School Choice: New open problems

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New Questions raised by school Choice

- How to do tie breaking?
- Tradeoffs between Pareto optimality, stability, strategy proofness—what are the 'costs' of each?
- Evaluating welfare from different points in time
- Restricted domains of preferences?

Matching with indifferences

- When we were mostly using matching models to think about labor markets, strict preferences didn't seem like too costly an assumption.
  - Strict preferences might be generic
- But that isn't the case with school choice
  - We already saw that one of the first NYC design decisions faced in 2003 was how to randomize to break ties.

New Theoretical Issues

- Erdil, Aytek and Haluk Ergin, What's the Matter with Tie-breaking? Improving Efficiency in School Choice , American Economic Review, 98(3), June 2008, 669-689.
- Abdulkadiroglu, Atila , Parag A. Pathak , and Alvin E. Roth, "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match ," American Economic Review, forthcoming.
- Featherstone, Clayton and Muriel Niederle, "Efficiency in School Choice Mechanisms under incomplete information: An Experimental Investigation", December 2008.

Other new issues we won't talk about in detail...

- Pathak, Parag and Tayfun Sönmez "Leveling the Playing Field: Sincere and Strategic Players in the Boston Mechanism" , American Economic Review, 98(4), 1636-52, 2008
- Ergin, Haluk and Tayfun Sonmez, Games of School Choice under the Boston Mechanism " , Journal of Public Economics , 90: 215-237, January 2006.
- Kesten, Onur "On Two Kinds of Manipulation for School Choice Problems" March, 2008.
- Kesten, Onur, " An Alternative Mechanism Design Approach to School Choice in the United States ," March, 2008.
- Abdulkadiroglu, Atila, Yeon-Koo Che, and Yosuke Yasuda, "Expanding 'Choice' in School Choice" November, 2008.
- Abdulkadiroglu, Atila, Yeon-Koo Che, and Yosuke Yasuda, "Resolving Conflicting Preferences in School Choice: The 'Boston' Mechanism Reconsidered" June 2009.
- Miralles, Antonio, "School choice: the case for the Boston Mechanism", working paper November 2008.

Matching with Indifference

- I: a finite set of students (individuals) with (strict) preferences P_i over school places.
- S: a finite set of schools with responsive weak preferences / priorities R_s over students (i.e. can include indifferences: P_s ($\succ_s$) is the asymmetric part of R_s).

As before:

- $q = (q_s)_{s \in S}$ a vector of quotas ($q_s \geq 1$, integer).

A matching is a correspondence $\mu : I \cup S \rightarrow I \cup S$ satisfying:
(i) For all $i \in I : \mu(i) \in S \cup \{i\}$
(ii) For all $s \in S : |\mu(s)| \leq q_s$, and $i \in \mu(s)$ implies $\mu(i) = s$.
We'll mostly concentrate on student welfare and student strategy, and regard R_s as fixed.

Matchings and student welfare

A matching μ is **individually rational** if it matches every $x \in I \cup S$ with agent(s) that is(are) acceptable for x.
A matching μ is **blocked** by (i, s) if $sP_i\mu(i)$, and either $[|\mu(s)| < q_s$ and $i \succ_s s]$ or $[i \succ_s i'$ for some i' $\in \mu(s)]$. μ is stable if μ is individually rational and not blocked by any student-school pair (i, s).
A matching μ **dominates** matching if $\mu(i)R_i(i)$ for all $i \in I$, and $\mu(i)P_i(i)$ for some $i \in I$. (Weak Pareto domination for students.)
A stable matching μ is **a student-optimal stable matching** if it is not dominated by any other stable matching.
"**A**" not "**the**": When school preferences aren't strict, there won't generally be a unique optimal stable match for each side, rather there will be a non-empty set of stable matches that are weakly Pareto optimal for agents on that side.

Example: Tie breaking does not always yield student-optimal stable matching:

Student Pref	School Pref
$s_2 P_{i_1} s_1 P_{i_1} s_3$	$i_1 \sim_{s_1} i_2 \sim_{s_1} i_3$
$s_1 P_{i_2} s_2 P_{i_2} s_3$	$i_2 \succ_{s_2} i_1 \succ_{s_2} i_3$
$s_1 P_{i_3} s_2 P_{i_3} s_3$	$i_3 \succ_{s_2} i_1 \succ_{s_2} i_2$

The stable matchings are

$$\mu_1 = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{pmatrix}, \mu_2 = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{pmatrix}, \mu_3 = \begin{pmatrix} i_1 & i_2 & i_3 \\ s_3 & s_2 & s_1 \end{pmatrix}$$

μ_1, μ_2 and μ_3 are produced by student proposing DA when s_1's indifference is broken as $i_1 \succ_{s_1} i_3 \succ_{s_1} i_2$, $i_2 \succ_{s_1} i_x \succ_{s_1} i_y$ and $i_3 \succ_{s_1} i_x \succ_{s_1} i_y$ respectively. But μ_2 dominates μ_1, hence *DA* need not produce a student-optimal stable match even if ties are broken the same way.

