Supplement to
"Strategic Trading in Informationally Complex Environments"

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Abstract

This supplement contains additional results and proofs omitted from the main body of the paper.

Contents

1 Zero Aggregate Demand in Equilibrium

In this section, we formally derive equilibria in the special examples in which aggregate demand ends up being zero on the equilibrium path (footnote 12 in Section 2.2 and footnote 20 in Section 5 in the main body of the paper). In Section 1.1 we work out the example with one informed trader, corresponding to footnote 12, and in Section 1.2 we work out the example with multiple informed traders, corresponding to footnote 20.

1.1 One Informed Trader

Example OA.1 *The value of the security, v, is distributed normally with mean zero and variance one. There is one strategic trader with signal θ_1 who observes the value perfectly: $\theta_1 = v$. The demand of liquidity traders is $u = -v$. The market maker does not observe any signals beyond the aggregate demand.*

This example satisfies Assumptions 1 and 2 in Section 3, and thus we can use the closed form solutions derived in the proof of Theorem 1 and presented in Section 4.2 of the paper (for the special case $k_M = 0$). Because we have only one strategic trader in the example, many of the matrices become scalars, simplifying the calculation.

Specifically, $\Sigma_{\theta\theta} = \Sigma_{diag} = 1$, and therefore $\Lambda = 2$ and $\Lambda^{-1} = 1/2$. Next, $\Sigma_{\theta v} = 1$, while $\Sigma_{\theta u} = -1$. Thus, $A_v = 1/2$ and $A_u = -1/2$. The coefficients of the quadratic equation in γ are: $a = -1/4$, $b = (1/2) \cdot 4 \cdot (-1/2) + 1 = 0$, $c = Var(-\theta/2 - u) = Var(v/2) = 1/4$. Therefore, $\gamma = 1$, $\beta_D = 1/\gamma = 1$, and $\alpha = \gamma A_v - A_u = 1$. Thus, on the equilibrium path, aggregate demand is equal to $D = \alpha\theta + u = v - v = 0$.

1.2 Multiple Informed Traders

Example OA.2 *The value of the security, v, is distributed normally with mean zero and variance one. There are m strategic traders with the same signal $\theta_1 = v$. The demand of liquidity traders is $u = -v$. The market maker does not observe any signals beyond the aggregate demand.*

This example also satisfies Assumptions 1 and 2 in Section 3, and thus we can use the closed form solutions derived in the proof of Theorem 1 and presented in Section 4.2 of the paper (again, for the special case $k_M = 0$). We now have multiple strategic traders, so the calculations involve matrix manipulations.

Specifically, $\Sigma_{\theta\theta}$ is an m-dimensional matrix whose elements are all equal to one, while Σ_{diag} is an m-dimensional identity matrix. We thus have

$$\Lambda = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \quad \text{and} \quad \Lambda^{-1} = \frac{1}{m+1} \begin{pmatrix} m & -1 & \cdots & -1 \\ -1 & m & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & m \end{pmatrix}.$$

Next,

$$\Sigma_{\theta v} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \Sigma_{\theta u} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}, \quad A_v = \begin{pmatrix} 1/(m+1) \\ \vdots \\ 1/(m+1) \end{pmatrix}, \quad A_u = \begin{pmatrix} -1/(m+1) \\ \vdots \\ -1/(m+1) \end{pmatrix}.$$

The coefficients of the quadratic equation on γ are therefore $a = -m/(m+1)^2$, $b = -(m-1)/(m+1)^2$, and $c = 1/(m+1)^2$, which in turn gives us $\gamma = 1/m$ and $\beta_D = m$. Thus,

$$\alpha = \gamma A_v - A_u = \frac{1}{m} \begin{pmatrix} 1/(m+1) \\ \vdots \\ 1/(m+1) \end{pmatrix} + \begin{pmatrix} 1/(m+1) \\ \vdots \\ 1/(m+1) \end{pmatrix} = \frac{1}{m} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and so on the equilibrium path, aggregate demand is equal to $D = \alpha^T\theta + u = (1/m) \cdot m \cdot v - v = 0$. Thus, for every m, the market price on the equilibrium path is also always equal to zero, not revealing any information contained in the signals of the strategic traders and in the demand coming from the liquidity traders.

2 Zero Intercepts in Equilibrium

In this section we prove the statement made informally in footnote 13 in Section 3.2 of the main body of the paper that in our setting, linear equilibria with nonzero intercepts do not exist.

