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On rational belief equilibria ⋆

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Summary. We study equilibria in which agent's belief are rational in the sense of Kurz [1994]. The market is formulated by specifying a stochastic demand function and a continuum of producers, each with a quadratic cost function who must select their output before knowing prices. Holding Rational Beliefs about future prices, producers maximize expected profits. In a Rational Belief Equilibrium (RBE) agents select diverse forecast functions but each one is rational in the sense that it is based on a theory which cannot be rejected by the data. It is shown that there exists a continuum of RBE's and they could entail very different patterns of time series for the economy and consequently different aggregate levels of longterm volatility. Since the model contains exogenously specified random variables, the difference in the level of long-term volatility of prices among the different RBE's arises endogenously as an "amplification" of the volatility of exogenous variables. The paper derives exact bounds on the possible levels of such "amplification."
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1. Introduction

The theory of Rational Expectations has been the dominant doctrine employed in the construction of equilibria of models which deviate from the standard, complete market, general equilibrium model. Although Rational Expectations take different forms in different models of the economy (stochastic vs. deterministic, with or without securities, with or without complete information etc.) the common element of all these forms is that agents need to know demand and supply functions, they need to be able to compute general equilibrium and, using the stochastic or deterministic law of motion for the economy (which is known to the agents) they make forecasts of future economic variables to be the forecasts implied by the equilibrium law of motion of the economy. For ease of reference we shall call this kind of knowledge "Structural Knowledge."

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The idea of Rational Expectations clearly provides a logically consistent way for *a model builder* to close his model. However, is it sensible to expect humans to perform the kind of feats which Rational Expectations requires them to perform? Moreover, is it reasonable to require that the rationality of economic agents be conditioned on their possession of structural knowledge and skills that none of us possess? In an earlier paper (Kurz [1994]) we have introduced a new approach to the theory of expectations formation. This approach proposes to study the formation of expectations in a dynamic context in which ample past data on the performance of the system is available. It is postulated that agents possess no structural knowledge and their expectations should be based only on the knowledge of observable past *data*. To distinguish our approach we use in Kurz [1994] the term "Rational Beliefs" to describe those expectations which are based only on the observed data. The concept of "Rationality" employed is expressed in certain axioms which postulate that rational beliefs cannot be contradicted by the observed data.

In our earlier paper we made no attempt to incorporate the idea of Rational Beliefs into an equilibrium concept. Moreover, due to the inherent complexity of the issues involved we do not wish to follow a general and abstract conceptualization of "Rational Belief equilibrium." Instead, we shall study this type of equilibrium in different specific models and this paper is the first such application.

In the present paper we consider a simple model of stochastic price determination where producers or firms must commit (to investment or output) before observing prices. This is the basic set-up of most Rational Expectations Models (see for example Muth [1961], Lucas and Prescott [1971] and Sargent [1979]). In the version used here producers commit to output only and no investment is involved. In the early days of Rational Expectations this type of model was used to give a convincing argument why a permanent cobweb cycle with a regular amplitude is not a sensible equilibrium for a price dynamics of a market. This same model was used later to show how a process of learning by agents will converge to a Rational Expectations equilibrium (see for example Townsend [1978], Bray and Savin [1986] and, in a variant which includes investment, Townsend [1983]). Indeed, we have selected this type of model because it has been successfully employed in support of the theory of Rational Expectations.

In the development below, we will study a variant of this market model and examine its behavior under Rational Expectations as well as under Rational Beliefs. We then show that Rational Belief equilibria may exhibit behavioral patterns which are very different from Rational Expectations equilibria. To illustrate, we briefly state here three of these differences:

(a) In Rational Expectations equilibria all agents know the true probability distribution of prices and therefore have the same expectations. In a Rational Belief equilibria no one knows the true distribution of prices and each agent must form his own belief about it.
(b) A *permanent* cobweb cycle of fixed amplitude cannot arise in a Rational Belief equilibrium. However, any pattern of short term fluctuations, serial correlation and cobweb cycles can arise as long as these temporary patterns are sufficiently

different from each other so that they get averaged out over time and the long term patterns of prices and quantities are compatible with a certain stationary probability which is commonly known by all the agents since they can learn it from the data.

(c) The volatility of prices in any Rational Belief equilibrium is greater than the volatility in the Rational Expectations equilibrium of the model. The main reason for this excess volatility is the effect of non-stationary price forecasts of the agents on the equilibrium distribution of prices. Since no agent knows the true distribution, all agents make incorrect forecasts. (The term "incorrect forecast" does not refer to the existence of a "forecast error" which arises since agents do not know the future). The fact that the agents make incorrect, non-stationary, forecasts at each date introduces a non-stationary component of volatility into the equilibrium market prices. This, in turn, influences the long run average (and stationary) distribution of market prices and therefore on the Rational Beliefs of the agents. No such equilibrium feedback is present in a Rational Expectations equilibrium.

The market under study is a rather simple one. Nevertheless, all the essential differences between Rational Expectations equilibria and Rational Belief equilibria clearly come into focus. Moreover, this comparison allows us to draw very general conclusions about the vital properties which Rational Belief equilibria will have in other applications as well.

2. Rational beliefs

We provide here a brief summary of the theory proposed in Kurz [1994]. The economy is represented by a stable dynamical system in which agents do not know the true probability Q under which the data is generated. In all economic applications it is the *conditional* probabilities and expectations which represents the causal structure of the economy. Thus, modeling the agents as not knowing the true unconditional probability Q of the dynamical system is simply the formal way of stating our view that agents have no knowledge of the true *structure* of the economy.