Weak Pareto optimality generalizes...
Proposition 1. If μ is a student-optimal stable matching, there is no individually rational matching ν (stable or not) such that $\nu(i) P_i \mu(i)$ for all $i \in I$.

(terminology: a student optimal stable matching is weakly Pareto optimal because it can't be strictly Pareto dominated, but the outcome of student proposing deferred acceptance algorithm might not be strongly Pareto optimal, i.e. might not be student optimal, because it can be weakly Pareto dominated)

Tie breaking

A tie-breaker is a bijection $r : I \longrightarrow N$, that breaks ties at school s by associating R_s with a strict preference relation P_s :

$$iP_sj \Leftrightarrow [(i \succ_s j) \textit{ or } (i \sim_s j \text{ and } r(i) < r(j))].$$

Basic Deferred Acceptance (Gale and Shapley 1962)

- Step 0: arbitrarily break all ties in preferences
- Step 1: Each student "proposes" to her first choice. Each school tentatively assigns its seats to its proposers one at a time in their priority order. Any remaining proposers are rejected.

  ...

- Step k: Each student who was rejected in the previous step proposes to her next choice if one remains. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time in priority order. Any remaining proposers are rejected.

The algorithm terminates when no student proposal is rejected, and each student is assigned her final tentative assignment.

Deferred acceptance algorithm with tie breaking: DA^{τ}

A single tie breaking rule uses the same tie-breaker $r_s = r$ at each school, while a multiple tie breaking rule may use a different tie breaker r_s at each school s.
For a particular set of tie breakers $\tau = (r_s)_{s \in S}$, let the mechanism DA^{τ} be the student-proposing deferred acceptance algorithm acting on the preferences (P_I, P_S), where P_s is obtained from R_s by breaking ties using r_s, for each school s.

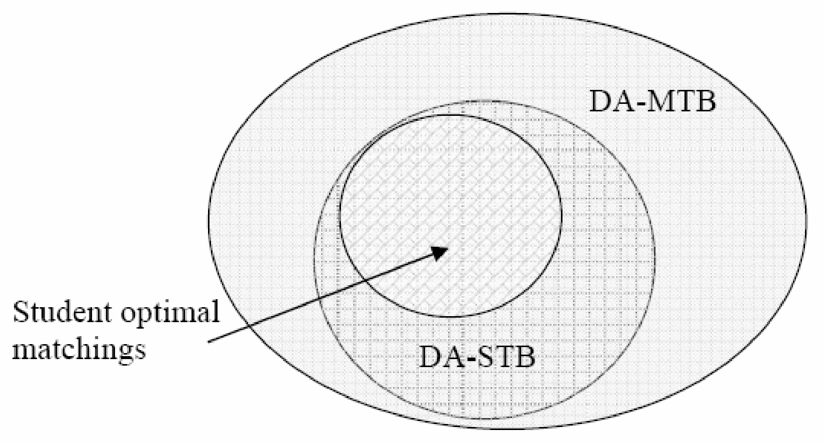
Single and Multiple tie breaking

The dominant strategy incentive compatibility of the student-proposing deferred acceptance mechanism for every student implies that DA^{τ} is strategy-proof for any τ.
But the outcome of DA^{τ} may not be a student optimal stable matching.
We already saw this is true even for single tie breaking.

Single versus multiple tie breaking (NYC Grade 8 applicants in 2006-07)

Choice	Deferred Acceptance Single Tie-Breaking DA-STB (1)	Deferred Acceptance Multiple Tie-Breaking DA-MTB (2)
1	32,105.3 (62.2)	29,849.9 (67.7)
2	14,296.0 (53.2)	14,562.3 (59.0)
3	9,279.4 (47.4)	9,859.7 (52.5)
4	6,112.8 (43.5)	6,653.3 (47.5)
5	3,988.2 (34.4)	4,386.8 (39.4)
6	2,628.8 (29.6)	2,910.1 (33.5)
7	1,732.7 (26.0)	1,919.1 (28.0)
8	1,099.1 (23.3)	1,212.2 (26.8)
9	761.9 (17.8)	817.1 (21.7)
10	526.4 (15.4)	548.4 (19.4)
11	348.0 (13.2)	353.2 (12.8)
12	236.0 (10.9)	229.3 (10.5)
unassigned	5,613.4 (26.5)	5,426.7 (21.4)

Proposition: For any (P_I, R_S), any matching that can be produced by deferred acceptance with multiple tie breaking, but not by deferred acceptance with single tie breaking is not a student-optimal stable matching.