Proposition OA.1 *Suppose there exists an equilibrium of the form:*

- $d_i(\theta_i) = \delta_i + \alpha_i^T \theta_i$;

- $P(\theta_M, D) = \beta_0 + \beta_M^T \theta_M + \beta_D D$.

Then $\beta_0 = 0$ *and for all* i, $\delta_i = 0$.

Proof. Consider a particular realization of θ_i's, θ_M, and u.

Then in this equilibrium, the realized price will be given by

$$\begin{aligned} P = P(\theta_M, D) &= \beta_0 + \beta_M^T \theta_M + \beta_D D \\ &= \beta_0 + \beta_M^T \theta_M + \beta_D u + \beta_D \sum_{i=1}^{n} \delta_i + \beta_D \sum_{i=1}^{n} \alpha_i^T \theta_i. \end{aligned} \tag{OA.1}$$

By the definition of equilibrium, for every realization of θ_M and D, the price set by the market maker is equal to the expected value of the security conditional on θ_M and D:

$$P(\theta_M, D) = E[v|\theta_M, D].$$

Integrating over all possible realizations of θ_M and D, we thus get the following for the unconditional expectation of the price:

$$E[P] = E[v].$$

Since by assumption, $E[v] = 0$, and also $E[\theta_M]$, $E[u]$, and $E[\theta_i]$ (for all i) are equal to zero, by taking the unconditional expectation of Equation (OA.1), we get

$$0 = \beta_0 + \beta_D \sum_{i=1}^{n} \delta_i. \tag{OA.2}$$

Now, as in Step 2 of the proof of Theorem 1 in the paper, consider the expected payoff of strategic trader i from submitting demand d after observing signal $\widetilde{\theta}_i$. It is equal to

$$E\left[d \left(v - \beta_0 - \beta_M^T \theta_M - \beta_D \left(d + \sum_{j \neq i} \delta_j + \sum_{j \neq i} \alpha_j^T \theta_j + u \right) \right) \middle| \theta_i = \widetilde{\theta}_i \right].$$

Except for the presence of constants β_0 and δ_j, this is the same expression as in Step 2 of the proof of Theorem 1. By the same logic as in that step, it has to be the case that $\beta_D > 0$ and the unique optimal demand d^* is given by:

$$d^* = \frac{1}{2\beta_D} E\left[v - \beta_0 - \beta_M^T \theta_M - \beta_D \left(\sum_{j \neq i} \delta_j + \sum_{j \neq i} \alpha_j^T \theta_j + u \right) \middle| \theta_i = \widetilde{\theta}_i \right]$$

$$= -\frac{1}{2\beta_D} \left(\beta_0 + \beta_D \sum_{j \neq i} \delta_j \right) + \frac{1}{2\beta_D} \left(\Sigma_{iv}^T - \beta_M^T \Sigma_{iM}^T - \beta_D \left(\sum_{j \neq i} \alpha_j^T \Sigma_{ij}^T + \Sigma_{iu}^T \right) \right) \Sigma_{ii}^{-1} \widetilde{\theta}_i,$$

By assumption, we also have $d^* = \delta_i + \alpha_i^T \widetilde{\theta}_i$. Since the equalities above have to hold for all realizations $\widetilde{\theta}_i$, it has to be the case that

$$\delta_i = -\frac{1}{2\beta_D} \left(\beta_0 + \beta_D \sum_{j \neq i} \delta_i \right),$$

which can be rearranged as

$$\beta_0 + \beta_D \sum_{j \neq i} \delta_j + 2\beta_D \delta_i = 0. \qquad \text{(OA.3)}$$

Combining equations (OA.2) and (OA.3), we see that for every i, $\beta_D \delta_i = 0$, and thus $\delta_i = 0$, and therefore we also have $\beta_0 = 0$. ∎

3 Proof of Proposition 1

We first prove the following Lemma.

Lemma OA.1 *Consider a market with at least two strategic traders, and suppose the signals of traders 1 and 2 can be represented as* $\theta_1 = (\theta^1; \theta_C)$ *and* $\theta_2 = (\theta^2; \theta_C)$*, respectively, in such a way that random vector* θ_C *has dimension of at least 1, random vector* θ^1 *is orthogonal to* θ_C*, and random vector* θ^2 *is also orthogonal to* θ_C*.*[1] *Represent trader 1's and trader 2's equilibrium strategies as vectors* $\alpha_1 = (\alpha^1; \alpha_{1C})$ *and* $\alpha_2 = (\alpha^2; \alpha_{2C})$*, respectively, with dimensions corresponding to those of* $(\theta^1; \theta_C)$ *and* $(\theta^2; \theta_C)$*. Then,* $\alpha_{1C} = \alpha_{2C}$*.*

[1] We maintain the assumption that matrices $Var((\theta^1; \theta_C))$ and $Var((\theta^2; \theta_C))$ are full rank.