The concept of "stability" requires the relative frequencies, at which the system visits any finite dimensional event, to converge. This implies that by computing relative frequencies from past data agents, in fact, do learn a probability m which is a stationary probability. This probability is then known to all observing agents. A crucial observation of the theory is that m may not be the same as Q and even if $m = Q$, agents do not know that the dynamical system is stationary and may not believe that $m = Q$. The main theorem of our earlier paper provides a characterization of all rational beliefs. To illustrate how to use the theorem suppose the observation at date τ is $y_\tau \in Y \subseteq \mathbb{R}^N$ and past data is $I_t = (y_0, y_1, \ldots, y_{t-1})$ representing the information at t. Suppose also that agent k adopts a rational belief Q_k with which he computes his forecast $- y^e_{kt}(I_t) -$ of y_t given I_t. Then, the main theorem states that his forecast takes the form

$$y^e_{kt}(I_t) = \lambda_k y^m(I_t) + (1 - \lambda_k) y^{\perp}_{kt}(I_t)$$

where

$y^m(I_t)$ – the conditional forecast of y_t under the stationary probability m known to all agents,

$y^{\perp}_{kt}(I_t)$ – the conditional forecast of y_t under a non-stationary probability $Q^{\perp}_k$ which is singular with m but is stable, with m as its stationary (or invariant) probability. This imposes severe restrictions on the asymptotic properties of $y^{\perp}_{kt}$,

λ_k – a constant with $0 < \lambda_k \leq 1$ representing the weight given by the agent to the possibility that the true dynamical system is stationary.

To interpret this theorem it is useful to think of a rational belief as a "theory" which an agent has about the system which he observes. With a probability λ_k he believes that the system is stationary and under stationarity his forecast is $y^m(I_t)$; this forecast is common to all observers. However, the agent gives weight $(1 - \lambda_k)$ to the possibility that the system is non-stationary. This would entail a forecast $y^{\perp}_{kt}$ under the probability $Q^{\perp}_k$ which represents the agent's theory about the nature of non-stationarity. Given the information I_t the stationary forecast can be interpreted as the "normal," long term, forecast given I_t. On the other hand $y^{\perp}_{kt}(I_t)$ represents the forecast based on the theory of agent k that at date t specific considerations are present such that given the information I_t the likelihood of future events may be sharply different from their normal, long term, stationary probabilities under m. In short, $y^{\perp}_{kt}(I_t)$ incorporates the belief of agent k in any *unusual* circumstances which prevail at t. The theorem shows that agents may differ in the pair $(\lambda_k, Q^{\perp}_k)$ which they select subject to strict limitations on the asymptotic properties of the system under Q_k (and hence under $Q^{\perp}_k$).

The theory of Rational Beliefs thus identifies a set $B(Q)$ of beliefs which are compatible with the data generated under Q: members of $B(Q)$ are theories which cannot be *rejected* by the data. On the other hand, a selection of $Q_k \in B(\Pi)$ by agent k is an act of *accepting* a theory among competing hypotheses in $B(Q)$. There are many considerations a prudent agent may have in adopting one selection criterion or another. For example he may minimize a loss function or insist on some specified level of confidence in choosing among members of $B(Q)$. Needless to say the agent has available to him a vast statistical literature on this question. Keep in mind that utilization of the wrong probability will lead the agent to employ suboptimal decision rules since the adoption of a belief is equivalent to the adoption of a sequence of decision rules. This obviously leads to the possibility that an investor in the stock market could adopt a most unusual belief Q_k which satisfies all the asymptotic properties needed in order to belong to $B(Q)$ but in the short run forecasts a decline of the Dow-Jones average by more than 500 points during the first week of October in each of the next 10 years. From the perspective of our theory if an investor deploys all of his assets to optimize relative to this belief he should not be declared irrational. We propose that the term of "Rationality" be reserved only to the identification of members of $B(Q)$ which cannot be rejected by the data, and *not be applied to the specific selection criteria in $B(Q)$ employed by the agents.* But, given the natural diversity in the criteria employed by a population of agents to select from among members of $B(Q)$, our theory suggests that the center of investigation should be shifted to the *distribution* of beliefs in the population rather than focus on the evaluation of the belief of any one particular agent. This, in fact, is true in the model presented below where it is the *distribution* of forecasts in the population which has significant economic implications rather than the forecasts of any particular agent.

3. A market model

We employ here a variant of the model used by Bray and Savin [1986] and others. The demand side of the market is represented by a stochastic demand function

$$d_t = \alpha_t - \beta p_t + v_t, \quad \alpha_t > 0, \quad \beta > 0 \tag{1}$$

where the exogenous shocks of demand are represented by (α_t, v_t). α_t, $t = 0, 1, 2, \ldots$ is a sequence of real numbers which are, asymptotically, not correlated with any past data (including past α_t) and v_t, $t = 0, 1, 2, \ldots$ is a sequence of random variables with mean zero. As in Kurz [1994] we consider only stable systems which imply, among other requirements, that the following limits exist:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \alpha_t = \alpha, \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} v_t = 0 \quad \text{a.e.}$$

Note that $\alpha_t \neq \alpha$ is equivalent to modifying the assumption that v_i has a mean zero. This is so since (1) can be rewritten as

$$d_t = \alpha - \beta p_t + [v_t + \alpha_t - \alpha]$$

and the mean of $(v_t + \alpha_t - \alpha) \neq 0$. In the form (1) the exogenous unobserved error has a mean zero. The key advantage of (1) is the emphasis on the fact that v_t is a *random* sequence of unobserved noise with mean 0 while α_t is an unobserved *deterministic* sequence of parameters causing demand to be non-stationary. This sequence is not required to converge and therefore non-stationarity is not required to vanish.