Dominating stable matchings

Lemma: Suppose μ is a stable matching, and ν is some matching (stable or not) that dominates μ. Then the same set of students are matched in both ν and μ

- **Proof:** If there exists a student who is assigned under μ and unassigned under ν, then $\nu(i) = iP_i\mu(i)$, which implies that μ is not individually rational, a contradiction. So every i assigned under μ is also assigned under ν. Therefore $|\nu(S)| \geq |\mu(S)|$.
- If $|\nu(S)| > |\mu(S)|$ then there exists some $s \in S$ and $i \in I$ such that $|\nu(s)| > |\mu(s)|$ and $\nu(i) = s \neq \mu(i)$. This implies there is a vacancy at s under μ and i is acceptable for s. Furthermore, $sP_i\mu(i)$ since ν dominates μ. These together imply that μ is not stable, a contradiction. So $|\nu(S)| = |\mu(S)|$.
- Then the same set of students are matched in both ν and μ since $|\nu(S)| = |\mu(S)|$ and every student assigned under μ is also assigned under ν.

Stable Improvement Cycles (Erdil and Ergin, 08)

Fix a stable matching μ w.r.t. given preferences P and priorities R.
Student i **desires** s if $sP_i\mu(i)$.
Let B_s = the set of highest R_s-priority students among those who desire school s.

Definition: A stable improvement cycle C consists of distinct students $i_1, ..., i_n = i_0$ $(n \geq 2)$ such that
(i) $\mu(i_k) \in S$ (each student in the cycle is assigned to a school),
(ii) i_k desires $\mu(i_{k+1})$, and
(iii) $i_k \in B_{\mu(i_k+1)}$, for any $k = 0, ..., n-1$.

Given a stable improvement cycle define a new matching μ' by:

$\mu'(j) = \mu(j)$ if j is not one of $\{i_1, ..., i_n\}$
$\mu'(j) = \mu(i_{k+1})$ *if* $j = i_k$.

Proposition: μ' is stable and it (weakly) Pareto dominates μ.

Improving on DA^{τ}

Theorem (Erdil and Ergin, 2008): Fix P and R, and let μ be a stable matching. If μ is Pareto dominated by another stable matching, then μ admits a stable improvement cycle.

Algorithm for finding a student optimal matching: start with a stable matching. Find and implement a stable improvement cycle, as long as one exists.

Outline of proof

Fix P and R. Suppose μ is a stable matching Pareto dominated by another stable matching ν.

- *Simplifying assumption: Each school has one seat.*

1. $I' := \{i \in I \mid \nu(i) P_i \mu(i)\} = \{i \in I | \nu(i) \neq \mu(i)\}$.
2. All students in I' are matched to a school at ν.
3. $S' := \nu(I') = \mu(I')$.

Hence, I [S] can be partitioned into two subsets I' and $I \setminus I$' [S' and $S \setminus S$'] such that

- those in $I \setminus I$' [$S \setminus S$'] have the same match under μ and ν.
- the matches of those in I' [S'] have been "shuffled" among themselves to obtain ν from μ.

4. For all $s \in S$':
$I'_s := (i \in I' | i$ desires s at μ, and no $j \in I$' desires s at μ and $jP_s i)$ is nonempty;.

5. Construct a directed graph on S':

- For each $s \in S$', arbitrarily choose and fix $i_s \in I'_s$.
- $i_s \in B_s$: i.e., i_s desires s at μ, and there is no $j \in I$ who desires s at μ and $jP_s i$. (from stability of ν)
- For all $s, t \in S$', let $t \to s$ if $t = \mu(i_s)$.

6. The directed graph has a cycle of $n \geq 2$ distinct schools:
$s_1 \to s_2 \to \cdots \to s_n \to s_1$
7. The students $i_{s_1}, i_{s_2}, \ldots, i_{s_n}$ constitute a stable improvement cycle at μ.

How much room is there to improve on deferred acceptance?

Are there costs to Pareto improvements in welfare?

Strategy-proof mechanisms

- A direct mechanism ϕ is a function that maps every (P_I, R_S) to a matching.
- For $x \in I \cup S$, let $\phi_x(P_I; R_S)$ denote the set of agents that are matched to x by ϕ.

