Probability Elicitation for Agents with Arbitrary Risk Preferences

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Abstract

A principal solicits an agent for his probability assessment of a random event and offers compensations for the service provided. The agent is probabilistically sophisticated with Anscombe-Aumann preferences. The principal does not have precise information on the agent's preferences. He only knows they belong do some general class. I consider two such classes. One class represents individuals who strictly prefer larger monetary payoffs. The other class represents individuals who, in addition, are averse to risk. Karni (2009) introduces an elegant mechanism that induces truthful reports from any agent represented by any of these classes. The main result of this paper is a simple characterization of all such (strictly) incentive compatible mechanisms, for each of the two classes being considered. I apply the result to the two mechanisms most commonly encountered: Those whose prospects are deterministic, and those whose prospects take the form of lottery tickets. I show that the first type can only elicit which of the event or its complement is most likely, while the second type comprises the only mechanisms incentive compatible among those that randomize payoffs over at most two values.

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1 Introduction

This paper considers a simple principal-agent setting, in which the principal solicits the agent's subjective probability for a random event. For example, a firm may ask its marketing department for the likelihood of a sales increase the upcoming quarters. The agent is subsequently rewarded for the service provided. The principal has full bargaining power and decides on the compensation structure, while the agent strategically decides on his report.

To induce honest assessments, the compensation structure must be designed with an appropriate degree of care. If the agent maximizes expected value, it is well-established that, to induce unbiased reports, payments must take the form of proper scoring rules (Brier, 1950; McCarthy, 1956; Winkler, 1969; Savage, 1971; Good, 1997; Gneiting and Raftery, 2007). If, however, the agent is not neutral to risk, proper scoring rules generally yield biased probability estimates. For example Winkler and Murphy (1970) and Kadane and Winkler (1988) show that when the principal compensates the agent according to the Brier score, risk-averse agents tend to overestimate the degree of uncertainty of the event.

This paper departs from the assumption that the agent seeks to maximize expected value. The agent has preferences that belong to some general class, which includes as special case the expected utility theory and the cumulative prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). The principal need not know the preferences of the participating agent. I study the problem of designing compensation structures that induce truthful reports from any individual whose preferences belong to the class.

If the agent reveals his preferences or if they are otherwise known, it is usually possible to obtain truthful estimates. One can either reverse the bias induced by the proper scoring rule, or correct the scoring rule to satisfy the incentive compatibility constraint. This approach is considered in Winkler and Murphy (1970). When the preferences are not known, Offerman et al. (2009) propose a two-stage mechanism. In the first stage, the agent participates in a series of experiments designed to learn his preferences; in the second stage, he is rewarded with a Brier score, his report is then post-processed to correct a possible bias. Similar procedures are explored in Jaffray and Karni (1999) for agents with state-dependent utilities.

In contrast, as in Karni (2009), the present work focuses on single-stage mechanisms. The agent's only task consists in submitting a probability report. Karni suggests a remarkably simple and elegant mechanism that is strategyproof. The mechanism does not require an estimation of the agent's preferences. Other instances of single-stage mechanisms include notably Roth and Malouf (1979), Grether (1981), Allen (1987), Koszegi and Rabin (2008), Holt and Smith (2009), Schlag and van der Weele (2010) and Hossain and Okui (2010). This vast and growing literature offers various methods to eliciting probabilities. The purpose of this paper is to unify these methods and offer a simple, complete characterization of all the *incentive compatible compensation structures*—the structures that induce truthful reports from any agent represented by the class of interest.

I derive results for two classes of preferences, one more general than the other. Both lie within the framework of Anscombe and Aumann (1963). They represent probabilistically sophisticated agents, that is, agents who hold subjective beliefs regarding the event's probability. The most general class is that of the *first order stochastically monotone preferences*. It captures individuals who strictly prefer larger monetary payoffs. The other class is that of *second order stochastically monotone preferences*. It represents individuals who, in addition, have strict preference for less risky prospects. The preferences under consideration are fairly general, they encompass expected and non-expected utility theories as a special case.

The characterization makes use of proper scoring rules. For the class of first order stochastically monotone preferences, the compensation schemes that satisfy incentive compatibility are described by a family of bounded proper scoring rules, indexed by real values, that is point-wise weakly decreasing. The scoring rule at a particular index value gives the probability that the payment exceeds the value. Strictly proper scoring rules yield strict incentive-compatibility. For second order stochastically monotone preferences, incentive compatible compensation structures take the form of a family of negative proper scoring rules, indexed on the real line, that is point-wise differentiable and with decreasing and bounded marginals. The scoring rule at a particular index value is a function of the expected payment to the agent, conditional on that payment not exceeding the index value. Strictly proper scoring rules also yield strict incentive-compatibility.