Proof. Consider equilibrium condition (8) from the proof of Theorem 1 in the main body of the paper, and rewrite it for trader 1 as follows

$$\begin{aligned} 2\begin{pmatrix} Var(\theta^1) & 0 \\ 0 & Var(\theta_C) \end{pmatrix}\begin{pmatrix} \alpha^1 \\ \alpha_{1C} \end{pmatrix} &= \frac{1}{\beta_D}\left(\begin{pmatrix} Cov(\theta^1, v) \\ Cov(\theta_C, v) \end{pmatrix} - \begin{pmatrix} Cov(\theta^1, \theta_M)\beta_M \\ Cov(\theta_C, \theta_M)\beta_M \end{pmatrix}\right) \\ &\quad - \begin{pmatrix} Cov(\theta^1, \theta^2) & 0 \\ 0 & Var(\theta_C) \end{pmatrix}\begin{pmatrix} \alpha^2 \\ \alpha_{2C} \end{pmatrix} \\ &\quad - \sum_{j>2}\begin{pmatrix} Cov(\theta^1, \theta_j)\alpha_j \\ Cov(\theta_C, \theta_j)\alpha_j \end{pmatrix} - \begin{pmatrix} Cov(\theta^1, u) \\ Cov(\theta_C, u) \end{pmatrix}. \end{aligned}$$

Restricting attention to the bottom block of rows (corresponding to the signal θ_C), we get

$$\begin{aligned} 2Var(\theta_C)\alpha_{1C} &= \frac{1}{\beta_D}\left(Cov(\theta_C, v) - Cov(\theta_C, \theta_M)\beta_M\right) \\ &\quad - Var(\theta_C)\alpha_{2C} \\ &\quad - \sum_{j>2} Cov(\theta_C, \theta_j)\alpha_j - Cov(\theta_C, u). \end{aligned} \tag{OA.4}$$

The corresponding equation for trader 2 is

$$\begin{aligned} 2Var(\theta_C)\alpha_{2C} &= \frac{1}{\beta_D}\left(Cov(\theta_C, v) - Cov(\theta_C, \theta_M)\beta_M\right) \\ &\quad - Var(\theta_C)\alpha_{1C} \\ &\quad - \sum_{j>2} Cov(\theta_C, \theta_j)\alpha_j - Cov(\theta_C, u). \end{aligned} \tag{OA.5}$$

Subtracting Equation (OA.5) from Equation (OA.4), we get

$$2Var(\theta_C)(\alpha_{1C} - \alpha_{2C}) = Var(\theta_C)(\alpha_{1C} - \alpha_{2C}),$$

and so $Var(\theta_C)(\alpha_{1C} - \alpha_{2C}) = 0$. Since $Var(\theta_C)$ is full rank, we get $\alpha_{1C} = \alpha_{2C}$. ■

We can now finish the proof of Proposition 1. Without loss of generality, assume that θ_A and θ_B are orthogonal (this can always be achieved by a change of basis). Slightly abusing notation, let $(\alpha_A; \alpha_B)$ be the equilibrium strategy of trader A (and thus, by Lemma OA.1, the equilibrium strategy of trader B is α_B).

The expected profit of trader A is equal to $\beta_D\left(\alpha_A^T Var(\theta_A)\alpha_A + \alpha_B^T Var(\theta_B)\alpha_B\right)$, and the expected profit of trader B is equal to $\beta_D\alpha_B^T Var(\theta_B)\alpha_B$. Since matrix $Var(\theta_A)$ is positive semidefinite, $\alpha_A^T Var(\theta_A)\alpha_A \geq 0$, and thus the expected profit of trader A is at least as high as the expected profit of trader B.

4 Proof of Proposition 2

Some parts of the proof (the lower bound on $Var(p)$, the lower bound on $Var(D)$ for class C_1, and the upper bound on the expected loss of liquidity traders for class C_3) rely on Theorem 1. Some parts (the upper bound on $Var(p)$, the (lack of) upper bound on $Var(D)$, the lower bound on $Var(D)$ for classes C_2 and C_3, the upper bound on β_D for classes C_2 and C_3, the lower bound on β_D, and the lower bound on the expected loss of liquidity traders) rely on Theorem 2 or parts of its proof. The remaining parts of the proof of Proposition 2 are self-contained.