There is a continuum of firms in the market; they are indexed by $k\varepsilon[0, 1]$ and are uniformly distributed on the unit interval. They are all identical with cost function

$$c(q_{kt}) = \frac{1}{2\gamma} q_{kt}^2 \quad \gamma > 0. \tag{2}$$

Firms must decide on their optimal output *before* they know the price p_t. They do get to observe at date t a vector $x_t = (x_{1t}, x_{2t}, \ldots, x_{Nt})$ of N observable variables representing the exogenous environment. Given a probability belief Q_k of firm k on (p_t, x_t), a maximization of expected profits leads to the optimal output of firm k defined by

$$\gamma p_{kt}^e \quad k \in [0, 1]$$

where $I_t = ((\mathbb{R}_+, x_0), (p_0, x_1), (p_1, x_2), \ldots, (p_{t-1}, x_t))$ and

$$p_{kt}^e = E_{Q_k}(p_t | I_t).$$

The market aggregate supply is

$$s_t = \gamma \int_0^1 p^e_{kt}(I_t)dk + x_t'\mu + \varepsilon_t$$

where ε_t is an unobserved random variable with mean 0. The integral in the definition of s_t is assumed to exist. In fact, all functions of the index k which appear in this paper are assumed uniformly bounded and measurable with respect to the Borel σ-field on $[0, 1]$. The term $(x_t'\mu + \varepsilon_t)$ is a random shock to supply representing random factors which vary the amounts effectively available to the market relative to what the producers planned to have available. x_t and ε_t are independent for all t. We assume that the process $\{(x_t, \varepsilon_t), t = 0, 1, 2, \ldots\}$ is stable as well. Consequently, among other conditions we require that the following limits exist a.e.

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} x_t = \bar{x}, \quad \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \varepsilon_t = 0$$

Market clearing requires $d_t = s_t$ at all t and hence p_t must satisfy

$$\alpha_t - \beta p_t + v_t = \gamma \int_0^1 p^e_{kt}(I_t) + x_t'\mu + \varepsilon_t$$

which can then be written as

$$p_t = (\delta_t + x_t'b) + a \int_0^1 p^e_{kt}(I_t)dk + u_t \tag{3}$$

where

$$\delta_t = \frac{\alpha_t}{\beta} > 0 \quad a = -\frac{\gamma}{\beta} < 0$$

$$b = -\frac{1}{\beta}\mu \quad u_t = \frac{1}{\beta}(v_t - \varepsilon_t).$$

For simplicity of exposition we assume that δ_t are neither autocorrelated nor correlated with any past observed data. This assumption can easily be relaxed but will entail more complex computations.

All these structural equations are not known to the agents and they certainly do not know the stochastic properties of x_t, v_t, ε_t or u_t. On the other hand these agents know the entire history of the system and therefore they have a large data bank of past prices, quantities and exogenous variables x_t since the beginning of time. In addition, agents have unlimited computing ability and they are rational in terms of using probability and statistical methods to draw inferences from the data at their disposal. These assumptions are extreme since data storage, recall and processing is difficult and expensive. We make these assumptions to maintain theoretical simplicity and avoid the technicalities of approximation procedures. However, allowing for cost of data processing and difficulties of recall will only introduce added complexity to the resulting "Rational Belief equilibrium"; its break with Rational Expectations will become even more drastic.

In order to discuss the subject of equilibrium we need to be more specific about the probability spaces involved. Since

$$p_t \in \mathbb{R}_+, \quad x_t \in X \subseteq \mathbb{R}^N$$

we take the measurable space $(\Omega, \mathscr{F})$ to be the coordinate space:

$$\Omega = (\mathbb{R}_+ \times X)^\infty$$

$$\mathscr{F}_t = \sigma((\mathbb{R}_+, x_0), (p_0, x_1), (p_1, x_2), \ldots, (p_{t-1}, x_t))$$

$$\mathscr{F} = \sigma\left(\bigcup_{t=0}^{\infty} \mathscr{F}_t\right).$$

Ω is the space of infinite sequences $(p, x) = ((p_0, x_0), (p_1, x_1), \ldots)$, and $\mathscr{F}$ is the σ-field generated by the data. The information available to the agents at date t includes x_t but excludes p_t. Now, for any infinite sequence y let $y^t = (y_t, y_{t+1}, y_{t+2}, \ldots)$ and define the shift operator T by

$$Ty^t = y^{t+1}.$$

For any probability belief Q_k of agent k we can think of $(\Omega, \mathscr{F}, Q_k, T)$ as the dynamical system as viewed by agent k. We recall from Kurz [1994] that if a dynamical system $(\Omega, \mathscr{F}, Q, T)$ is *stable* then it has a stationary (or invariant) mean probability which we shall denote by m_Q.

4. Equilibrium concepts

We now return to our model to discuss the issue of equilibrium. We note first that in our development above we wrote

$$p^e_{kt}(I_t) = E_{Q_k}(p_t \mid I_t)$$

to indicate that in forming their forecasts agents can use only the information available to them. However, in a *Rational Expectations Equilibrium* agents are assumed to know Q – the true (endogenously determined) probability of (p, x). Thus, taking expectations of (3) and specifying that $Q_k = Q$ for all k we have that

$$p^e_t = E_Q(p_t \mid I_t) = (\delta_t + x'_t b) + a p^e_t$$

and hence Q is defined by the restrictions

$$(4) \qquad \begin{cases} p_t = p^e_t(x_t, \delta_t) + u_t \\ p^e_t(x_t, \delta_t) = (\delta_t + x'_t b)(1-a)^{-1}. \end{cases}$$

Definition: A *Rational Expectations equilibrium* is an infinite sequence (p, x) of random variables under the true probability Q and a set of agent probabilities Q_k, $k \in [0, 1]$ such that

(i) $Q = Q_k$ for all $k \in [0, 1]$,
(ii) Under $Q, (p, x)$ satisfy the market clearance conditions at all t and therefore

$$p_t = p^e_t(x_t, \delta_t) + u_t$$
$$E_Q(p_t \mid I_t) = p^e_t(x_t, \delta_t) = (\delta_t + x'_t b)(1-a)^{-1}.$$

Note that in this model there is no conceivable way in which the agents can know the parameters δ_t. Yet, a Rational Expectations equilibrium requires the function $p^e_t(x_t, \delta_t)$ to be known to all. It also requires the stochastic properties of the sequence (p, x) to be commonly known even though they cannot possibly be discovered by the agents.

