Algebraic K-theory and Chow groups

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I made these notes after I gave a talk of the same title at Stanford's Student Algebraic Geometry Seminar. In these notes, we explain localisation of abelian categories, define Quillen-style higher algebraic K-theory, and prove a connection between Algebraic K-theory and Chow groups.

1 Localisation of abelian categories

We begin by discussing the localisation (or quotient) of an abelian category by a Serre subcategory. The main result, that we later use in showing that certain categories are quotient categories, can be found in [2].

Let $\mathcal{A}$ be an abelian category with $\mathcal{B}$ a Serre subcategory, that is, a nonempty full subcategory such that for a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$, the object M is in $\mathcal{B}$ if an only only if M' and M'' are in $\mathcal{B}$, i.e. $\mathcal{B}$ is closed under taking subobjects, quotient objects, and extensions in $\mathcal{A}$. We want to be able to define a *quotient category* $\mathcal{A}/\mathcal{B}$ that is abelian, together with a *localising functor* $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ that is exact, such that for any abelian category $\mathcal{C}$ and exact functor $Q : \mathcal{A} \to \mathcal{C}$ such that $F(B)$ is a zero object of $\mathcal{C}$ for every object B of $\mathcal{B}$, there exists a unique functor $\overline{F} : \mathcal{A}/\mathcal{B} \to \mathcal{C}$ such that $G = \overline{F} \circ Q$.

Lemma 1.1. *Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$, and let S be the class of morphisms in $\mathcal{A}$ whose kernel and cokernel are both in $\mathcal{B}$. Then S is a multiplicative system.*

Proof. All identity maps have zero kernel and cokernel, and so belong to S. Since $\mathcal{B}$ is a Serre subcategory, if $A' \to A \to A''$ is an exact sequence in $\mathcal{A}$ with A', A'' in $\mathcal{B}$, then A is also in $\mathcal{B}$. Suppose that f, g are maps in $\mathcal{B}$ such that gf exists. We have the exact sequences

$$0 \to \ker f \to \ker gf \to \ker g,$$
$$\operatorname{coker} f \to \operatorname{coker} gf \to \operatorname{coker} g \to 0,$$

so that $\ker gf$ and $\operatorname{coker} gf$ are in $\mathcal{B}$. Hence gf is in S, so S is a wide subcategory of $\mathcal{A}$.

Next, suppose that we have maps $g : A \to B$, $t : A \to C$ in $\mathcal{A}$ with $t \in S$. Since $\mathcal{A}$ is abelian, we have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{f} & B \amalg_A C \end{array}$$

with the map $\operatorname{coker} t \to \operatorname{coker} s$ being an isomorphism and $\ker t \to \ker s$ being surjective. Hence s has both kernel and cokernel in $\mathcal{B}$, and so is in S. Hence the left Ore condition is satisfied. A dual argument using pullbacks gives us the right Ore condition.

Finally, let $f, g : A \to B$ be maps in $\mathcal{A}$, with a map $s : X \to A$ in S such that $fs = gs$. This means that $(f - g)s = 0$, so we can take the induced map $\overline{f - g} : A/\operatorname{im} s \to B$. Hence $\operatorname{im}(f - g)$ is a quotient of the cokernel of s, and so is in $\mathcal{B}$. Define the quotient map $t : B \to B/\operatorname{im}(f - g)$. Its kernel is $\operatorname{im}(f - g)$ and its cokernel is zero, which are both in $\mathcal{B}$, so t is in S. By definition, $t(f - g) = 0$, so $tf = tg$ and we have left cancellability. A dual argument gives us right cancellability and so S is a multiplicative system. □

Now that we know that S is a multiplicative system, we can form the two-sided localisation $Q : \mathcal{A} \to S^{-1}\mathcal{A}$, whose objects are those of $\mathcal{A}$ and whose morphism $A \to A'$ are diagrams of the form

$$A \xleftarrow{s:\ker s, \operatorname{coker} s \in \mathcal{B}} A'' \xrightarrow{f} A'$$

which we write as fs^{-1}. By definition, any functor $F : \mathcal{A} \to \mathcal{C}$ that inverts all maps in S factors uniquely through this localisation. It remains to prove that $S^{-1}\mathcal{A}$ is an abelian category, that the localising functor Q is exact, and that an exact functor $F : \mathcal{A} \to \mathcal{C}$ inverts all maps in S if and only if it sends objects of $\mathcal{B}$ to zero objects in $\mathcal{C}$.

Lemma 1.2. *The category $S^{-1}\mathcal{A}$ is abelian.*

Proof. First, we note that $S^{-1}\mathcal{A}$ inherits additivity from $\mathcal{A}$. Every morphism in $S^{-1}\mathcal{A}$ is of the form fs^{-1} for morphisms s, f in $\mathcal{A}$ with $s \in S$. Since s is an isomorphism, in the quotient, fs^{-1} has a cokernel in $S^{-1}\mathcal{A}$ if and only if f does. Since S is a left multiplicative system, the localisation functor Q commutes with all finite colimits. In particular, we have $\operatorname{coker} Qf = Q(\operatorname{coker} f)$. Hence f, and so our original map fs^{-1}, admits a cokernel in $S^{-1}\mathcal{A}$. Dually, since S is a right multiplicative system, Q commutes with all finite limits, which gives us kernels in the same way. Finally, we need to show that the natural map $\operatorname{coim} fs^{-1} \to \operatorname{im} fs^{-1}$ is an isomorphism. The map s^{-1} is an isomorphism and so has trivial kernel and cokernel, so it suffices to prove this for the map f. This is true for f because Q commutes with both kernels and cokernels, so this property is inherited from f, since $\mathcal{A}$ is abelian. Hence $S^{-1}\mathcal{A}$ is abelian. □

