Game Theory Refresher

Muriel Niederle *

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1. Definition of a Game

We start by first defining what a game is. A game consists of:

- A set of players (here for simplicity only 2 players, all generalized to N players).
- A set of possible strategies for each player;

  We denote a possible strategy for player $i = 1, 2$ as s_i, and the set of all possible strategies of player i as S_i.
- A payoff function that tells us the payoff each player receives as a function of the strategies of all players.

  We write payoffs directly as a function of the strategies. If player 1 uses strategy s_1 and player 2 s_2, then the payoff for each player i is $v_i(s_1, s_2)$.

  Payoffs should be interpreted as von Neumann-Morgenstern utilities, not as monetary outcomes. This is important, especially whenever there is uncertainty in the game.

Sometimes we write $v_i(s_i, s_{-i})$ to show that payoff for player i depends on his own strategy s_i and on his opponent's strategy $s_{-i} \in S_{-i}$.

We always assume that all players know the structure of the game, including the payoff of the opponent. This assumption is strong, and can be weakened, to games in which players have uncertainty about the type of the other players. Though here we assume that the structure is known.

*Department of Economics, Stanford University and NBER, http://www.stanford.edu/~niederle

We will distinguish between *normal-form* and *extensive form* games. In normal form games (the reason why they have this name will become clearer later on) the players have to decide simultaneously which strategy to choose. Therefore, timing is not important in this game, there is no first mover.

Sometimes we want to make timing more explicit, and acknowledge that one player moves after another. This will be the reason for modeling games in extensive form.

2. The Ultimatum game as a normal form game

Two players have to decide how to divide \$10. Player 1, the proposer, decides how much to pass on to player 2, the responder. Let x be the amount player 1 passes to player 2. Let us assume that player 1 has to choose $x \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Since player 1 can only divide the \$10, and neither destroy increase the amount of money, player 1 gets to keep $10 - x$.

Player 2, the responder has to decide whether to accept or reject the offer. If player 2 accepts the offer, the division is implemented, if he rejects the offer both he and player 1 receive 0.

The strategy of player 2 consists of a decision (accept, reject) for each possible division of the \$10, that is for each possible x he gets offered from payer 1.

For the payoff table below, we write the payoffs in \$. Note that this implies that either, the \$ amount equals the number of utils players receive from the joint actions, or that we indeed do not have a representation of the payoff matrix.

x	0	1	2	3	4	5	6	7	8	9	10
Accept	(10,0)	(9,1)	(8,2)	(7,3)	(6,4)	(5,5)	(6,4)	(3,7)	(2,8)	(1,9)	(0,10)
Reject	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

3. Solution Concepts for Normal Form Games

In this section we examine what will happen in equilibrium if we assume that both players are rational and choose their strategies to maximize their utility.

3.1. 2.1 Dominant strategies. A strategy s_i for player i is a weakly dominant strategy if for all $s_{-i} \in S_{-i}$ and all $\tilde{s}_i \in S_i$ we have $v_i(s_i, s_{-i}) \geq v_i(\tilde{s}_i, s_{-i})$.

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Note that the ultimatum game has a weakly dominant strategy for player 2. Accept always yields weakly higher payoffs than reject.

The strategy Accept is however not a strictly dominant strategy: If $s_1 = x = 0$, then $v_2(0,\text{Accept}) = v_i(0,\text{Reject})$.

When players have a strictly dominant strategy, we sometimes think they might play that strategy, it may be a good predictor for their behavior. Note, this may not necessarily be the case, see the Prisoners Dilemma Game.

3.2. Nash equilibrium. To predict the outcome of a game, Nash equilibrium is a concept that basically formalizes the idea that every player is doing the best possible given the behavior of the other player. That is, there is no room for unilateral deviation.

Strategies s_1 and s_2 form a Nash equilibrium, if

$$\begin{aligned} v_1(s_1, s_2) &\geq v_1(\tilde{s}_1, s_2) \text{ for all } \tilde{s}_1 \in S_1 \\ v_2(s_1, s_2) &\geq v_2(s_{1,}\tilde{s}_2) \text{ for all } \tilde{s}_2 \in S_2. \end{aligned}$$

Note that this definition basically assumes that player 1 knows what player 2 is going to do and the other way round.

Pure Strategy Nash equilibria:

Some games may not have a Nash equilibrium in pure strategies. Think for example of the game matching pennies: Player 1 and Player 2 each decide whether to say 0 or 1. If both players say the same number, player 1 receives a payoff of x utils, and if they say a different number, player 2 receives a payoff of x utils. It is easy to see, that there is no pure strategy Nash equilibrium. It may therefore be useful to allow players to randomize over possible strategies, and use a mixed strategy. A mixed strategy is simply a probability distribution over the player's pure strategies. Sometimes we will denote the set of all mixed strategies for some player i by Σ_i and a given mixed strategy by $\sigma_i \in \Sigma_i$. If there are only two pure strategies, a mixed strategy is just the probability to play the first pure strategy - it is just a number between zero and one.