We turn now to the process of forming Rational Beliefs. As was stressed in section 1 agents do not know Q. However, using past experience agents can discover the stationary probability m on $(\Omega, \mathscr{F})$ which represents the average frequency by which events were experienced in the past. This represents what agents perceive as the "normal course of events" and thus m is common knowledge among all agents.

In selecting the probability space $(\Omega, \mathscr{F})$ we elected to study infinite sequences (p, x) starting at $t = 0$ and going to $t = +\infty$. This means that the information vector $I_t = ((\mathbb{R}_+, x_0), \ldots, (p_{t-1}, x_t))$ changes dimension with t. This leads to the difficulty of needing to write $E_m(p_t | I_t)$ as a sequence of time dependent forecasts and consequently to some difficulties in writing out the stability conditions. To overcome this difficulty we shall distinguish blocks of information according to the dates at which the information becomes available. Thus, in order to represent a fixed block of information which is shifted in time consider such a finite block which we denote by I

$$I = ((\mathbb{R}_+, x_0), (p_0, x_1), (p_1, x_2), (p_2, x_3), \ldots, (p_{L-1}, x_L)). \tag{5}$$

Now we denote by $I_\sigma(I)$ the same information I shifted from date L to date $\sigma > L$ but *without any additional information from* 0 *to* $\sigma - L$. That is

$$\begin{aligned} I_\sigma(I) &= ((\mathbb{R}_+, X), (\mathbb{R}_+, X), \ldots, (\mathbb{R}_+, x_0), (p_0, x_1), \ldots, (p_{L-1}, x_L)) \\ &= (\underbrace{(\mathbb{R}_+, X), (\mathbb{R}_+, X), \ldots, (\mathbb{R}_+, X)}_{(\sigma - L)\text{ times}}, I). \end{aligned}$$

The idea of this notation is to express the possibility that a given information becomes available at different dates. We can use the stationarity of m to define $p^m(I)$ using the notation

$$p^m(I) = E_m(p_\sigma | I_\sigma(I)). \tag{6}$$

The theory of Rational Beliefs says that each agent k will select a probability Q_k with which he will compute his forecast $p^e_{kt}(I_t)$. This forecast must satisfy, among others, two conditions:

(7a) (i) $$p^e_{kt}(I_t) = \lambda_k p^m(I_t) + (1 - \lambda_k) p^{\perp}_{kt}(I_t)$$

(ii) for any fixed block of information I as defined in (5)

$$\frac{\lim_{T \to \infty} \frac{1}{T} \sum_{\sigma = L}^{L+T-1} p^e_{k\sigma}(I_\sigma(I)) Q_k(I_\sigma(I))}{\lim_{T \to \infty} \frac{1}{T} \sum_{\sigma = L}^{L+T-1} Q_k(I_\sigma(I))} = p^m(I) \tag{7b}$$

where $Q_k(I_\sigma(I))$ is the Q_k probability of $I_\sigma(I)$.

Now let all the firms adopt Rational Beliefs. Inserting (7a) into (3) we have that the price P_t is an induced random variable defined by

$$p_t = (\delta_t + x_t'b) + a\int_0^1 (\lambda_k p^m(I_t) + (1-\lambda_k)p_{kt}^{\perp}(I_t))dk + u_t. \tag{8}$$

This will induce a "true" probability Q of (p, x). In equilibrium the true probability Q and the Rational Beliefs Q_k of the agents *do not need to be the same*; they are only required to be compatible! This means that if we think of $(\Omega, \mathscr{F}, Q, T)$ as the true dynamical system and of $(\Omega, \mathscr{F}, Q_k, T)$ as the dynamical systems as viewed by the agents, then these dynamical systems must all be stable and generate data with the same asymptotic properties. Since every stable system generates a stationary mean probability, the compatibility condition calls for all the stationary probabilities to be the same as the one computed from the data. That is

$$m = m_Q = m_{Q_k} \quad \text{for all} \quad k\in[0,1].$$

To translate these to the forecasts in (7a) take the fixed information block I_t (and hence a fixed vector $x_L = x_t$). Then we require for all $k\in[0,1]$ that $p^m(I_t)$ be defined as in (7b) and (δ_t, u_t) satisfy the following stability conditions

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \delta_t = \bar{\delta}$$

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} u_t = 0 \quad \text{a.e.}$$

We can therefore finally compute the conditional forecast under the stationary measure, $p^m(I_t)$ to be

$$p^m(I_t) = (\bar{\delta} + x_t'b)(1-a)^{-1}. \tag{9}$$

It is important to see that although an agent knows nothing about the structure of the economy, using past data he can discover the stationary measure m and consequently he can discover the parameters a, b and $\bar{\delta}$ used in the stationary forecast function (9). The fact that all the agents know these parameters *does not mean that they know how to use these parameters to construct, at each date, the true forecast function* $\mathbf{p}_t^Q(\mathbf{I}_t)$ under the true equilibrium probability Q. These conditional expectations may have time varying parameters. In their disagreement the agents adopt diverse beliefs Q_k which will then induce the endogenously determined equilibrium price function and hence Q. To see how the market calculates this function we return to equation (8). Let the non-stationary component of the forecast of agents be defined by

$$p_t^{\perp}(I_t) = \int_0^1 (1-\lambda_k)p_{kt}^{\perp}(I_t)dk$$

and hence we have the stochastic price equation

$$p_t(I_t) = (\bar{\delta} + x_t'b)(1-a)^{-1}(1-a(1-\lambda)) + (\delta_t - \bar{\delta}) + ap_t^{\perp}(I_t) + u_t \tag{10}$$