The second part of the paper explores some implications of the main result. It first investigates the deterministic scoring rule method, arguably the best known procedure of probability elicitation. Deterministic scoring rules reward the agent with non-random event-contingent payments. While the method can induce truthful probability estimates from agents whose preferences are known, it cannot do so without sufficiently precise knowledge of these preferences. The paper shows that, if an individual is only known to have first order stochastically monotone preferences, the only information that can be elicited is which of the event or its complement is most likely.

The mechanisms employed by Karni and other authors achieve incentive compatibility for a broad class of preferences through the linearization of the agent's preferences. The principal rewards the agent with two possible fixed monetary payoffs, effectively transferring some fixed amount of money upfront and, once the agent has delivered his probability assessment, rewarding him with a lottery ticket of fixed monetary prize. Although this paper implies many other schemes can be used, it also shows that, under some regularity condition, these types of schemes are the only ones to reach incentive compatibility among the schemes whose prospects can be decomposed as a combination of a deterministic event-contingent payment and an event-contingent lottery ticket.

The paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the classes of weakly and strictly incentive compatible compensation structures for agents with first order and second order stochastically monotone preferences. Section 4 refines the characterization for the most commonly used compensation structures: One in which the agent is compensated with deterministic event-contingent payments; the other in which the agent is rewarded with a combination of deterministic payments and lottery tickets. All proofs are relegated to the Appendix.

2 Setup

Throughout the paper, E denotes the event of interest, E^c the complementary event, and ω the state variable: $\omega = E$ if the event materializes, and $\omega = E^c$ otherwise.

2.1 Proper Scoring Rules

Scoring rules are functions used for the evaluation of probabilistic forecasts. They attribute a score to each pair of forecast and realization. Formally, a *scoring rule* $S : [0, 1] \times \{E, E^c\} \mapsto \mathbb{R}$ takes as input a probability estimate q that E occurs, a state

$\omega \in \{E, E^c\}$ that indicates the event's realization, and returns a score $S(q, \omega)$.

Accurate forecasts match empirical probabilities. Proper scoring rules ensure that the empirical average score is maximized under this condition. The scoring rule S is *proper* when, for all p, the expected score when E occurs with probability p is maximized for a report $q = p$:

$$pS(q, E) + (1 - p)S(q, E^c) \leq pS(p, E) + (1 - p)S(p, E^c) \qquad \forall p, q \ .$$

The scoring rule is strictly proper when the inequality is strict whenever $p \neq q$, that is, the expected score is strictly maximized for accurate predictions. Proper scoring rules and their constructions have been thoroughly studied. McCarthy (1956), Shuford et al. (1966), Hendrickson and Buehler (1971), Savage (1971), Friedman (1983), and Schervish (1989) offer various characterizations of proper and strictly proper scoring rules. Gneiting and Raftery (2007) provide a summary of the results.

2.2 The Anscombe-Aumann Framework

Agent preferences are defined within the framework of Anscombe and Aumann (1963). Anscombe and Aumann consider individuals with preferences over horse/roulette lotteries. In the context of this paper, a *horse/roulette lottery* is described by a function that associates a distribution of monetary payoffs to each state E or E^c. An agent who is given horse/roulette lottery L gets paid an amount drawn at random according to the distribution function $L(\omega)$, where ω is the state that materializes. If the event E occurs with probability p, a horse/roulette lottery reduces to a compound distribution over monetary payoffs given by

$$G_p^L := pL(E) + (1 - p)L(E^c) \ .$$

To allow the elicitation of subjective probabilities, the paper focuses on probabilistically sophisticated agents (Machina and Schmeidler, 1992). All agents form beliefs about the likelihood that E occurs, and rank horse/roulette lotteries accordingly. *Agent preferences* are defined by a pair (p, V), where p is the *subjective probability* of event E, and V is a real-valued function over distribution functions on monetary payoffs. The agent prefers (in a weak sense) horse/roulette lottery L^* to horse/roulette lottery L when

$$V(G_p^{L^*}) \geq V(G_p^L) \ .$$

If the inequality is strict, the agent strictly prefers L^* to L.