Lemma 1.3. *The functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ is exact.*

Proof. As above, since S is left multiplicative, Q commutes with cokernels, and since S is right multiplicative, Q commutes with kernels. Hence Q is exact. □

Lemma 1.4. *An exact functor $F : \mathcal{A} \to \mathcal{C}$ inverts all morphisms in S if and only if it sends objects of $\mathcal{B}$ to zero objects in $\mathcal{C}$.*

Proof. Suppose F inverts every morphism in S. This includes every map of the form $0 \to B$ and $B \to 0$ for every object B of $\mathcal{B}$, since their kernel and cokernels are 0 and B, which are objects of B. Hence $F(B)$ is isomorphic to a zero object in $\mathcal{C}$, and so is a zero object in $\mathcal{C}$.

Conversely, suppose that F sends every object in B to a zero object in $\mathcal{C}$. Since any morphism in S has both kernel and cokernel in $\mathcal{B}$, it follows that both the kernel and cokernel of its image under F will be zero objects and so this morphism will be invertible in $\mathcal{C}$. □

We now have the definition of $\mathcal{A}/\mathcal{B}$ that we need; this is the localisation $S^{-1}\mathcal{A}$. We need one more definition to state the main result nicely.

Definition 1.5. Let $F : \mathcal{A} \to \mathcal{C}$ be an exact functor between abelian categories. Then we define the *kernel* of F to be the full subcategory of $\mathcal{A}$ whose objects are mapped to zero objects by F.

Since F is exact, $\ker F$ will be a Serre subcategory of $\mathcal{A}$.

Lemma 1.6. *Let $F : \mathcal{A} \to \mathcal{C}$ be as in the definition above, and let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$ such that $\mathcal{B}$ is also a (Serre) subcategory of $\ker F$, so that F factors through $\overline{F} : \mathcal{A}/\mathcal{B} \to \mathcal{C}$. Then $\overline{F}$ is faithful if and only if $\ker F = \mathcal{B}$. In particular, F is faithful if and only if its kernel is trivial.*

Proof. Suppose that $\overline{F}$ is faithful. Let A be an object of $\mathcal{A}$ such that $F(A) = 0$ in $\mathcal{C}$. Since $\overline{F}$ is faithful, we have

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, A) \cong \mathrm{Hom}_{\mathcal{C}}(F(A), F(A)) = \mathrm{Hom}_{\mathcal{C}}(0, 0) = 0.$$

In particular, the identity morphism of A in $\mathcal{A}/\mathcal{B}$ is the zero map, hence A is a zero object in $\mathcal{A}/\mathcal{B}$ and so is an object of $\mathcal{B}$. Hence $\ker F$ is a Serre subcategory of $\mathcal{B}$, so $\ker F = \mathcal{B}$.

Conversely, suppose that $\ker F = \mathcal{B}$, so that $\ker \overline{F}$ is trivial. Let $f : A \to A'$ be a morphism in $\mathcal{A}/\mathcal{B}$ mapped to a zero morphism in $\mathcal{C}$. This means the image of $\overline{F}f$ is zero, so we have $\overline{F}(\mathrm{im}\, f) = \mathrm{im}\, \overline{F}f = 0$. Since $\ker \overline{F}$ is trivial, this means that $\mathrm{im}\, f = 0$. Hence $f = 0$, so $\overline{F}$ is faithful. □

This allows us to make the following statement, which is Theorem 5.11 in [2].

Theorem 1.7. *Let $F : \mathcal{A} \to \mathcal{C}$ be an exact covariant functor, and let $\mathcal{B} = \ker F$. We know that F factors uniquely through $\mathcal{A}/\mathcal{B}$, so we have a diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{Q} & \mathcal{A}/\mathcal{B} \\ & \searrow^{F} & \downarrow{\scriptstyle \overline{F}} \\ & & \mathcal{C} \end{array}$$

where $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is the localising functor. Suppose that the following conditions hold.

(i) For every object C of $\mathcal{C}$, there exists an object A of $\mathcal{A}$ such that $F(A) \cong C$.

(ii) For every morphism $f : F(A) \to F(A')$ in $\mathcal{C}$, there exists an object A'' and maps $h : A'' \to A$, $g : A'' \to A'$ in $\mathcal{A}$, such that we have the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} & F(A'') & \\ {\scriptstyle Fh,\ \cong}\swarrow & & \searrow{\scriptstyle Fg} \\ F(A) & \xrightarrow{\quad f \quad} & F(A') \end{array}$$

with Fh an isomorphism.

Then $\overline{F} : \mathcal{A}/\mathcal{B} \to \mathcal{C}$ is an equivalence of categories.