If players play mixed strategies they evaluate their utility according to the von-Neumann Morgenstern criterion. If player one has n_1 pure strategies and player 2 has n_2 pure strategies there are generally $n_1 * n_2$ possible outcomes - i.e. possible states of the world. The probabilities of these states are determined by the mixed strategies. We can write a player i's payoff (utility function) as a function $u_i(\sigma_1, \sigma_2)$. A Nash equilibrium in mixed strategies is then simply a profile of mixed strategies (σ_1, σ_2) (in the cases below these will just be two probabilities) such that

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &\geq u_1(\tilde{\sigma}_1, \sigma_2) \text{ for all } \tilde{\sigma}_1 \in \Sigma_1 \\ u_2(\sigma_1, \sigma_2) &\geq u_2(\sigma_1, \tilde{\sigma}_2) \text{ for all } \tilde{\sigma}_2 \in \Sigma_2. \end{aligned}$$

Example: Pure Strategy Nash Equilibria in the Ultimatum Game:

Given the definition of the ultimatum game above, where x is the amount passed to player 2, and $x \in \{0, 1, .., 10\}$, and the earnings of player 1 are $10 - x$. Let us assume that dollar earnings equal utils. Then there is a Nash equilibrium where player 1 receives 10 and player 2 receives 0.

Strategy of player 1 : Propose $x = 0$.

Strategy of Player 2 : Accept every proposal.

Can Player 1 gain from deviating to some other strategy, given players 2's strategy? Player 1 already achieves her highest possible payoff, she certainly cannot gain from taking another action.

Can player 2 gain from deviating to some other strategy given player $1's$ strategy? Given player 1 offers $x = 0$, player 2 will get 0, independently of whether he accepts or rejects, so player 2 cannot gain from deviation.

Here's a Nash equilibrium where player 1 receives 9 and player 2 receives 1

Strategy of player 1 : Propose $x = 1$.

Strategy of Player 2 : Accept every proposal with $x > 0$, reject a proposal with $x = 0$.

Can Player 1 gain from deviating to some other strategy, given players 2's strategy? Player 1 cannot gain by offering $x > 1$, as then her payoff decreases. Suppose player 1 offers $x = 0$, then player 2 rejects, so, player 1 doesn't gain from that deviation either.

Can player 2 gain from deviating to some other strategy given player $1's$ strategy? Given player 1 offers $x = 1$, player 2 will get 0 if he rejects that offer. Player 2 cannot gain by changing any response other than to an offer of 1, since that is player 1's strategy.

Similarly, we can have a Nash equilibrium where player 1, receives 8, $7, 6, 5, 4, 3, 2, 1$ and player 2 receives the remaining amount of money. Let me show you the one where player 1 receives less than one.

Here is a Nash equilibrium where player 1 receives 0 and player 2 receives 10.

Strategy of player 1 : Propose $x = 10$.

Strategy of Player 2 : Accept every proposal with $x > 9$, reject all other proposals.

Can Player 1 gain from deviating to some other strategy, given players 2's strategy? Player 1 cannot gain by offering $x < 10$: suppose player 1 offers $x = 9$ then player 2 rejects, so, player 1 doesn't gain from that deviation either.

Can player 2 gain from deviating to some other strategy given player $1's$ strategy? Given player 1 offers $x = 10$, player 2 will get 0 if he rejects that offer. Player 2 cannot gain by changing any response other than to an offer of 1, since that is player 1's strategy.

A Nash equilibrium where both players get 0

Strategy of player 1 : Propose $x = 0$.

Strategy of Player 2 : Reject every proposal.

Given the strategy of player 2, it does not matter what proposal player 1 makes, she will receive 0 no matter what.

Given the strategy of player 1, offering 0 to player 2, player 2 cannot gain from deviating and accepting the offer of 1. Clearly, since player 1 makes no other offers, player 2 cannot gain from changing his strategy to any other proposal either.

4. Extensive Form Games

Now we will consider situations in which one player moves first, the other player observes what the first player did and then decides on which action to take.

To capture the sequential structure of the game, we will depict sequential games by using game trees. What is a strategy for a player in extensive form games? A strategy for a player who moves second will be a contingent plan: for all possible actions of the first player, the second player needs to specify his action.