Bounds on $Var(p)$. As $p = E[v|D]$, we have $0 \leq Var(p) \leq \sigma_{vv}$. The fact that the upper bound can be approached arbitrarily closely (in class C_1, and therefore also in classes C_2 and C_3) follows directly from Theorem 2 for the case $Cov(u, v|\theta, \theta_M) = 0$. Note also that the (lack of) upper bound on $Var(D)$, the lower bound on β_D, and the lower bound on the expected loss of liquidity traders also follow directly from this case of Theorem 2.

To see that the lower bound on $Var(p)$ can also approached arbitrarily closely, consider a market with two strategic traders (and liquidity demand u independent of all other variables). Trader 1 observes $v + \epsilon$, where ϵ is distributed normally with mean zero and variance $\sigma_{\epsilon\epsilon}$, independently of u and v. Trader 2 observes $v - \epsilon$. The resulting covariance matrix is

$$\Omega = \left(\begin{array}{c:cc:c} \sigma_{vv} & \sigma_{vv} & \sigma_{vv} & 0 \\ \hdashline \sigma_{vv} & \sigma_{vv} + \sigma_{\epsilon\epsilon} & \sigma_{vv} - \sigma_{\epsilon\epsilon} & 0 \\ \sigma_{vv} & \sigma_{vv} - \sigma_{\epsilon\epsilon} & \sigma_{vv} + \sigma_{\epsilon\epsilon} & 0 \\ \hdashline 0 & 0 & 0 & \sigma_{uu} \end{array}\right).$$

In this case,

$$\Lambda = \Sigma_{diag} + \Sigma_{\theta\theta} = \begin{pmatrix} 2(\sigma_{vv} + \sigma_{\epsilon\epsilon}) & \sigma_{vv} - \sigma_{\epsilon\epsilon} \\ \sigma_{vv} - \sigma_{\epsilon\epsilon} & 2(\sigma_{vv} + \sigma_{\epsilon\epsilon}) \end{pmatrix},$$

$$\Lambda^{-1} = \frac{1}{3\sigma_{vv}^2 + 10\sigma_{vv}\sigma_{\epsilon\epsilon} + 3\sigma_{\epsilon\epsilon}^2} \begin{pmatrix} 2(\sigma_{vv} + \sigma_{\epsilon\epsilon}) & -\sigma_{vv} + \sigma_{\epsilon\epsilon} \\ -\sigma_{vv} + \sigma_{\epsilon\epsilon} & 2(\sigma_{vv} + \sigma_{\epsilon\epsilon}) \end{pmatrix},$$

$$A_v = \Lambda^{-1}\Sigma_{\theta v} = \frac{1}{3\sigma_{vv}^2 + 10\sigma_{vv}\sigma_{\epsilon\epsilon} + 3\sigma_{\epsilon\epsilon}^2} \begin{pmatrix} (\sigma_{vv} + 3\sigma_{\epsilon\epsilon})\sigma_{vv} \\ (\sigma_{vv} + 3\sigma_{\epsilon\epsilon})\sigma_{vv} \end{pmatrix} = \frac{\sigma_{vv}}{3\sigma_{vv} + \sigma_{\epsilon\epsilon}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Next, using the compact formula for the case when liquidity demand u is independent of the other variables in the model, we get

$$\beta_D = \sqrt{\frac{A_v^T \Sigma_{diag} A_v}{\sigma_{uu}}} = \frac{\sigma_{vv}}{3\sigma_{vv} + \sigma_{\epsilon\epsilon}} \sqrt{\frac{2(\sigma_{vv} + \sigma_{\epsilon\epsilon})}{\sigma_{uu}}}$$

and

$$\alpha = \frac{1}{\beta} A_v = \sqrt{\frac{\sigma_{uu}}{2(\sigma_{vv} + \sigma_{\epsilon\epsilon})}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Finally,