Proof. Condition (i) states that F is essentially surjective, and therefore so is $\overline{F}$. Since $\ker F = \ker Q = \mathcal{B}$, by Lemma 1.6, the functor $\overline{F}$ is faithful. From the construction of the quotient category, we know that a morphism $A \to A'$ in $\mathcal{A}/\mathcal{B}$ is a diagram of the form

$$\begin{array}{ccc} & A'' & \\ {\scriptstyle h:\ker h, \operatorname{coker} h \in \mathcal{B}}\swarrow & & \searrow{\scriptstyle g} \\ A & & A' \end{array}$$

We also know that Fh is an isomorphism, since h is in the multiplicative system corresponding to $\mathcal{B} = \ker F$. Hence condition (ii) states given a morphism in $\mathcal{C}$ and preimages of its source and target, the morphism is the image of a morphism between the preimages in $\mathcal{A}/\mathcal{B}$, so that the functor $\overline{F}$ is full and therefore is an equivalence of categories. □

2 Quillen K-theory

Here we outline Quillen's construction of the K-theory of an exact category, as well as stating the main theorems we will be using. For the full treatment, see Quillen's original paper in [1].

2.1 Definitions and theorems

We start by defining the notion of an exact category.

Definition 2.1. An *exact category* is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is a full additive subcategory of a small abelian category $\mathcal{A}$ that is closed under extensions, i.e. if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$ with A, C in $\mathcal{C}$, then B is also in $\mathcal{C}$; and $\mathcal{E}$ is the class of all sequences $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ that are short exact sequences in $\mathcal{A}$. We call these the short exact sequences of $\mathcal{C}$.

As for abelian categories, an *exact functor* between exact categories is one that preserves short exact sequences.

Remark. We can define an exact category $(\mathcal{C}, \mathcal{E})$ axiomatically, where $\mathcal{C}$ is an additive category and $\mathcal{E}$ is a class of sequences $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ that we call short exact sequences, modeled on the above definition. However, there is a theorem, the Gabriel-Quillen embedding theorem, that states that every such exact category can be embedded as a full additive subcategory of an abelian category $\mathcal{A}$, where the sequences in $\mathcal{E}$ are precisely those that are short exact sequences in $\mathcal{A}$. This means that the two definitions are equivalent.

The exact categories we are most interested in involve modules over rings or schemes. Let R be a commutative noetherian ring. The category of finitely generated R-modules is abelian; we denote this by $\mathcal{M}(R)$. The category of finitely generated projective modules forms an exact subcategory of $\mathcal{M}(R)$, which we denote $\mathcal{P}(R)$. Similarly, for a noetherian scheme X, we have the abelian category $\mathcal{M}(X)$ of coherent sheaves on X and the exact subcategory $\mathcal{P}(X)$ of locally free finite-rank sheaves on X.

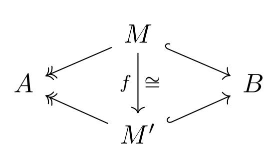
We now turn to the task of defining the Quillen K-theory of an exact category $(\mathcal{C}, \mathcal{E})$. We begin with Quillen's Q-construction.

Definition 2.2. Let $(\mathcal{C}, \mathcal{E})$ be an exact category. We define a new category $Q\mathcal{C}$ whose objects are those of $\mathcal{C}$ and whose morphisms $A \to B$ are equivalence classes diagrams of the form

$$A \twoheadleftarrow M \hookrightarrow B$$

where the map $M \twoheadrightarrow A$ is the second arrow in some short exact sequence in $\mathcal{E}$, and similarly $M \hookrightarrow B$ is the first arrow in some short exact sequence in $\mathcal{E}$.

Two diagrams $A \twoheadleftarrow M \hookrightarrow B$, $A \twoheadleftarrow M' \hookrightarrow B$ are equivalent if there exists an isomorphism $f : M \to M'$ in $\mathcal{C}$ such that the following diagram commutes:



Note that since an exact category need not be abelian, not every monomorphism in $\mathcal{C}$ is necessarily the first morphism in a short exact sequence, and similarly not every epimorphism is the second morphism in a short exact sequence. We call a monomorphism that appears as the first map in a short exact sequence an *admissible* monomorphism, and define admissible epimorphisms in a similar way.

We can now use this category to define the Quillen K-theory of a small exact category.

Definition 2.3. Let $(\mathcal{C}, \mathcal{E})$ be a small exact category, with the category $Q\mathcal{C}$ constructed as above. Define $BQ\mathcal{C}$ to be the geometric realisation of the nerve of $Q\mathcal{C}$ (also called the *classifying space* of $Q\mathcal{C}$); we call this the *K-theory space* of $(\mathcal{C}, \mathcal{E})$ and denote it by $K(\mathcal{C}, \mathcal{E})$. We then define the K-theory groups by

$$K_n(\mathcal{C}, \mathcal{E}) := \pi_{n+1}\left(K(\mathcal{C}, \mathcal{E}), 0\right).$$

In cases where the class $\mathcal{E}$ of short exact sequences is obvious, we simply write $K\mathcal{C}$, $K_n\mathcal{C}$.