I consider two main classes of preferences. The first class of preferences describes experts who prefer more money to less. A preference (p, V) is *first order stochastically monotone* if V satisfies $V(F) > V(G)$ whenever F first order stochastically dominates G, i.e., whenever $F(x) \leq G(x)$ for all x, and $F \neq G$. The second class of preferences describe experts who, in addition, are averse to risk. A preference (p, V) is *second order stochastically monotone* if, for all $F \neq G$, both with finite absolute first moment, V satisfies $V(F) > V(G)$ whenever F second order stochastically dominates G, that is, whenever G is a mean-preserving spread of F, or equivalently $\int_{-\infty}^{x} F(t)\mathrm{d}t \leq \int_{-\infty}^{x} G(t)\mathrm{d}t$, for all x. First order stochastically monotone preferences include, as a special case, in increasing order of generality, expected-value maximizers, expected utility maximizes, and individuals who rank prospects based on the cumulative prospect theory.

2.3 Compensation Structures

The agent is compensated for his probability assessment. As agents hold Anscombe-Aumann preferences, it is without loss of generality that I restrict the study to compensations expressible via horse/roulette lotteries. Throughout, a *compensation structure* is summarized by a function Φ that takes as input a likelihood estimate for E and returns a horse/roulette lottery. For notational convenience, I also introduce the distribution function $F^{\Phi}(\cdot|p, \omega)$, which provides the implied distribution of monetary payoffs when the agent reports p and state ω materializes.

The principal must be able to incite honest behavior without precise knowledge of the agent's preference. A compensation structure Φ is said to be *incentive compatible* for a class $\mathcal{C}$ of probabilistically sophisticated preferences if, for all $(p, V) \in \mathcal{C}$, and all $q \neq p$,

$$V(G_p^{\Phi(p)}) \geq V(G_p^{\Phi(q)}) \ .$$

If the inequality is strict, the compensation structure Φ is strictly incentive compatible.

3 Incentive Compatible Compensation Structures

This section presents the main result of the paper. It characterizes the incentive compatible compensation structures, for first and second order stochastically monotone preferences. In both cases, the characterization is expressed in terms of scoring rules.

Theorem 1. *A compensation structure* Φ *is incentive compatible for the class of first order stochastically monotone preferences if and only if, for all* x*, the probability* S_x *of getting a payoff greater than* x*, defined by*

$$S_x(p, \omega) := 1 - F^{\Phi}(x \mid p, \omega) \tag{1}$$

is a proper scoring rule. The compensation structure Φ *is strictly incentive compatible if and only if, in addition, for all* $p \neq q$*, there exists* x *such that*

$$pS_x(p, E) + (1 - p)S_x(p, E^c) > pS_x(q, E) + (1 - p)S_x(q, E^c) \ . \tag{2}$$

In particular, if S_x *is strictly proper for some* x*,* Φ *is strictly incentive compatible.*

The theorem asserts that any incentive compatible compensation structure Φ is described by a family of proper scoring rules $\mathcal{F} := \{S_x : x \in \mathbb{R}\}$. By (1) $\mathcal{F}$ has the following properties:

1. S_x takes value in $[0, 1]$.
2. The family $\mathcal{F}$ is weakly decreasing: if $x \leq y$, $S_x \geq S_y$.
3. The family $\mathcal{F}$ is point-wise right continuous, i.e., $x \mapsto S_x(p, \omega)$ is right continuous, and

$$\lim_{x \to -\infty} S_x(p, \omega) = 0 \ ,$$
$$\lim_{x \to +\infty} S_x(p, \omega) = 1 \ .$$

The principal then compensates the agent by random payments distributed according to $F^{\Phi}(x \mid p, \omega) = 1 - S_x(p, \omega)$. Incentive compatibility becomes strict when the family $\mathcal{F}$ allows to distinguish between two probability assessments.

Theorem 1 demonstrates that the mechanisms proposed in the literature are part of a much larger class of incentive compatible compensation structures. Here incentive compatibility is also achieved, by necessity, through a randomization of monetary payoffs. The mechanisms in the literature, such as those of Allen (1987) and Karni (2009), hold two payoffs fixed once and for all, but distribute the actual payoff such that the higher payoff is awarded with a probability determined by a proper scoring rule. Instead, in Theorem 1, the compensation structures fix a continuum of payoffs. The distributions of payoffs are such that, for any cutoff point x, the probability that a payoff greater than x is awarded is determined by a proper scoring rule.

However requiring incentive compatibility for such a large class of preferences generates strong constraints on the compensation structures. As an example, if one limits incentive compatibility to hold only for expected value maximizers, the scoring rules defined by (1) are merely required to be proper on average. One may wonder if excluding the non-expected utility preferences would introduce some flexibility. It turns out not to be the case. This owes to the well known result that a distribution F first order stochastically dominates a distribution G if and only if all expected utility maximizers with strictly increasing utility strictly prefer F to G (see, for example, Mas-Colell et al., 1995).