where

$$\lambda = \int_0^1 \lambda_k dk.$$

Note that for simplicity of exposition we have postulated that the δ_t are asymptotically uncorrelated with past observations. Consequently $\bar{\delta}$ is the optimal forecast of δ_t under the stationary probability. If we allowed δ_t to be serially correlated or even correlated with past values of x_t then past data of prices and x_t would provide some information about δ_t and therefore we would replace $\bar{\delta}$ with a stationary prediction $\delta^m(I_t)$ which is the optimal forecast of δ_t under the stationary measure. Replacing $\bar{\delta}$ with $\delta^m(I_t)$ does not change the substance of the argument below. We are now ready to introduce our new equilibrium concept.

Definition: A *Rational Belief equilibrium* is an infinite sequence (p, x) of random variables under the true probability Q and a set of Rational Beliefs $Q_k, k \in [0, 1]$ such that

(1) $(\Omega, \mathscr{F}, Q, T)$ and $(\Omega, \mathscr{F}, Q_k, T)$ for all $k \in [0, 1]$ are stable with m as the common stationary mean probability,
(2) under Q, (p, x) satisfy the market clearance conditions at all t such that for all t

$$p_t = p_t^e(I_t, \delta_t) + u_t$$
$$p_t^e(I_t, \delta_t) = (\bar{\delta} + x_t'b)(1-a)^{-1}(1 - a(1-\lambda)) + (\delta_t - \bar{\delta}) + a p_t^{\perp}(I_t).$$

Let us now examine some properties of this equilibrium. To do this let

p_t^R = the equilibrium price in a Rational Expectations equilibrium.
p_t^B = the equilibrium price in a Rational Belief equilibrium.

We know that

(11a) $$p_t^R = (\delta_t + x_t'b)(1-a)^{-1} + u_t$$

(11b) $$p_t^B = (\bar{\delta} + x_t'b)(1-a)^{-1}(1 - a(1-\lambda)) + a\int_0^1 (1-\lambda_k) p_{kt}^{\perp}(I_t) dk + (\delta_t - \bar{\delta}) + u_t.$$

First note that it follows from (11a)–(11b) and the stability conditions that $\bar{p} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} p_t^R = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} p_t^B = (\bar{\delta} + \bar{x}'b)(1-a)^{-1}$. This means that the long run *averages* of prices and quantities are the same under the two equilibrium concepts.

The existence of a Rational Belief equilibrium is established by simply exhibiting one. To do this consider the case where $Q_k = m$ for all k so that all agents believe that the environment is stationary. In this case $\lambda_k = 1$ for all k hence $\lambda = 1$ and consequently the equilibrium Q is defined by the following stochastic price equation

$$p_t^B = (\bar{\delta} + x_t'b)(1-a)^{-1} + (\delta_t - \bar{\delta}) + u_t.$$

In this case the difference between the two functions p_t^R and p_t^B becomes

$$p_t^R - p_t^B = a(\delta_t - \bar{\delta})(1-a)^{-1}.$$

This difference reflects the effect of the true non-stationarity represented by $(\delta_t - \bar{\delta})$.

Is the Rational Expectations equilibrium a Rational Belief equilibrium? Since agents do not know δ_t and the sequence is not learnable the unique Rational Expectations equilibrium can be established only if almost all the agents "accidentally" chose to believe in the Q which is defined by p_t^R. If $\lim_{t\to\infty} \delta_t$ does not exist such a belief will violate Axiom 2 in Kurz [1994] and hence the Rational Expectations equilibrium is not a Rational Belief equilibrium.

An important property of Rational Belief equilibria is seen by considering the special case when the environment is truly stationary and $\delta_t = \bar{\delta}$. In this case

$$p_t^B = (\bar{\delta} + x_t'b)(1-a)^{-1}(1 - a(1-\lambda)) + a\int_0^1 (1-\lambda_k)p_{kt}^{\perp}(I_t)dk + u_t$$

and consequently

$$p_t^R - p_t^B = (\bar{\delta} + x_t'b)(1-a)^{-1}a(1-\lambda) - a\int_0^1 (1-\lambda_k)p_{kt}^{\perp}(I_t)dk. \tag{12}$$

In (12) *a stationary environment may induce a non-stationary price structure via the effect of beliefs on prices.* The fact that market prices may be a non-stationary process even in a stationary environment has been recognized in the learning literature (see for example Townsend [1978] and [1983]).

5. Characterization of the stationary limit price distribution induced by Rational Belief equilibria

In our model, non-stationarity is a permanent fixture and the amount of non-stationarity does not vanish. However, a central component of our theory is that by computing empirical distributions of the data generated by the economy, agents discover the stationary mean probability m. We have already seen that at any date t the conditional forecast of p_t^B under m is

$$E_m(p_t^B | x_t) = (\bar{\delta} + x_t'b)(1-a)^{-1}.$$

Now we want to investigate the behavior of the variance of p_t^B under m. It is obviously of interest to compare it with the corresponding variance of p_t^R in the Rational Expectations equilibrium since this last variance reflects only the "fundamentals" of the economy. However, we are also interested in the subtle question of how changes in the variances of p_t^B due to *non-stationarity at all dates* $t = 0, 1, 2, \ldots$ translate into changes in the variance of p_t^B under the *stationary* mean probability m. This is what we called earlier the "feedback effect" of beliefs on the stationary mean probability of equilibrium prices.