For a noetherian ring R, the groups $K_0\mathcal{P}(R)$, $K_1\mathcal{P}(R)$, and $K_2\mathcal{P}(R)$ are isomorphic to the classical algebraic K-theory groups $K_0(R)$, $K_1(R)$, and $K_2(R)$. For this reason, we use the notation $K(R)$ for $K\mathcal{P}(R)$ and $K_n(R)$ for $K_n\mathcal{P}(R)$. Similarly, we define $K(X) = K\mathcal{P}(X)$ and $K_n(X) = K_n\mathcal{P}(X)$. For the abelian categories of finitely generated modules over a noetherian ring R, or coherent sheaves on a noetherian scheme X, we call the K-theory *G-theory*, with the notation $G(R)$, $G(X)$ for the K-theory spaces, and $G_n(R)$, $G_n(X)$ for the K-theory groups. Pullback of locally free sheaves makes K-theory contraviariant for morphisms of schemes, and G-theory contravariant for flat morphisms.

Quillen proved the following theorems for the K-theory that he defined:

Theorem 2.4 (Resolution)**.** *Let $\mathcal{M}$ be an exact category, with $\mathcal{P}$ a full subcategory, closed under taking extensions and kernels of admissible surjections in $\mathcal{M}$, such that every object in $\mathcal{M}$ admits a finite resolution by objects in $\mathcal{P}$. Explicitly, for every object M of $\mathcal{M}$, there exist objects $P_0, P_1, \ldots, P_n$ of $\mathcal{P}$ and an exact sequence*

$$P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Then the inclusion functor $\mathcal{P} \to \mathcal{M}$ induces a homotopy equivalence $K\mathcal{P} \simeq K\mathcal{M}$ between K-theory spaces and hence isomorphisms $K_n\mathcal{P} \cong K_n\mathcal{M}$ between K-theory groups.

As a corollary, for a regular noetherian ring R, we have $K_n(R) \cong G_n(R)$, and for a regular separated noetherian scheme X, we have $K_n(X) \cong G_n(X)$.

Theorem 2.5 (Devissage)**.** *Let $\mathcal{A}$ be an abelian category, and $\mathcal{B}$ a full abelian subcategory closed under taking subobjects, quotient objects, and finite products in $\mathcal{A}$ (so the inclusion functor is exact). Suppose that every object A of $\mathcal{A}$ has a finite filtration $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ such that every subquotient A_{i+1}/A_i is isomorphic to an object of $\mathcal{B}$. Then the inclusion functor $\mathcal{B} \to \mathcal{A}$ induces a homotopy equivalence $K\mathcal{B} \simeq K\mathcal{A}$ between K-theory spaces and hence isomorphisms $K_n\mathcal{B} \cong K_n\mathcal{A}$ between K-theory groups.*

Theorem 2.6 (Resolution)**.** *Let $\mathcal{A}$ be an abelian category, with $\mathcal{B}$ a Serre subcategory, so there exists a quotient category $\mathcal{A}/\mathcal{B}$ as in Section 1. Then the sequence of exact functors $\mathcal{B} \xrightarrow{i} \mathcal{A} \xrightarrow{loc} \mathcal{A}/\mathcal{B}$ induces a homotopy fibration $K\mathcal{B} \to K\mathcal{A} \to K(\mathcal{A}/\mathcal{B})$, and hence a long exact sequence in K-theory groups:*

$$\cdots \to K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_n\mathcal{B} \xrightarrow{i} K_n\mathcal{A} \xrightarrow{loc} K_n(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_{n-1}\mathcal{A} \to \dots$$
$$\to K_0\mathcal{B} \xrightarrow{i} K_0\mathcal{A} \xrightarrow{loc} K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

2.2 K_0 of an exact category

Without considering Quillen's Q-construction, there is a natural way to define K_0 of an exact category, following Grothendieck. Let $(\mathcal{C}, \mathcal{E})$ be an exact category. We define $K_0(\mathcal{C}, \mathcal{E})$ to be the group generated by isomorphism classes $[C]$ of objects in $\mathcal{C}$, with the relation $[C] = [C'] + [C'']$ for every short exact sequence $C' \hookrightarrow C \twoheadrightarrow C''$ in the class $\mathcal{E}$. In particular, if we take $\mathcal{C}$ to be the exact category of finitely generated projective modules $\mathcal{P}(R)$ over a noetherian ring R; this is the same as the classical definition of $K_0(R)$.

As already stated, the K-theory groups defined above coincide with the classical K-theory groups $K_0(R)$, $K_1(R)$, and $K_2(R)$. We now prove this for K_0. First, we note that for an admissible monic $C' \hookrightarrow C$ in $\mathcal{C}$, we can use the isomorphism $C' \to C'$ to get the morphism $C' \twoheadleftarrow C' \hookrightarrow C$ from C' to C in $Q\mathcal{C}$, which we denote simply by $C' \hookrightarrow C$. Similarly, for an admissible epic $C \hookrightarrow C''$, we can use the isomorphism $C'' \to C''$ to get the morphism $C'' \twoheadleftarrow C \hookrightarrow C$ from C'' to C in $Q\mathcal{C}$, which we denote by $C'' \hookleftarrow C$. In particular, for an object C, we have the maps $0 \hookrightarrow C$ and $0 \twoheadleftarrow C$ in $Q\mathcal{C}$. This gives us a loop $0 \twoheadleftarrow C \hookleftarrow 0$ in the geometric realisation $K\mathcal{C}$ of the nerve of $Q\mathcal{C}$ for each object C, which we denote $[C]$. We claim that these loops generate the fundamental group, subject to the relation $[C] = [C'][C'']$ for every short exact sequence $C' \hookrightarrow C \twoheadrightarrow C''$.