The next result extends the characterization to risk-averse agents. This case restricts the class of allowed preferences sufficiently to relax some design constraints.

Theorem 2. *A compensation structure Φ is incentive compatible for the class of second order stochastically monotone preferences if and only if, for all x, the function defined by*

$$S(p, \omega) := -\int_{-\infty}^{x} F^{\Phi}(t \mid p, \omega)\mathrm{d}t \tag{3}$$

is a proper scoring rule. The compensation structure Φ is strictly incentive compatible if and only if, in addition, for all $p \neq q$, there exists x such that

$$pS_x(p, E) + (1-p)S_x(p, E^c) > pS_x(q, E) + (1-p)S_x(q, E^c) \ . \tag{4}$$

In particular, if S_x is strictly proper for some x, Φ is strictly incentive compatible.

Theorems 1 and 2 are similar in spirit. The latter shows that the structures that are incentive compatible for all risk-averse agents are shaped by a family of proper scoring rules $\mathcal{F} := \{S_x : x \in \mathbb{R}\}$. This family is must compel to the following conditions:

- S_x is weakly negative.

- The function $x \mapsto S_x(p,\omega)$ is differentiable and its derivative $g(x)$ is weakly decreasing, takes value in $[-1,0]$, and

$$\lim_{x\to-\infty} g(x) = 0 \ ,$$
$$\lim_{x\to+\infty} g(x) = -1 \ .$$

The family $\mathcal{F}$ specifies the distribution of the monetary payoffs according to the equality $S_x = P(X \leq x)(\mathsf{E}[X \mid X \leq x] - x)$. As previously, focusing on individuals who maximize expected utility does not offer increased flexibility in the design of the compensations. This owes to the result that a distribution F second order stochastically dominates a distribution G is and only if all expected utility maximizers with strictly increasing and strictly concave utility strictly prefer F to G (see, for example, Mas-Colell et al., 1995).

4 Implications for Common Methods

The preceding section describes the entire family of incentive compatible compensation structures. This section restricts attention to two special cases of compensation structures, that include those commonly encountered in practice and in the literature.

The first type is the scoring rule method. The principal decides on monetary payoffs as a function of the agent's report and of the materialization of the event. The payoffs involve no randomness. With the second type, the agent is offered one of two possible monetary payoffs. The payoffs are allocated at random, with a distribution determined by the agent's report and the materialization of the event.

4.1 Deterministic Payoffs

In many practical settings the rewards are deterministic monetary payoffs. These rewards generally depend on the realized state and the agent's response, but are not random. This corresponds to paying the agent according to a deterministic scoring rule: The principal pays the agent an amount $S(p,\omega)$ when the agent announces p while the true state is ω. In the current framework, the principal uses a compensation structure Φ, with an associated distribution function $F^{\Phi}(\cdot \mid p,\omega)$ that is a point mass Dirac distribution, $F^{\Phi}(x \mid p,\omega) = \mathbb{1}\{x \geq S(p,\omega)\}$. Such compensations are said to be *deterministic*.

Theorem 1 implies that, for all the scoring rules that are incentive compatible for the whole class of first order stochastically monotone preferences, for each cutoff payoff x, the probability that the agent gets at least least x is a proper scoring rule limited to take value 0 or 1. These are known as binary proper scoring rules, and characterized in the following lemma.

Lemma 1. *Let S be a scoring rule that takes value 0 or 1. Then S is proper if and only if either $S(\cdot, \omega) = 0$, or $S(\cdot, \omega) = 1$, or*

$$S(p, \omega) = \begin{cases} \mathbb{1}\{\omega = E^c\} & \text{if } p > \frac{1}{2} \ , \\ \mathbb{1}\{\omega = E\} & \text{if } p < \frac{1}{2} \ . \end{cases}$$

Lemma 1, together with Theorem 1, directly imply the characterization of the incentive compatible deterministic compensation structures.

Theorem 3. *A deterministic compensation structure Φ is incentive compatible for first order stochastically monotone preferences if and only if one of the following is true:*

- *$\Phi(p)$ gives an identical payoff regardless of p.*
- *For some $M > m$, Φ gives payoff M if E and m if E^c whenever $p < 1/2$, and gives payoff m if E and M of E^c whenever $p > 1/2$.*

This result essentially asserts that without randomization of the payoffs, and without knowledge of the agent's preferences, the only information that can be elicited with strict incentives is whether event E is more likely than E^c. It agrees with the result of Schlag and van der Weele (2010), who show that no strictly proper scoring rules is strictly incentive compatible for all expected utility maximizers.