A mechanism ϕ is **dominant strategy incentive compatible** (DSIC) for $i \in I$ if for every (P_I, R_S) and every P_i' ,

$$\phi_i(P_I; R_S) R_i \phi_i(P_i', P_{-i}; R_S).$$

A mechanism will be called **strategy-proof** if it is DSIC for all students.

Pareto improvement and strategy proofness

Fix R_S. We say that a mechanism ϕ **dominates** ψ if

- for all $P_I : \phi_i(P_I; R_S) R_i \psi_i(P_I; R_S)$ for all $i \in I$, and
- for some $P_I : \phi_i(P_I; R_S) P_i \psi_i(P_I; R_S)$ for some $i \in I$.

Theorem (Abdulkadiroglu, Pathak, Roth): For any tie breaking rule τ, there is no mechanism that is strategy-proof and dominates DA^{τ}.

Proof:

Suppose that there exists a strategy-proof mechanism φ and tie-breaking rule r such that φ dominates DA^{τ}. There exists a profile P_I such that

$$\varphi_i(P_I; R_S) R_i DA^{\tau}(P_I; R_S) \text{ for all } i \in I, \text{ and}$$
$$\varphi_i(P_I; R_S) P_i DA^{\tau}(P_I; R_S) \text{ for some } i \in I.$$

Let $s_i = DA_i^{\tau}(P_I; R_S)$ and s_i'$= \varphi_i(P_I; R_S)$ be i's assignment under $DA^{\tau}(P_I; R_S)$ and $\varphi(P_I; R_S)$, respectively, where s_i'$P_i s_i$.

Consider profile $P_I' = (P_i', P_{-i})$, where P_i' ranks s_i' as the only acceptable school. Since DA^τ is strategy-proof, $s_i = DA_i^\tau(P_I; R_S) R_i DA_i^\tau(P_I\prime; R_S)$, and since $DA_i^\tau(P_I\prime; R_S)$ is either s'_i or i, we conclude that $DA_i^\tau(P_I\prime; R_S) = i$. Then the Lemma implies $\varphi_i(P_I\prime; R_S) = i$.

Now let $(P_I\prime; R_S)$ be the actual preferences. In this case, i could state P_i and be matched to $\varphi_i(P_I; R_S) = s_i'$, which under P_i' she prefers to $\varphi_i(P_I\prime; R_S) = i$.

So φ is not strategy-proof.

Let's look at some data

We can't tell what preferences would have been submitted with a different (non strategy-proof) mechanism, but we can ask, given the preferences that were submitted, how big an apparent welfare loss there might be due to not producing a student optimal stable matching.

Inefficiency in the NYC match (cost of strategy-proofness)

Table 1— Tie-breaking for Grade 8 Applicants in NYC in 2006-07

Choice	Deferred Acceptance Single Tie-Breaking DA-STB (1)	Deferred Acceptance Multiple Tie-Breaking DA-MTB (2)	Student-Optimal Stable Matching (3)	Improvement from DA-STB to Student-Optimal	Number of Students (4)
1	32,105.3 (62.2)	29,849.9 (67.7)	32,701.5 (58.4)	+1	633.2 (32.1)
2	14,296.0 (53.2)	14,562.3 (59.0)	14,382.6 (50.9)	+2	338.6 (22.0)
3	9,279.4 (47.4)	9,859.7 (52.5)	9,208.6 (46.0)	+3	198.3 (15.5)
4	6,112.8 (43.5)	6,653.3 (47.5)	5,999.8 (41.4)	+4	125.6 (11.0)
5	3,988.2 (34.4)	4,386.8 (39.4)	3,883.4 (33.8)	+5	79.4 (8.9)
6	2,628.8 (29.6)	2,910.1 (33.5)	2,519.5 (28.4)	+6	51.7 (6.9)
7	1,732.7 (26.0)	1,919.1 (28.0)	1,654.6 (24.1)	+7	26.9 (5.1)
8	1,099.1 (23.3)	1,212.2 (26.8)	1,034.8 (22.1)	+8	17.0 (4.1)
9	761.9 (17.8)	817.1 (21.7)	716.7 (17.4)	+9	10.2 (3.1)
10	526.4 (15.4)	548.4 (19.4)	485.6 (15.1)	+10	4.7 (2.0)
11	348.0 (13.2)	353.2 (12.8)	316.3 (12.3)	+11	2.0 (1.1)
12	236.0 (10.9)	229.3 (10.5)	211.2 (10.4)		
unassigned	5,613.4 (26.5)	5,426.7 (21.4)	5,613.4 (26.5)	Total:	1,487.5