To get the generators and relations that we want in the fundamental group , we define the graph whose vertices are objects in $Q\mathcal{C}$ (which are just objects in $\mathcal{C}$) and whose edges are the maps $0 \hookrightarrow A$ for each object. This is a maximal tree in the (small, connected) category $Q\mathcal{C}$. Van Kampen's theorem then tells us that the fundamental group is generated by morphisms $A \to B$ in $Q\mathcal{C}$ (corresponding to loops of the form $0 \hookrightarrow A \to B \hookleftarrow 0$), with operation given by composition of maps, subject to the relations that the class of every identity morphism, and every $0 \hookrightarrow A$, is trivial.

Since every $[0 \hookrightarrow A]$ corresponds to the trivial element in the fundamental group, so does every $[A \to B]$ in $Q\mathcal{C}$, since we have $[0 \hookrightarrow A \hookrightarrow B] = [0 \hookrightarrow A][A \hookrightarrow B]$, with $[0 \hookrightarrow A \hookrightarrow B] = [0 \hookrightarrow B]$ and $[0 \hookrightarrow A]$ trivial. Hence given a morphism $A \to B = A \twoheadleftarrow C \hookrightarrow B$ in QC, the class $[A \twoheadleftarrow C \hookrightarrow B]$ is equal to $[A \twoheadleftarrow C]$, so the fundamental group is in fact generated by oppositely-oriented admissible epics in $Q\mathcal{C}$. Similarly, looking at the composition $[0 \twoheadleftarrow A \twoheadleftarrow B]$, we have $[0 \twoheadleftarrow B] = [0 \twoheadleftarrow A \twoheadleftarrow B] = [0 \twoheadleftarrow A][A \twoheadleftarrow B]$, giving $[A \twoheadleftarrow B] = [0 \twoheadleftarrow A]^{-1}[0 \twoheadleftarrow B]$, so oppositely-oriented admissible epics in $Q\mathcal{C}$, and hence the corresponding classes in the fundamental group, are generated by the maps $[0 \twoheadleftarrow A]$ for the objects A in $Q\mathcal{C}$ (and therefore $\mathcal{C}$). In particular, if A is isomorphic to B, the isomorphism is an admissible monic as well as an admissible epic, so we have $[0 \twoheadleftarrow A]^{-1}[0 \twoheadleftarrow B] = [A \twoheadleftarrow B] = [A \hookrightarrow B]$, which is trivial, so $[0 \twoheadleftarrow A] = [0 \twoheadleftarrow B]$. Hence we can think of the fundamental group of KC as being generated by isomorphism classes of objects of C. In addition, we have $[0 \twoheadleftarrow A \hookleftarrow 0] = [0 \twoheadleftarrow A][0 \hookrightarrow A]^{-1} = [0 \twoheadleftarrow A]$, so the generators correspond precisely to the loops we defined earlier and called $[A]$.

Now we turn our attention to the relations in the fundamental group. Let $C' \hookrightarrow C \twoheadrightarrow C''$ be a short exact sequence in the original exact category $\mathcal{C}$. Composing the paths corresponding to $0 \hookrightarrow C'' = 0 \twoheadleftarrow 0 \hookrightarrow C''$ and $C'' \twoheadleftarrow C = C'' \twoheadleftarrow C \hookrightarrow C$, we obtain the map $0 \twoheadleftarrow 0 \times_{C''} C \hookrightarrow C = 0 \twoheadleftarrow C' \hookrightarrow C$. So we have $[C'' \twoheadleftarrow C] = [0 \hookrightarrow C''][C'' \twoheadleftarrow C] = [0 \hookrightarrow C'' \twoheadleftarrow C] = [0 \twoheadleftarrow C' \hookrightarrow C] = [0 \twoheadleftarrow C'][C' \hookrightarrow C] = [0 \twoheadleftarrow C']$. Hence $[0 \twoheadleftarrow C] = [0 \twoheadleftarrow C''][C'' \twoheadleftarrow C] = [0 \twoheadleftarrow C''][0 \twoheadleftarrow C']$. This is the relation $[C] = [C''][C']$ for short exact sequences that we require. In particular, for objects A and B, we have $[A][B] = [A \oplus B] = [B \oplus A] = [B][A]$.

As an important example, for a field k, we have $K_0(k) \cong \mathbb{Z}$, where the image of a vector space is given by its dimension.

2.3 K_1 and the boundary map of the localisation sequences

We define a particular type of element in K_1. Let $\mathcal{C}$ be an exact category, and suppose we have an automorphism f of an object C in this category. We then have the commutative diagram

$$\begin{array}{ccccc} 0 & \hookrightarrow & C & \twoheadrightarrow & 0 \\ \| & & \downarrow{\scriptstyle f} & & \| \\ 0 & \hookrightarrow & C & \twoheadrightarrow & 0 \end{array}$$

This defines a continuous map $[0,1]^2 \to BQ\mathcal{C}$. By identifying the top and bottom edges of the diagram, and observing that the left and right edges map to the basepoint, this gives us a continuous map $S^2 \to BQ\mathcal{C}$ and hence an element of $\pi_2 K\mathcal{C} = K_1(\mathcal{C})$, which we denote $[f]$.

