$

(51)

$$u_2^k(x_0^k, \tilde{x}_1^{2k}) \geq u_2^k(x_0^k, x_1^{2k}), \quad k = 1, \ldots, K,$$

*where $p_{t+1}^c x_{t+1}^{2k}(j) = (\theta_t^k + z_t^k(p_{t+1}(j)))(p_{t+1}(j) + p_{t+1}^c R)$, with a strict inequality in (51) for some k at some date t. x_0^k is some initial consumption for the old which may be set equal to 0 if the function is bounded. The (*) symbol indicates, as before, price ratio. The allocation is said to be Risk Allocation Pareto Optimal if, in addition, the alternative feasible allocation must satisfy*

$$\sum_{k=1}^{K} \tilde{x}_t^{2k} = \sum_{k=1}^{K} x_t^{2k} \qquad t = 1, 2, \ldots \tag{52}$$

Comment. Restriction (52) aims to focus on the risk allocation of the sequence of markets and to avoid the intergenerational efficiency problem which is different in nature.

Theorem 2. *An RBE with a complete structure of price contingent contracts is Pareto Optimal.*

Proof. The proof of the Risk Allocation optimality of an RBE is standard. To see that, suppose that an RBE $(x_t^{1k}, x_t^{2k}, \theta_t^k, z_t^k)$ $t = 1, 2, \ldots, k = 1, \ldots, K$ is not R.A.P.O., then there exists a feasible allocation $(\tilde{x}^{1k}, \tilde{x}^{2k}, \tilde{\theta}_t^k, \tilde{z}_t^k)$ $k = 1, \ldots, K$ such that

$$\sum_{j=1}^{M} u^k(\tilde{x}_t^{1k}, \tilde{x}_{t+1}^{2k}(j)) f_t^k(p_{t+1}(j)) \geq \sum_{j=1}^{M} u_2^k(x_t^{1k}, \tilde{x}_{t+1}^{2k}(j)) f_t^k(p_{t+1}(j)), \quad k = 1, \ldots, K.$$

$u^k(x_0^k, \tilde{x}_1^{2k}) \geq u^k(x_0^k, x_1^{2k})$, $k = 1, \ldots, K$, with a strict inequality for some agent k. Hence

$$p_t^c \tilde{x}_t^{1k} + \sum_{j=1}^{M} \beta_t(p_{t+1}(j))(\tilde{\theta}_t^k + \tilde{z}_t^k(p_{t+1}(j)))$$

$$\geq p_t^c x_t^{1k} + \sum_{j=1}^{M} \beta_t(p_{t+1}(j))(\theta_t^k + z_t(p_{t+1}(j))), \qquad k = 1, \ldots, K.$$

and $\tilde{x}_1^{2k} \geq x_1^{2k}$, with some inequality strict. Adding over k, using (52), the positivity of p_t^c and the definition of feasibility lead to $\sum_{k=1}^{K} \tilde{x}_t^{1k} + \sum_{k=1}^{K} \tilde{x}_t^{2k} > \sum_{k=1}^{K} x_t^{1k} + \sum_{k=1}^{K} x_t^{2k} = \sum_{k=1}^{K} \omega_t^{1k} + R$, contradicting the feasibility of $(\tilde{x}_t^{1k}, \tilde{x}_t^{2k}, \tilde{\theta}_t^k, \tilde{z}_t^k)$.

To prove that an RBE is Pareto Optimal we use Peled's [1982, Appendix] rather lengthy argument which we omit. It utilizes the observation that if the old are made better off at some date then the young of that date must be compensated at the following date. Since our RBE has a positive interest rate in all states, this compensation is positively compounded over time and after a finite number of steps in time the allocation becomes infeasible. □

6 Final remarks

This paper integrates two main ideas. The first proposes that agents without structural knowledge adopt rational beliefs and second, that endogenous uncertainty is traded with PCC rather than with the Arrow-Debreu state contingent contracts. The result is a new equilibrium concept in which price uncertainty is treated by the optimizing agents in the same way in which they treat the traditional exogenous uncertainty. Moreover, equilibrium fluctuations of allocations and

prices in the economy over time are partly endogenously propagated. The existence proof of an RBE is in the spirit of Arrow-Debreu theory and depends crucially upon the construction of the finite price state space which, in turn, results from the use of the rationality conditions. The importance of the rationality restrictions points the way to future research which, we hope, would generalize the results of the paper in two ways: (i) introduce multiple consumption goods and (ii) introduce multiple but finite number of periods and more complex financial assets.

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