Let us start by examining price variability under Rational Expectations. Use (11a) to conclude that

$$(p_t^R - \bar{p}) = \left(\frac{1}{1-a}\right)[(\delta_t - \bar{\delta}) + (x_t - \bar{x})'b] + u_t. \tag{13}$$

Since δ_t, x_t and ε_t are uncorrelated, the average volatility of prices over time is measured by the limits

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}(p_t^R-\bar{p})^2=\left(\frac{1}{1-a}\right)^2$$

$$\left[\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}(\delta_t-\bar{\delta})^2+\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}((x_t-\bar{x})'b)^2\right]+\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}u_t^2.$$

Stability ensures that all these limits exist a.e. and we denote them by $\sigma_{pR}^2, \sigma_\delta^2, \sigma_x^2$ and σ_u^2 respectively. Hence, under Rational Expectations we have the following variance which represents the fundamentals of the economy

$$\sigma_{pR}^2=\left(\frac{1}{1-a}\right)^2[\sigma_\delta^2+\sigma_x^2]+\sigma_u^2. \tag{14}$$

In the discussion below we also use the notation $\sigma_{z_k}^2, \sigma_{\eta_k}^2, \sigma_Z^2, \sigma_\eta^2, \sigma_{pB}^2$ and some covariances of these variables. All are limits of the means sum of squares of the relevant variables. The existence of these limits follows from the conditions of stability.

We turn now to Rational Beliefs. Given his belief Q_k agent k calculates the forecast $p_{kt}^{Be}=E_{Q_k}(p_t^B|I_t)$. We then define η_{kt} to satisfy

$$p_t^B=p_{kt}^{Be}+\eta_{kt}. \tag{15a}$$

Since at date t agents observe only x_t, a reasonably general class of forecast functions p_{kt}^{Be} can be expressed by

$$p_{kt}^{Be}=\frac{1}{1-a}[z_{kt}+x_t'b]. \tag{15b}$$

The functions z_{kt} contain the non-stationary component of the forecasts and they may depend upon all information available at date t (including past prices, quantities and x's). It is through these non-stationary forecasts that short term cyclical patterns and serial correlations are introduced into market equilibrium. These patterns reflect the "theories" of traders about the causes of price movements. In a Bayesian setting, when agents are assumed to know more about the structure of the economy, the functions z_{kt} may be interpreted to contain a component of Townsend's "Forecasting the Forecasts of Others" (see Townsend [1978], [1983]). In our non-stationary setting z_{kt} is more complex and we return to this issue below.

The conditions of stability impose strong restrictions on z_{kt} and η_{kt} in (15a)–(15b). If Q is the true equilibrium probability of sequences (p, x) agents use the wrong probabilities Q_k to arrive at the forecasts z_{kt} and the forecast errors η_{kt}. The conditions of stability require that the following limits exist Q a.e.

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}z_{kt}=\bar{\delta}\quad \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\eta_{kt}=0\quad \text{all } k\in[0,1]. \tag{16}$$

Similarly, if we denote by $\sigma_{z_k}^2$ and $\sigma_{\eta_k}^2$ the limits of the corresponding mean finite sums then $\sigma_{z_k}^2$ and $\sigma_{\eta_k}^2$ exist Q a.e. for all k and also

$$\operatorname{cov}(z_k, x)=0\quad \text{for almost all realizations } x, \tag{17a}$$

(17b) $$\operatorname{cov}(z_k, \delta) = 0,$$

(17c) $$\operatorname{cov}(z_k, u) = 0 \quad \text{for almost all realizations } u.$$

The conditions in (17a) specify that z_k is asymptotically uncorrelated with x. (17b)–(17c) express the assumption made earlier that the sequences u and δ are unpredicted. Obviously there are z_k sequences which violate these conditions (e.g. $z_{kt} = \delta_t$) but we disregard them since the probability of their discovery is zero. Keeping these facts in mind, use (3) and (15b) to compute the price function for equilibrium under Rational Beliefs to be

(18) $$p_t^B = \frac{1}{1-a}[\delta_t + x_t'b] + \frac{a}{1-a}\int_0^1 (z_{kt} - \delta_t)dk + u_t$$

and we then define

(19) $$Z_t = \int_0^1 z_{kt}dk.$$

It follows from (18) and (19) that stability requires the limit of the means of the market aggregate Z_t to exist a.e. and for σ_Z^2 to exist a.e. as well. Hence, computing the mean sum of squares of $(p_t^B - \bar{p})$ in (18) and taking limits we have

(20) $$\sigma_{pB}^2 = \left\{\left(\frac{1}{1-a}\right)^2 [\sigma_\delta^2 + \sigma_x^2] + \sigma_u^2\right\} + \left(\frac{a}{1-a}\right)^2 \sigma_Z^2 + \frac{a(a-2)}{(1-a)^2}\sigma_\delta^2.$$

Comparison of (14) and (20) shows that

$$\sigma_{pB}^2 - \sigma_{pR}^2 = \left(\frac{a}{1-a}\right)^2 \left[\sigma_Z^2 + \left(1 - \frac{2}{a}\right)\sigma_\delta^2\right] > 0$$

and the inequality holds since $a < 0$ and hence $\left(1 - \frac{2}{a}\right) > 0$. We have then demonstrated the following result:

Proposition 1: Price volatility under the stationary mean probability induced by a Rational Belief equilibrium is greater than the corresponding asymptotic volatility of prices in the Rational Expectations equilibrium and hence, permanent price volatility in Rational Belief Equilibria exceeds the volatility induced by the fundamentals of the economy.

We stress that the short term variance of p_t^B *at any date t* under Q is essentially unrestricted. By this we mean that price volatility like in an October 1987 crash is entirely compatible with equilibrium as long as such events are sufficiently infrequent and different from each other so that when averaged out over time they are compatible with the variance of p_t^B under m. We formalize this as follows:

Proposition 2: The variance $\sigma^2_{p_t^B}$ of p_t^B under Q at date t in a Rational Belief equilibrium is unrestricted. Sharp price fluctuations due to the non-stationary component of beliefs $p_t^\perp(I_t)$ are possible as long as they are infrequent and when averaged over time they are compatible with the variance σ_{pB}^2 under m.