For such an element of K_1, we can actually compute its image under the boundary map of the localisation sequence explicitly. Suppose we have an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{B}$, with an automorphism $\alpha : A \to A$ in $\mathcal{A}/\mathcal{B}$; recall that this corresponds to a map in $\mathcal{A}$ whose kernal and cokernel are both in $\mathcal{B}$.

Proposition 2.7. *For the boundary map* $\partial : K_1(\mathcal{A}/\mathcal{B}) \to K_0(\mathcal{B})$*, we have* $\partial([\alpha]) = [\operatorname{coker} \alpha] - [\ker \alpha]$.

Proof. As above, we have the diagram

$$\begin{array}{ccccc} 0 & \hookrightarrow & A & \twoheadrightarrow & 0 \\ \| & & \downarrow{\scriptstyle \alpha} & & \| \\ 0 & \hookrightarrow & A & \twoheadrightarrow & 0 \end{array}$$

in $\mathcal{A}/\mathcal{B}$ for the automorphism α, which gives us the corresponding element in $\pi_2 K(\mathcal{A}/\mathcal{B}) = K_1(\mathcal{A}/\mathcal{B})$. Lifting this diagram to the category $\mathcal{A}$ gives us the diagram

$$\begin{array}{ccccc} \ker\alpha & \hookrightarrow & A & \twoheadrightarrow & 0 \\ \uparrow & & \downarrow{\scriptstyle \alpha} & & \uparrow \\ 0 & \hookrightarrow & A & \twoheadrightarrow & \operatorname{coker}\alpha. \end{array}$$

Reading off this diagram, we have the corresponding map $[0,1]^2 \to K\mathcal{A}$, and we can also see that the boundary map $\partial\left([0,1]^2\right) \to K\mathcal{B}$ is just $0 \hookrightarrow \operatorname{coker}\alpha \twoheadrightarrow 0$ followed by the reverse of $0 \hookrightarrow \ker\alpha \twoheadrightarrow 0$. Hence we have $\partial([\alpha]) = [\operatorname{coker}\alpha] - [\ker\alpha]$. ☐

3 The coniveau spectral sequence

In this section, we derive a spectral sequence for the G-theory of a noetherian scheme that we will eventually relate to its Chow groups. Let X be a noetherian scheme. Let $\mathcal{M}^i(X)$ be the Serre subcategory of $\mathcal{M}(X)$ of coherent sheaves whose supports have codimension at least i; we will generally suppress the X in this notation. We then have a filtration

$$\mathcal{M}(X) = \mathcal{M}^0 \supset \mathcal{M}^1 \supset \mathcal{M}^2 \supset \dots$$

which terminates if X is of finite dimension; this filtration is called the Gersten filtration, or, more descriptively, the coniveau filtration. By the localisation theorem, we have homotopy fibrations of the form $K\mathcal{M}^{i+1} \to K\mathcal{M}^i \to K\left(\mathcal{M}^i/\mathcal{M}^{i+1}\right)$ for every i, so our next task is to find a simpler form for the quotient category $\mathcal{M}^i/\mathcal{M}^{i+1}$.

Proposition 3.1. *Let X^i be the set of points in X of codimenion i. Let $\mathrm{Modfl}(R)$ be the category of finite length modules over a ring R. The quotient category $\mathcal{M}^i/\mathcal{M}^{i+1}$ is equivalent as a category to the category*

$$\coprod_{x \in X^i} \mathrm{Modfl}(\mathcal{O}_{X,x}).$$

Proof. We define a functor $F : \mathcal{M}^i \to \coprod_{x\in X^i} \mathrm{Modfl}(\mathcal{O}_{X,x})$ and show that the hypotheses of Theorem 1.7 hold. First, we note that we have a natural functor $\mathcal{M}^i \to \coprod_{x\in X^i} \mathcal{M}(\mathcal{O}_{X,x})$ given by localising at each $x \in X^i$. A sheaf in $\mathcal{M}^{i+1}$ will localise trivially at each point of codimension i; conversely, a sheaf that localises trivially at each $x \in X^i$ must have codimension strictly greater that i. Hence the kernel of this functor is $\mathcal{M}^{i+1}$. Moreover, for a sheaf $\mathcal{F}$ in $\mathcal{M}^i$, if its localisation $\mathcal{F}_x$ at $x \in X^i$ is nonzero, its support as a module over $\mathcal{O}_{X,x}$ must consist only of the maximal ideal, since a smaller prime ideal in its support would correspond to a support of lower codimension in the original scheme X. We know that every finitely generated module over a noetherian ring has a finite filtration whose subquotients are isomorphic to $R/\mathfrak{p}$ for some associated prime ideal $\mathfrak{p}$, and associated primes are also primes in the support. Since the only prime in the support of $\mathcal{F}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$, there exists a finite filtration whose subquotients are isomorphic to $k(x)$, so that $\mathcal{F}_x$ is a module of finite length. Hence we have a functor $F : \mathcal{M}^i \to \coprod_{x \in X^i} \mathrm{Modfl}(\mathcal{O}_{X,x})$ with kernel $\mathcal{M}^{i+1}$.