Cost of stability in NYC

Table 2— Welfare Consequences of Stability for Grade 8 Applicants in in 2006-07

Choice	Student-Optimal Stable Matching (1)	Efficient Matching (2)	Improvement from Student-Optimal Stable Matching	Number (3)	k	Count of Students with k Blocking Pairs (4)
1	32,701.5 (58.4)	34,707.8 (50.5)	+1	1,819.7 (41.3)	1	22,287.5 (205.1)
2	14,382.6 (50.9)	14,511.4 (51.1)	+2	1,012.8 (26.4)	2	6,707.8 (117.9)
3	9,208.6 (46.0)	8,894.4 (41.2)	+3	592.0 (19.5)	3	2,991.0 (79.6)
4	5,999.8 (41.4)	5,582.1 (40.3)	+4	369.6 (16.0)	4	1,485.4 (56.5)
5	3,883.4 (33.8)	3,492.7 (31.4)	+5	212.5 (12.0)	5	716.6 (32.5)
6	2,519.5 (28.4)	2,222.9 (24.3)	+6	132.1 (9.1)	6	364.6 (22.9)
7	1,654.6 (24.1)	1,430.3 (22.4)	+7	77.0 (7.1)	7	183.1 (13.6)
8	1,034.8 (22.1)	860.5 (20.0)	+8	43.0 (5.6)	8	85.6 (10.9)
9	716.7 (17.4)	592.6 (16.0)	+9	26.3 (4.5)	9	44.7 (6.4)
10	485.6 (15.1)	395.6 (13.7)	+10	11.6 (2.8)	10	22.6 (4.9)
11	316.3 (12.3)	255.0 (10.8)	+11	4.8 (2.0)	11	9.9 (3.0)
12	211.2 (10.4)	169.2 (9.3)			12	3.2 (1.6)
unassigned	5,613.4 (26.5)	5,613.4 (26.5)	Total:	4,296.6		34,898.8

Comparison with Boston

Table 3— Tie-breaking for Elementary School Applicants in Boston in 2006-07

Choice	Deferred Acceptance Single Tie-Breaking DA-STB (1)	Deferred Acceptance Multiple Tie-Breaking DA-MTB (2)	Student-Optimal Stable Matching (3)	Improvement from DA-STB to Student-Optimal	Number of Students (4)
1	2,251.8 (8.4)	2,157.3 (13.4)	2,256.6 (8.2)	+1	4.6 (2.6)
2	309.8 (10.3)	355.5 (12.0)	307.4 (10.0)	+2	1.2 (1.1)
3	154.9 (7.9)	189.3 (10.1)	154.0 (7.7)	+3	0.5 (0.7)
4	59.7 (5.5)	76.1 (7.0)	58.7 (5.5)	+4	0.3 (0.5)
5	27.4 (4.5)	34.1 (4.8)	27.0 (4.4)	+5	0.0 (0.1)
6	4.9 (1.9)	6.0 (2.5)	4.9 (1.9)	+6	0.0 (0.1)
7	2.6 (1.4)	2.8 (1.6)	2.5 (1.4)	+7	0.0 (0.1)
8	1.9 (1.2)	0.9 (0.9)	1.9 (1.2)	+8	0.0 (0.1)
9	1.2 (1.1)	0.4 (0.6)	1.2 (1.0)	+9	0.0 (0.0)
10	0.3 (0.6)	0.1 (0.2)	0.3 (0.5)		
unassigned	112.4 (4.6)	104.6 (4.5)	112.4 (4.6)	Total:	6.5

Open questions

- (Equilibrium) misrepresentation in stable improvement cycles? (Can potential gains be realized?)
    - It appears there will be an incentive to raise popular schools in your preferences, since they become tradeable endowments...
- Restricted domains of preference?
    - Manipulation will be easier on some domains than others, and potential welfare gains greater on some domains than others.

Miralles, 2008, Abdulkadiroglu et al 2009

Suppose all students have the same ordinal preferences, but potentially different cardinal preferences.

Miralles, 2008, Abdulkadiroglu et al 2009

Suppose all students have the same ordinal preferences, but potentially different cardinal preferences.

Suppose cardinal preferences (or the distribution over them) is common knowledge.

Suppose there are no more school seats than students, then:

- No assignment that fills all school seats is in expectation inferior to DA
- Boston weakly dominates DA.

Are these the right costs of strategyproofness?
Ex post versus ex ante evaluation?