4.2 Binary Payoffs

To overcome the limitations of the deterministic payoffs, Karni (2009) randomizes the actual reward over two monetary payoffs, an approach common to several elicitation procedures. The implied distribution of monetary payoffs $F^{\Phi}(\cdot \mid p, \omega)$ is a point-mass distribution with at most two points. I refer to these structures as *binary compensation structures*.

Binary compensation structures essentially reward the agent through a combination of deterministic payment $S(p, \omega)$ and a lottery ticket whose prize $L(p, \omega)$ and winning probability $\alpha(p, \omega)$ may depend on the expert's report p and the realized state ω. If the winning probability is maintained constant at 1 or 0, compensations involve no randomness and reduces to the previous case. On the other extreme, S and L are maintained constant, and the only variable part concerns the winning probability.

This latter design is common in the literature. The principal randomizes agent payoffs between two fixed monetary values, effectively neutralizing attitudes toward risk. Instead of giving a payment shaped according to a proper scoring rule, the principal randomizes over two fixed payments, with a distribution shaped according to a proper scoring rule. The following result shows that, subject to some regularity conditions, those schemes form the only incentive compatible binary compensation structure for first order stochastically monotone preferences.

The regularity conditions restrict the compensations to have smooth variations and eliminates degenerate cases. I say that a binary compensation structure is *regular* when the associated functions S, L, α are continuous in their first argument, L is strictly positive and $0 < \alpha(p, \omega) < 1$ for $p \in (0, 1)$.

Theorem 4. *A regular binary compensation structure is incentive compatible for first order stochastically monotone preferences if and only if, its deterministic payment S and lottery prize L are constant, and the winning probability $\alpha(p, \omega)$ is a proper scoring rule continuous in its first argument.*

Appendix

Proof of Theorem 1

If $S_x(p, \omega)$ defined by (1) is a proper scoring rule, then, for all p, q, and all x,

$$\begin{aligned} G_p^{\Phi(p)}(x) &= pF^{\Phi}(x \mid p, E) + (1-p)F^{\Phi}(x \mid p, E^c) \\ &\leq pF^{\Phi}(x \mid q, E) + (1-p)F^{\Phi}(x \mid q, E^c) \\ &= G_p^{\Phi(q)}(x) \ . \end{aligned}$$

Hence either $G_p^{\Phi(p)} = G_p^{\Phi(q)}$, or $G_p^{\Phi(p)}$ first order stochastically dominates $G_p^{\Phi(q)}$. So any individual with first order stochastically monotone preferences weakly prefers prize

$\Phi(p)$ to prize $\Phi(q)$, which yields incentive compatibility. If, in addition, for S_x satisfies inequality (2) for some x, then $G_p^{\Phi(p)}(x) < G_p^{\Phi(q)}(x)$ so $G_p^{\Phi(p)}$ first order stochastically dominates $G_p^{\Phi(q)}$ and the individual strictly prefers $\Phi(p)$ to $\Phi(q)$.

The converse makes use of the following lemma:

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a right-continuous, non-negative bounded function such that $\int f < +\infty$. For each $n \in \mathbb{N}$, let $w_n : \mathbb{R} \to \mathbb{R}$ be a continuous positive function such that $\int w_n = 1$, and*

$$\int_x^{x+\frac{1}{n}} w_n \to 1$$

as $n \to +\infty$. Then $\int w_n f < +\infty$ and $\int w_n f \to f(x)$.

Proof. That $\int w_n f < +\infty$ owes to the boundedness of f. Besides,

$$\int w_n f = \int_x^{x+\frac{1}{n}} w_n f + o(1) \ .$$

Then, by decomposing the integral on the right hand side,

$$\int_x^{x+\frac{1}{n}} w_n f = \int_x^{x+\frac{1}{n}} w_n (f - f(x)) + f(x) \int_x^{x+\frac{1}{n}} w_n \ .$$

The result obtains by the assumption that $\int_x^{x+\frac{1}{n}} w_n \to 1$ and that, by right continuity of f,

$$\int_x^{x+\frac{1}{n}} w_n |f - f(x)| \leq \sup_{y \in \left(x, x+\frac{1}{n}\right)} |f(y) - f(x)| \to 0 \ .$$