The extra term in (20), $\left(\frac{a}{1-a}\right)^2\left[\sigma_Z^2+\left(1-\frac{2}{a}\right)\sigma_\delta^2\right]$, is called *the permanent amplification effect of beliefs on price volatility.* We use this term because, under the condition that agents know δ_t and thus need not form any beliefs, all the "fundamental" sources of price volatility are already included in the expression $\left(\frac{1}{1-a}\right)^2[\sigma_\delta^2+\sigma_x^2]+\sigma_u^2$. The extra volatility in (20) is due to the term $\frac{a}{1-a}(Z_t-\delta_t)$ in (18) which exists only because the δ_t are not known and agents must form rational beliefs about prices. To clarify this further recall first the definition of the stationary forecast $p^m(I_t)$ in (9) and combine it with (18) to conclude that

(21) $$p_t^B = p^m(I_t) + \left(\frac{a}{1-a}\right)\int_0^1 (z_{kt}-\bar{\delta})dk + (\delta_t - \bar{\delta}) + u_t.$$

Equation (21) clarifies the non-stationary forecasting problem of the agent: he knows $p^m(I_t)$ but needs forecasts of δ_t as well as Z_t. It is therefore useful to distinguish between the "forecast error" of agent k and his incorrect forecast. To explain further, express the price equations under Q and under Q_k as follows:

(22a) $$p_t^B = p_t^{Be} + u_t \quad \text{under} \quad Q$$

(22b) $$p_t^B = p_{kt}^{Be} + \eta_{kt} \quad \text{under} \quad Q_k.$$

Under Q the market price forecast is (from (18))

(23) $$p_t^{Be} = \frac{1}{1-a}[\delta_t + x_t'b] + \frac{a}{1-a}(Z_t - \delta_t)$$

whereas the forecast error under Q is u_t. Since agent k does not know Q his forecast error is η_{kt}. The incorrect forecasting of agent k arises because $p_{kt}^{Be} \neq p_t^{Be}$ and if we denote his mistake by $M_{kt} = p_{kt}^{Be} - p_t^{Be}$ we have from (15b) and (18) that

$$M_{kt} = \frac{1}{1-a}[z_{kt} - \delta_t] - \frac{a}{1-a}[Z_t - \delta_t].$$

Consequently the aggregate forecasting mistake is

$$\int_0^1 M_{kt}dk = Z_t - \delta_t.$$

In attempting to approximate p_t^{Be} in (23) an agent aims to forecast not only the exogenous non-stationary parameter δ_t but also the aggregate forecasting mistake $(Z_t-\delta_t)$ of the market. In fact, the agents never succeed in making the correct forecast and therefore the aggregate forecasting mistake $(Z_t-\delta_t)$ persists in the price equation; this incorrect forecasting is the root cause of the permanent amplification in the volatility of prices.

The economic interpretation of the volatility amplification is rather immediate. Note first that if agents knew δ_t they would have selected $z_{kt}=\delta_t$ and the second term in (18) would have vanished. The result would be a Rational Expectations equilibrium. On the other extreme suppose the agents believed that the market is stationary and then selected the mistaken forecast $z_{kt} = \bar{\delta}$ for all k and all t. In this case $\sigma_Z^2 = 0$ but the volatility amplification conclusion remains true. In essence, price volatility increases simply because fluctuations of demand are not anticipated correctly by the producers so that the fluctuations in demand and supply do not match. The consequence is the excess volatility. This provides an intuitive explanation for the fact that the Rational Expectations equilibrium is the equilibrium with the *lowest level of price volatility*. It also explains why any model in which agents do not have structural knowledge will result in a Rational Belief equilibrium in which price volatility will exceed the price volatility of a Rational Expectations Equilibrium.

What is the range of possible levels of the permanent volatility amplification which may occur in Rational Belief equilibria? To answer this we employ a simple consequence of the conditions of stability. In order to state it we recall first that if $Q_k = Q$ it follows from elementary considerations of conditional probability that the variance of the price forecast p_{kt}^{Be} of agent k is less than the variance of the variable forecasted, namely p_t^B. In our case where $Q_k \neq Q$ the conditions of stability imply a similar condition with respect to the asymptotic empirical variances. Thus, the condition becomes

$$(24) \qquad \sigma^2_{p_k^{Be}} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} (p_{kt}^{Be} - \bar{p}^B)^2 \leq \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} (p_t^B - \bar{p}^B) = \sigma^2_{pB}.$$

We can now calculate from (15b)

$$(25a) \qquad \sigma^2_{p_k^{Be}} = \left(\frac{1}{1-a}\right)^2 (\sigma_x^2 + \sigma_{z_k}^2)$$

and from (20)

$$(25b) \qquad \sigma^2_{pB} = \left(\frac{1}{1-a}\right)\sigma_x^2 + \sigma_\delta^2 + \sigma_u^2 + \left(\frac{a}{1-a}\right)^2 \sigma_z^2.$$

Consequently, (24) implies that

$$(26) \qquad \left(\frac{1}{1-a}\right)^2 \sigma_{z_k}^2 \leq \left(\frac{a}{1-a}\right)\sigma_Z^2 + (\sigma_\delta^2 + \sigma_u^2).$$

Integrating (26) with respect to k and using Jensen's inequality we have

$$\left(\frac{1}{1-a}\right)^2 \sigma_Z^2 \leq \left(\frac{1}{1-a}\right)^2 \int_0^1 \sigma_{z_k}^2 \, dk \leq \left(\frac{a}{1-a}\right)^2 \sigma_Z^2 + (\sigma_\delta^2 + \sigma_u^2).$$