4.1. The Ultimatum Game as an Extensive Form Game.

When looking at outcome if all part

In order to figure out how Nash-equilibria look like, we want to ask, what are the possible strategies in this game. Obviously player 1's strategies are $S_1 = \{0,1,2,3,4,5,6,7,8,9,10\}$. Naively one would think that Player 2's strategies are $S_2 = \{\text{Accept, Reject}\}$. However, this is

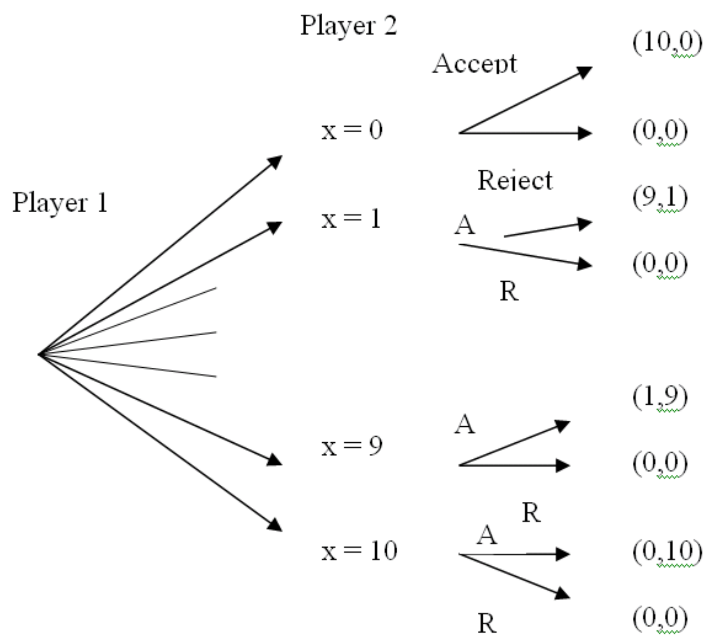

Figure 1: Game Tree of the Ultimatum Game

false. Player 2 knows what player 1 has done when it is his turn to move. So his actual strategy has to specify what he does in each possible situation. His strategies can differ depending on player 1's action.

We will see below why it is important to treat this issue carefully and why this formulation gives us some problems with the concept of Nash equilibrium.

When we think of the Nash equilibria of the ultimatum game in this extensive form game description, we see immediately what the "problem" of some of the Nash equilibria we found above are.

Take for instance the Nash equilibrium where the responder, player 2, receives 9, and player 1, the proposer receives 1. Intuitively, player 2 threatens to reject all other offers from player 1. Player 1 thinks that the threat is credible and therefore offers 9 to player 2. Note, however, that the threat of 2 to reject if 1 chooses to offer less than 9 (say only 4) is not credible. Once 1 has chosen to only offer 4 to player 2, player 2 will understand that he hurts himself by choosing to reject that offer and that he would do better by choosing not to reject it but rather accept it. Hence, this Nash equilibrium is not convincing.

In order to rule out these types of unconvincing Nash equilibria we require that in a sequential game an equilibrium has to be "subgame perfect".

Definition 1 (Subgame perfect equilibrium) A Nash equilibrium is subgame perfect, if the strategies of all players form a Nash equilibrium not only in the game as a whole, but also in every subgame of the game. That is, after every possible history of the game the strategies of the players have to be mutually best responses.

One way to solve for the subgame perfect equilibrium is by backward induction. We first ask, for player 2, what is the optimal strategy at each possible node. Then, given the strategies of player 2, we can ask about player 1's optimal strategy.

What are possible strategies of player 2 in the ultimatum game that satisfy that they are a best response to the strategy of player 1 at every possible node?

Consider offers of player 1 in which $x > 0$. What is the payoff maximizing strategy of player 2? If player 2 accepts, he receives x. If player 2 rejects, he receives 0. Since $x > 0$, the best response of player 2 to any offer $x > 0$ is to accept that offer. When $x = 0$, then player 2 receives 0, whether he accepts or rejects.

There are therefore two strategies in which player 2 plays a payoff maximizing strategy at every possible node:

Strategy 1: Player 2 accepts every offer $x \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Strategy 2 : Player 2 accepts any offer $x > 0$, and rejects an offer of $x = 0$.

Suppose player 2 plays strategy 1. What is the best response of player 1? If player 1 offers $x = 0$, then player 2 accepts, player 1 receives 10, her highest possible payoff. Hence one subgame perfect equilibrium is for player 1 to offer $x = 0$ and for player 2 to accept.

Suppose player 2 plays strategy 2. What is the best response of player 1? If player 1 offers $x = 0$, then player 2 rejects and player 1 receives 0. What is player 1 offers $x = 1$. Then player 2 accepts that proposal, player 1 receives 9, and player 2 receives 1. Player 1 has no strategy that gives her a payoff higher than 9, as player 2 rejects an offer of 10, hence this is a subgame perfect equilibrium.

Since there were only two possible strategies of player 2, that fulfill that player 2 plays a best response at every node, we found the two subgame perfect equilibria of the game.