□

Suppose that the mechanism Φ is incentive compatible for first order stochastically monotone preferences. Take any x and consider the weight function

$$w_n(t) = \frac{n^2}{e^{-n^2(t-x)+n} + 2 + e^{n^2(t-x)-n}} \ .$$

Observe that the weight function satisfies the conditions of Lemma 2. Any (p, V_n) where V_n is defined by $V_n(F) = 1 - \int w_n F$ is a well-defined first order stochastically monotone preference. It corresponds to that of an expected utility maximizer, with utility

$$u_n(t) = \frac{1}{1 + e^{-n^2(t-x)+n}} \ .$$

By incentive compatibility,

$$V_n(G_p^{\Phi(p)}) \geq V_n(G_p^{\Phi(q)}) \ ,$$

so

$$\begin{aligned} & p\left(1 - \int_{-\infty}^{+\infty} w_n(t)F(t \mid p, E)\mathrm{d}t\right) + (1-p)\left(1 - \int_{-\infty}^{+\infty} w_n(t)F(t \mid p, E^c)\mathrm{d}t\right) \\ & \geq p\left(1 - \int_{-\infty}^{+\infty} w_n(t)F(t \mid q, E)\mathrm{d}t\right) + (1-p)\left(1 - \int_{-\infty}^{+\infty} w_n(t)F(t \mid q, E^c)\mathrm{d}t\right) \ . \end{aligned} \quad (5)$$

Any distribution function F is right continuous, and Lemma 2 implies that

$$\int w_n F \mathrm{d}t \to F(x) \ .$$

Hence (5) implies

$$\begin{aligned} & p(1 - F^\Phi(x \mid p, E)) + (1-p)(1 - F^\Phi(x \mid p, E^c)) \\ & \qquad \geq p(1 - F^\Phi(x \mid q, E)) + (1-p)(1 - F^\Phi(x \mid q, E^c)) \ , \end{aligned}$$

and so S_x defined by (1) is a proper scoring rule. If, by contradiction, Φ is strictly incentive compatible but for some $p \neq q$, inequality (2) is an equality for all x, then for all x

$$\begin{aligned} & p(1 - F^\Phi(x \mid p, E)) + (1-p)(1 - F^\Phi(x \mid p, E^c)) \\ & \qquad = p(1 - F^\Phi(x \mid q, E)) + (1-p)(1 - F^\Phi(x \mid q, E^c)) \ . \end{aligned}$$

By linearity, for all $n > 0$, the inequality (5) becomes an equality, which implies $V_n(G_p^{\Phi(p)}) \geq V_n(G_p^{\Phi(q)})$, and contradicts strict incentive compatibility.

Proof of Theorem 2

The proof is similar to that of Theorem 1. Suppose that S_x defined by (3) is indeed a proper scoring rule. Then,

$$\begin{aligned}\int_{-\infty}^{x} G_p^{\Phi(p)} &= p\int_{-\infty}^{x} F^{\Phi}(\cdot \mid p, E) + (1-p)\int_{-\infty}^{x} F^{\Phi}(\cdot \mid p, E^c)\\ &\leq p\int_{-\infty}^{x} F^{\Phi}(\cdot \mid q, E) + (1-p)\int_{-\infty}^{x} F^{\Phi}(\cdot \mid q, E^c)\\ &= \int_{-\infty}^{x} G_p^{\Phi(q)} \ .\end{aligned}$$

Therefore it is either the case that $G_p^{\Phi(p)} = G_p^{\Phi(q)}$, or that $G_p^{\Phi(p)}$ second order stochastically dominates $G_p^{\Phi(q)}$. So any individual with second order stochastically monotone preferences weakly prefers prize $\Phi(p)$ to prize $\Phi(q)$, which yields incentive compatibility. If, in addition, S_x satisfies inequality (4) for some x, then $G_p^{\Phi(p)} \neq G_p^{\Phi(q)}$ so $G_p^{\Phi(p)}$ second order stochastically dominates $G_p^{\Phi(q)}$ and the individual strictly prefers $\Phi(p)$ to $\Phi(q)$.

Conversely, suppose that Φ is incentive compatible for second order stochastically monotone preferences. Let $x \in \mathbb{R}$, and consider again the weight function

$$w_n(t) = \frac{n^2}{e^{-n^2(t-x)+n} + 2 + e^{n^2(t-x)-n}} \ .$$

We focus on preferences of the form (p, V_n), where

$$V_n(F) = -\int_{-\infty}^{+\infty} w_n(t)\left(\frac{1}{|t|+1}\int_{-\infty}^{t} F\right) \mathrm{d}t \ .$$

Note that V_n is well defined for all distribution functions F, is second order stochastically monotone, and that as $n \to +\infty$, by Lemma 2

$$V_n(F) \to -\frac{1}{|x|+1}\int_{-\infty}^{x} F \ .$$

By incentive compatibility, for all n,

$$V_n(G_p^{\Phi(p)}) \geq V_n(G_p^{\Phi(q)}) \ ,$$

therefore,

$$p(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid p, E)) + (1-p)(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid p, E^c))$$
$$\geq p(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid q, E)) + (1-p)(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid q, E^c)) \ ,$$

so S_x is a proper scoring rule.