Now we can check the two conditions in Theorem 1.7. Let $\mathcal{F}_x$ be a module over $\mathcal{O}_{X,x}$ for some $x \in X^i$, and let Z_x be the closure of x in X. We can extend $\mathcal{F}_x$ to an open neighbourhood of x in Z_x by clearing denominators, and since X is noetherian, we can extend this to a coherent sheaf on Z_x, which we can then extend by zero outside Z_x to a coherent sheaf on X. Now suppose that we have an object $(\mathcal{F}_x)_{x \in X^i}$ in $\coprod_{x \in X^i} \mathrm{Modfl}(\mathcal{O}_{X,x})$. For each of the (finitely many) $x \in X^i$ such that $\mathcal{F}_x$ is nonzero, we can produce a coherent sheaf supported on Z_x whose stalk at x is $\mathcal{F}_x$, as above, and take the direct sum. This will localise to $\mathcal{F}_x$ at every $x \in X^i$, since if x, x' are distinct points in X^i, we have $x' \notin Z_x$ and $x \notin Z_{x'}$. This means that if we are given an object in $\coprod_{x \in X^i} \mathrm{Modfl}(\mathcal{O}_{X,x})$, we can find an object in $\mathcal{M}^i$ whose image under F is isomorphic to the chosen object, so condition (i) holds.

Now suppose that we have two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, with a map $f : (\mathcal{F}_x) \to (\mathcal{G}_x)$ in $\coprod_{x \in X^i} \mathrm{Modfl}(\mathcal{O}_{X,x})$. As above, for each $x \in X^i$, we can clear denominators to extend both $\mathcal{F}_x$ and the map f to an open neighbourhood of x in Z_x; and again, since X is noetherian, we can extend these both to Z_x, and then by zero outside of Z_x to the whole of X. Taking the direct sum over all $x \in X^i$ then gives us a coherent sheaf $\mathcal{F}'$ and a map $g : \mathcal{F}' \to \mathcal{G}$ such that $Fg = f$. Finally, we take the identity map $(\mathcal{F}_x) \to (\mathcal{F}_x)$ and extend it to a map $\mathcal{F}' \to \mathcal{F}$ on an open neighbourhood of each $x \in X^i$. Sinxe X is noetherian, there exists a coherent subsheaf $\mathcal{F}''$ of $\mathcal{F}'$, agreeing with $\mathcal{F}'$ on this open set, to which we can extend this map, which we call h. Note that $\mathcal{F}''_x = \mathcal{F}_x$, and $Fg = f$ still holds for this subsheaf. This means that we have the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{F}''_x & & \\ & \swarrow^{Fh=\mathrm{id}} & & \searrow^{Fg=f} & \\ \mathcal{F}_x & & \xrightarrow{\quad f \quad} & & \mathcal{G}_x \end{array}$$

Hence condition (ii) is satisfied and

$$\overline{F} : \mathcal{M}^i/\mathcal{M}^{i+1} \to F : \mathcal{M}^i \to \coprod_{x \in X^i} \mathrm{Modfl}(\mathcal{O}_{X,x})$$

is an equivalence of categories. □

We have an explicit description of the quotient category $\mathcal{M}^i/\mathcal{M}^{i+1}$, and we can use this to express its K-theory in terms of the K-theory of the residue fields of X.

Proposition 3.2. *We have isomorphisms*

$$K_n\left(\coprod_{x\in X^i}\operatorname{Modfl}(\mathcal{O}_{X,x})\right)\cong\bigoplus_{x\in X^i}K_nk(x)$$

for all n, where $k(x)$ is the residue field at x.

Proof. Consider the inclusion of $\mathcal{P}(k(x))$ into $\operatorname{Modfl}(\mathcal{O}_{X,x})$. Since modules in the latter category are of finite length, the devissage theorem tells us that this inclusion induces a homotopy equivalence of K-theory spaces. Since K-theory commutes with taking direct sums, we therefore have $K_n\left(\coprod_{x\in X^i}\operatorname{Modfl}(\mathcal{O}_{X,x})\right)\cong\bigoplus_{x\in X^i}K_nk(x)$ as required. □

We now have long exact sequences of the form

$$\cdots\to K_n(\mathcal{M}^{i+1})\to K_n(\mathcal{M}^i)\to\bigoplus_{x\in X^i}K_nk(x)\to\dots$$

for all i. These fit into an exact couple and give us the cohomology spectral sequence that is the main result of this section.

Theorem 3.3 (The coniveau spectral sequence)**.** *Let X be a noetherian scheme of finite dimension, and let X^p be the set of points of codimension p in X. Then there is a cohomology spectral sequence $E_r^{p,q}$ with $p\geq 0$, $p+q\leq 0$,*

$$E_1^{p,q}(X)=\bigoplus_{x\in X^p}K_{-(p+q)}k(x)\implies G_{-n}(X).$$

This spectral sequences relates the G-theory of a noetherian scheme X (or ring R, in the same way) to the K-theory of its residue fields. Since flat morphisms preserve codimension, this is contravariant for flat morphisms. Note that if X is regular and separated, every coherent sheaf admits a finite resolution by locally free sheaves, so by the resolution theorem, this sequence converges to the K-theory of X. Finally, if $\dim X=\infty$, we can still define this spectral sequence, but it will now converge to the limit $\lim\limits_{\leftarrow}K_{-n}\left(\mathcal{M}(X)/\mathcal{M}^i\right)$.