This implies that

$$(27) \qquad \left(\frac{1+a}{1-a}\right)\sigma_Z^2 \leq \sigma_\delta^2 + \sigma_u^2.$$

Since $a < 0$, it follows from (27) that if $1 + a < 0$ then this inequality places no restrictions on σ_Z^2. On the other hand, if $1 + a > 0$ then we have

$$(28) \qquad \sigma_Z^2 \leq \left(\frac{1-a}{1+a}\right)(\sigma_\delta^2 + \sigma_u^2).$$

This leads to our third conclusion:

Proposition 3: If $1 + a > 0$ then among all Rational Belief equilibria the uniform bound on permanent volatility amplification satisfies

$$\left(\frac{a}{1-a}\right)^2\left[\sigma_Z^2 + \left(1 - \frac{2}{a}\right)\sigma_\delta^2\right] \leq \left(\frac{a}{1-a}\right)^2\left(\frac{1}{1+a}\right)\left[-\frac{2}{a}\sigma_\delta^2 + (1-a)\sigma_u^2\right].$$

If $1 + a < 0$ no finite uniform bound on the volatility amplification exists.

Proposition 3 becomes clearer if we recall from (3) that the parameter a measures the effect of variations in the aggregate mean forecast of the agents on the equilibrium price. Hence, if $|a| > 1$ then a \$1 change in the mean forecast of prices changes market price by more than \$1. This instability can generate unbounded, self fulfilling long term volatility of prices. This conclusion should, however, be viewed with the proper perspective. Each Rational Belief equilibrium will always have a bounded volatility amplification even if $1 + a < 0$. Moreover, there are other practical reasons (which are not explicitly handled in the model) that would put bounds on price volatility. These include capacity limitations on production, non negativity of economic variables such as p_t^B etc. The condition $1 + a > 0$ simply enables us to establish a *uniform bound on all possible equilibria* and this bound is determined only by the variables specified in the model.

One essential feature of the amplification is that its effect on the distribution of prices depends on the aggregate distributional properties of the individual forecasts z_{kt}. Keeping in mind that statements about the independence of a continuum of random variables require some care, we can note that if the z_{kt} are, in some sense, sufficiently diverse then σ_Z^2 would be very small. Also, if fads and other forms of "public opinions" have a strong impact on individual beliefs then σ_Z^2 would be large. This means that in our model *market risks are determined endogenously by the distribution of individual beliefs in the economy.* Since our agents view market prices as objects of uncertainty, changes in the distribution of beliefs change the riskiness of market prices to all agents. This is completely analogous to the effect of change in the distribution of tastes on equilibrium prices in a static model. There is, however, a crucial difference: changes in the volatility of equilibrium prices *keeping the mean function unchanged* means that the different equilibria which are induced in this manner would be Pareto ranked in an economy with risk averse agents. This is clearly not the case in the model of this paper. However, in a model with risk aversion such a comparison will open the door to some serious questions about the potential uses of public policy in an economy with Rational Belief equilibria.

To conclude this discussion of volatility we need to make a comment about *short term vs. long term price volatility.* Although the long term volatility measures provided here are very important for understanding the nature of Rational Belief equilibria, they have little relevance to short term volatility. That is, if an econometrician examines a sample of observations over a short span of time, the theory at hand provides little restrictions on the amount of volatility that he may find in the data. Moreover, agents with a positive discount rate have little interest in the long term averages of the economy. These agents form beliefs on the basis of which they take short term actions in the market. Short term fluctuations may contain cycles, short term serial correlations and other short term patterns. The theories embodied in the beliefs of some agents could turn out to be very profitable for them, particularly if their propensity to take risk allows them to take prompt action; these short term profits will not last for long. A similar argument applies to cobweb cycles. The theory of Rational expectations claims that a cobweb process will *never* arise at any time since agents know the true probability of prices. A *permanent* cobweb cycle with a constant amplitude cannot arise in our equilibrium either and on this point both equilibrium theories agree. However, periods of cobweb-like behavior may frequently arise in our equilibrium in the sense that the "market" (represented by $p_t^e(I_t)$) expects higher prices but experiences lower prices (and the converse). The amplitudes of these cobweb-like periods cannot repeat often. On the other hand there may be many periods in which the market expects *small* increases (or decreases) in prices and experiences *big* increases (or decreases) in prices due to the configurations of $p_t^e(I_t)$, δ_t and u_t. As agents average past data, such averages will show no cobweb pattern.

A final remark is appropriate regarding the data requirements of Rational Belief equilibria. The results of this paper in general and of this section in particular depend upon the assumption that the stationary probability m is known to all agents. This, in turn, is based on the idealization that agents have a very large amount of past data, unlimited capacity for storage and information processing and perfect analytical skills needed to study the data. This is clearly too strong and in reality the stationary probability itself may not be common knowledge and hence the agents may exhibit disagreement regarding the nature of the long run, stationary, forecast function. Such an eventuality only increases the diversity among agents and strengthens the main argument of this paper. We briefly sketch how to handle this case. The rationality condition requiring beliefs to be compatible with the data only enlarges the allowable set $B(Q)$. Each agent forms a belief which consists of a stationary component which is computed from the data and a non-stationary component which is stable but orthogonal with the stationary component. Different agents may adopt different stationary components of belief. The formal definition of Rational Belief Equilibrium remains essentially the same but permits the stationary components of beliefs to be different across agents. The set of equilibria is only enlarged but certain convergence properties of such a system needs to be studied.

The above argument shows that, conceptually speaking, *a Rational Belief Equilibrium does not require the agents to have this extraordinary amount of data and processing ability*. The concept remains viable even if the available data is not large. The idealization adopted in this paper aims to achieve analytical simplicity free of the detailed technicalities of approximations and convergence. This is compatible with our restriction to stable dynamical systems.

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