If, by contradiction, Φ is strictly incentive compatible but for some $p \neq q$, inequality (4) is an equality for all x, then for all x,

$$p(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid p, E)) + (1-p)(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid p, E^c))$$
$$= p(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid q, E)) + (1-p)(-\int_{-\infty}^{x} F^{\Phi}(\cdot \mid q, E^c)) \ .$$

which implies that for all n, $V_n(G_p^{\Phi(p)}) = V_n(G_p^{\Phi(q)})$. This contradicts strict incentive compatibility.

Proof of Lemma 1

Schervish (1989) shows that $S(p, E)$ is weakly increasing in p. If $S(\cdot, E)$ is not constant, there is a threshold p_0 such that, if $p < p_0$, $S(p, E) = 0$ and if $p > p_0$, $S(p, E) = 1$. If S is proper, for all p, q,

$$pS(p, E) + (1-p)S(p, E^c) \geq pS(q, E) + (1-p)S(q, E^c) \ . \tag{6}$$

If $p < p_0$ and $q > p_0$, (6) gives

$$(1-p)S(p, E^c) \geq p + (1-p)S(q, E^c) \ ,$$

hence $S(p, E^c) = 1$ and $S(q, E^c) = 0$. Now (6) implies $1 - p \geq p$ for $p < p_0$ and $p \geq 1 - p$ for $p > p_0$, so $p_0 = 1/2$.

Proof of Theorem 4

The proof makes use of the following lemmas.

Lemma 3. *If S is a proper scoring rule, and if $S(\cdot, E)$ (resp. $S(\cdot, E^c)$) is constant on some range of probabilities (a, b), then $S(p, E^c)$ (resp. $S(p, E)$) is constant on (a, b).*

Proof. Assume for example that $S(\cdot, E)$ is constant on (a, b), and by contradiction that $S(p, E^c) > S(q, E^c)$ for some $p, q \in (a, b)$. Then $qS(q, E) + (1 - q)S(q, E^c) < qS(p, E) + (1 - q)S(p, E^c)$ and S is not proper. □

Lemma 4. *If S is a proper scoring rule that is non-negative and non-zero, then for $p \in (0, 1)$, either $S(p, E) > 0$ or $S(p, E^c) > 0$ or both.*

Proof. Suppose $S(p, E) = 0$ and $S(p, E^c) = 0$ for some $p \in (0, 1)$. As S a non-negative and non-zero, there exists q such that $s(q, E) > 0$ or $s(q, E^c) > 0$. Then $pS(q, E)+(1-p)S(q, E^c) > 0 = pS(p, E)+(1-p)S(p, E^c)$, and so S is not proper. □

We now return to the main proof. If a compensation structure has $S = m$ and $L = M$ both constant , with α a proper scoring rule, then

$$\begin{aligned}
1 - F^{\Phi}(x \mid p, \omega) &= 0 && \text{if } x \geq m + M \ , \\
1 - F^{\Phi}(x \mid p, \omega) &= 1 && \text{if } x < m \ , \\
1 - F^{\Phi}(x \mid p, \omega) &= \alpha(p, \omega) && \text{if } m \leq x < m + M \ .
\end{aligned}$$

In all three cases, the function $(p, \omega) \mapsto 1 - F^{\Phi}(x \mid p, \omega)$ is a proper scoring rule and incentive compatibility follows from Theorem 1.

Conversely, suppose a regular binary compensation structure Φ is incentive compatible for first order stochastically monotone preferences. Let $R = S + L$ be the sum of its deterministic payment S and lottery prize L, and α the winning probability. Denote by S_x the probability of getting an amount greater than x, defined by (1). By Theorem 1, S_x is a proper scoring rule for all x.

STEP 1. We start by observing that $R(\cdot, E)$ is weakly increasing on $(0, 1)$. Suppose there exists $0 < p_1 < p_2 < 1$ with $R(p_1, E) > R(p_2, E)$. With $x = R(p_2, E)$, $S_x(p_2, E) = 0$ while $S_x(p_1, E) \geq \alpha(p_1, E) > 0$. As S_x is a proper scoring rule, $f(\cdot, E)$ is weakly increasing by Schervish (1989), hence a contradiction. By a similar argument, $S(\cdot, E)$ is weakly increasing on $(0, 1)$, and $R(\cdot, E^c)$ and $S(\cdot, E^c)$ are weakly decreasing.