4 K-theory and Chow groups

Finally, we are able to derive the connection between K-theory and Chow groups that is the main result of these notes. We will show that if X is equidimensional, the diagonal terms $E_2^{p,-p}$ of the E_2-page in the above spectral sequence are naturally isomorphic to the chow groups $CH^p(X)$.

To compute $E_2^{p,-p}$, we need to compute the homology at

$$E_1^{p-1,-p}\to E_1^{p,-p}\to E_1^{p+1,-p}.$$

Using the definition of $E_1^{p,q}$ and the calculations from classical K-theory, we have

$$E_1^{p+1,-p} = 0$$
$$E_1^{p,-p} = \bigoplus_{x \in X^p} K_0 k(x) \cong \bigoplus_{x \in X^p} \mathbb{Z}$$
$$E_1^{p-1,-p} = \bigoplus_{x \in X^{p-1}} K_1 k(x) \cong \bigoplus_{x \in X^{p-1}} k(x)^*$$

This means that $E_1^{p,-p}$ is isomorphic to the abelian group $Z^p(X)$ of codimension p algebraic cycles, and $E_1^{p-1,-p}$ is isomorphic to the group $R^{p-1}(X)$ of rational functions on codimension $p-1$ cycles.

We approach this by taking components of d_1. Fix arbitrary $y \in X^{p-1}$, with Y its closure, and let $x \in X^{p-1}$. Let the component of d_1 corresponding to these points be $(d_1)_{xy}$. The closed immersion $Y \to X$ will send $M^j(Y)$ to $M^{j+p-1}(X)$, so we get the diagram

$$\begin{array}{ccc} E_1^{p,-p}(X) & \xrightarrow{d_1} & E_1^{p-1,-p}(X) \\ \uparrow & & \uparrow \\ E_1^{0,-1}(Y) & \xrightarrow{(d_1)_{xy}} & E_1^{1,-1}(Y) \end{array}$$

This means that if $x \notin Y$, we have $(d_1)_{xy} = 0$. If x is of codimension 1 in Y, we then have the flat inclusion $\mathrm{Spec}(\mathcal{O}_{Y,x}) \to Y$, so we have the diagram

$$\begin{array}{ccc} E_1^{0,-1}(Y) & \xrightarrow{(d_1)_{xy}} & E_1^{1,-1}(Y) \\ \downarrow & & \downarrow \\ E_1^{0,-1}(\mathcal{O}_{Y,x}) & \longrightarrow & E_1^{1,-1}(\mathcal{O}_{Y,x}) \end{array}$$

We have now reduced the problem to computing the map

$$(d_1)_{yx} : E_1^{0,-1}(\mathcal{O}_{Y,x}) \to E_1^{1,-1}(\mathcal{O}_{Y,x}).$$

This map is just the final boundary map in the long exact sequence for the localisation of $\mathcal{M}(\mathcal{O}_{Y,x})$ by the subcategory of finitely generated torsion modules, which we denote $\mathcal{M}_{tor}(\mathcal{O}_{Y,x})$. By the devissage theorem, we have isomorphism $K_n(k(x)) \cong K_n(\mathcal{M}_{tor}(\mathcal{O}_{X,x}$, and the quotient category is the category of finite-dimensional vector spaces over the field of fractions. We now have only to prove that the last boundary map in such a localisation sequence corresponds to the order map.

Lemma 4.1. *Let R be a noetherian local domain of dimension 1 (so that all finitely generated torsion modules have finite length) with residue field k and fraction field F. Consider the boundary map $\partial : K_1 F \to K_0 k$ in the localisation sequence; this is a map $\partial : F^* \to \mathbb{Z}$. This map is the homomorphism* $\mathrm{ord} : F^* \to \mathbb{Z}$*, where* ord *is the homomorphism such that* $\mathrm{ord}(a)$ *is the length of R/aR for all $a \in R$.*

Proof. Let a be a nonzero element in R; this corresponds to a morphism of R-modules, and and isomorphism of F-vector spaces, given by multiplication by a. By Proposition 2.7, its image in $K_0k \cong \mathbb{Z}$ under ∂ is $[\operatorname{coker} a] - [\ker a]$. The cokernel of this map is R/aR, and since a is a nonzero element of R (and so F), its kernel is trivial. We want to show that $[R/aR]$ corresponds to its length in $\mathbb{Z} \cong K_0(k)$.

We have $K_0(k) \cong \mathbb{Z}$, where the isomorphism is given by dimension. We have the devissage isomorphism $K_n(k) \cong K_n(\mathcal{M}_{tor}(R)$; we can compute this explicitly for K_0. Let M be a module of length n, so we have a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

where $M_i/M_{i+1} \cong R/\mathfrak{m} = k$, where $\mathfrak{m}$ is the maximal ideal of R, so we have the short exact sequences

$$M_i \to M_{i+1} \to k.$$

This gives us the relation $[M_{i+1}] = [M_i]+[k]$ in $K_0(\mathcal{M}_{tor}(R)$, so that $[M] = n[k]$. Since $[k]$ has dimension 1 over k, this means that the image ∂a is the length of R/aR, so that $\partial = \operatorname{ord}$ as required. □

We have now proved the following result.

Theorem 4.2. *Let X be an equidimensional noetherian scheme, and let $E_1^{p,q}$ be the first page of its coniveau spectral sequence. Then we have*

$$E_2^{p,-p}(X) \cong CH^p(X).$$

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