Step 2. This steps proves that for all p,

$$\max\{R(p,E),R(p,E^c)\} = M \ ,$$
$$\min\{S(p,E),S(p,E^c)\} = m \ ,$$

for some $m < M$.

By contradiction, suppose for example that $h(p) := \max\{R(p,E),R(p,E^c)\}$ is not constant. The case of $\min\{S(p,E),S(p,E^c)\}$ can be treated in a similar fashion. Observe that h is continuous by the regularity assumption. Therefore we can choose $p_L, p_H \in (0,1)$ such that $h(p_L) < h(p_H)$. Take any $x \in (p_L, p_H)$. As at least one of $S(p_L,E)$ or $S(p_L,E^c)$ is non-zero, so S_x is non-zero. However, both $S_x(p_H,E)$ and $S_x(p_H,E^c)$ are zero, which contradicts Lemma 4. Hence h is constant.

Step 3. This steps shows that $R(\cdot,E)$ is constant on $(0,1)$, and so, by continuity, on $[0,1]$. If not, let $\epsilon = \frac{1}{2}\min\{\inf L, R(1,E) - R(0,E)\}$. ϵ is strictly positive as L is strictly positive on its compact domain and $R(\cdot,E)$ is weakly increasing and not constant.

Let $q^\delta = \sup\{p \leq 1 \mid R(p,E) \leq M - \delta\}$. Note that q^δ is well-defined for $0 \leq \delta \leq \epsilon$, and that $q^\delta \to 1$ as $\delta \to 0$ by continuity of R. Besides, as $R(\cdot,E)$ is weakly increasing, $R(p,E) \leq M - \delta$ when $p \leq q^\delta$. So $S_{M-\delta}(p,E) = 0$ if $p \leq q^\delta$.

As $\max\{R(p,E),R(p,E^c)\} = M$, $R(p,E^c) = M$ if $p \leq q^\delta$, implying $S_{M-\delta}(p,E^c) = \alpha(p,E^c)$. S_x is a proper scoring rule, which by Lemma 3, and by continuity of R implies that $\alpha(p,E^c)$ is constant for $p \leq q^\delta$. As $q^\delta \to 1$ when $\delta \to 0$, $\alpha(p,E^c)$ is a constant α_0.

There are now three cases to consider.

Case 1: $S(0,E^c) \leq R(0,E)$. Since $R(\cdot,E)$ is not constant and is weakly increasing, there exists $p_1 \in (0,1)$ such that if $p \leq p_1$, $R(p,E) \leq R(p_1,E)$ and if $p > p_1$, $R(p,E) > R(p_1,E)$. Let $x = R(p_1,E)$. Then $S_x(p,E) = 0$ for $p \in (0,p_1)$, and $S_x(p,E) \geq \alpha(p,E) > 0$ for $p \in (p_1,1)$, while $f_x(p,E^c) = \alpha(p,E^c) = \alpha_0$ for $p \in (0,p_1)$. By Lemma 3, S_x cannot be proper.

Case 2: $S(1,E^c) > R(0,E)$. Define p_1 as in case 1. Then $S_x(p,E) = 0$ for $p < p_1$, $S_x(p,E) = \alpha(p,E)$ for $p > p_1$, and $S_x(p,E^c) = 1$ for all p. Hence by Lemma 3 S_x cannot be proper.

Case 3: $S(0,E^c) > R(0,E)$ and $S(1,E^c) > R(1,E)$. There must exist some interval (p_1,p_2) such that $R(\cdot,E)$ is constant on (p_1,p_2), and either $R(\cdot,E) > S(\cdot,E)$

on (p_1, p_2), or $R(\cdot, E) < S(\cdot, E^c)$ on (p_1, p_2). By a similar reasoning as case 1 and 2 respectively, S_x is not proper.

This shows that $R(\cdot, E)$ is constant on $(0, 1)$ and by a similar argument $R(\cdot, E^c)$, $L(\cdot, E)$, and $L(\cdot, E^c)$ are constant on $(0, 1)$.

Now suppose for example $R(\cdot, E) > R(\cdot, E^c)$. Let $x = (R(\cdot, E) + R(\cdot, E^c))/2$. Then $S_x(p, E) = \alpha(p, E)$ while $S_x(p, E^c) = 0$, and so S_x is not proper, and $R(\cdot, E) = R(\cdot, E^c)$. Similarly, $S(\cdot, E) = S(\cdot, E^c)$.

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