$$\begin{aligned} Var(p) &= Var\left(\beta_D(\alpha^T\theta + u)\right) = \beta_D^2 \left(\alpha^T \Sigma_{\theta\theta}\alpha + \sigma_{uu}\right) \\ &= 2\left(\frac{\sigma_{vv}}{3\sigma_{vv} + \sigma_{\epsilon\epsilon}}\right)^2 \frac{\sigma_{vv} + \sigma_{\epsilon\epsilon}}{\sigma_{uu}} \left(\frac{\sigma_{uu}}{2(\sigma_{vv} + \sigma_{\epsilon\epsilon})} \times 4\sigma_{vv} + \sigma_{uu}\right) \\ &= 2\frac{\sigma_{vv}^2}{(3\sigma_{vv} + \sigma_{\epsilon\epsilon})^2} \left(2\sigma_{vv} + \sigma_{vv} + \sigma_{\epsilon\epsilon}\right) \\ &= \frac{2\sigma_{vv}^2}{3\sigma_{vv} + \sigma_{\epsilon\epsilon}}. \end{aligned}$$

Thus, $Var(p)$ can be arbitrarily close to zero when $\sigma_{\epsilon\epsilon}$ is large.

Lower bounds on $Var(D)$. Consider first class C_1. As the signals of the strategic traders (and thus their demands) are uncorrelated with u, we have $Var(D) \geq \sigma_{uu}$, and thus it is sufficient to find a sequence of markets for which $Var(D) \to \sigma_{uu}$. Consider the same two-trader setup as the one we used to establish the infimum for $Var(p)$. In that example,

$$\begin{aligned} Var(D) &= Var\left(\alpha^T\theta + u\right) = \alpha^T\Sigma_{\theta\theta}\alpha + \sigma_{uu} \\ &= \frac{\sigma_{uu}}{2(\sigma_{vv} + \sigma_{\epsilon\epsilon})} \times 4\sigma_{vv} + \sigma_{uu} \\ &= \sigma_{uu}\frac{3\sigma_{vv} + \sigma_{\epsilon\epsilon}}{\sigma_{vv} + \sigma_{\epsilon\epsilon}}, \end{aligned}$$

and so as $\sigma_{\epsilon\epsilon}$ grows large, $Var(D)$ converges to σ_{uu}.

To establish that $\inf Var(D) = 0$ on class C_2 (and thus also C_3), consider a market with two groups of n strategic traders. The demand of liquidity traders u is uncorrelated with the value of the asset v. In the first group, all traders observe $\theta_1 = v$ perfectly. In the second group, all traders observe $\theta_2 = u$ perfectly.

When a market contains several groups of identical traders, it is convenient to use the "hat" notation and closed-form expressions from the proof of the special case of Theorem 2. (This proof is in Appendix B of the main body of the paper.) Using that notation, we have

$$\Sigma_{\theta\theta} = \begin{pmatrix} \sigma_{vv} & 0 \\ 0 & \sigma_{uu} \end{pmatrix} \quad \text{and} \quad \widehat{\Sigma}_{diag} = \frac{1}{n} \begin{pmatrix} \sigma_{vv} & 0 \\ 0 & \sigma_{uu} \end{pmatrix},$$

and then

$$\widehat{\Lambda} = \Sigma_{\theta\theta} + \widehat{\Sigma}_{diag} = \frac{n+1}{n} \begin{pmatrix} \sigma_{vv} & 0 \\ 0 & \sigma_{uu} \end{pmatrix}$$

and

$$\widehat{A}_v = \widehat{\Lambda}^{-1}\Sigma_{\theta v} = \frac{n}{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \widehat{A}_u = \widehat{\Lambda}^{-1}\Sigma_{\theta u} = \frac{n}{n+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then, applying the closed-form expression from Step 1 of Appendix B of the main body of the paper, we obtain $\beta_D = \sqrt{n}\sqrt{\frac{\sigma_{vv}}{\sigma_{uu}}}$ and the vector of multipliers for the two groups

$$\alpha = \widehat{A}_v/\beta_D - \widehat{A}_u = \frac{1}{n+1} \begin{pmatrix} \sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}\sqrt{n} \\ -n \end{pmatrix}.$$

Thus, in this market, the aggregate demand is

$$D = \alpha^T\theta + u = \frac{\sqrt{n}}{n+1}\sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}v - \frac{n}{n+1}u + u,$$

and so

$$Var(D) = \frac{n}{(n+1)^2}\sigma_{uu} + \frac{1}{(n+1)^2}\sigma_{uu} = \frac{\sigma_{uu}}{n+1}.$$

In particular, $Var(D)$ converges to zero as n grows.

Upper bounds on $-E[u(v-p)]$. First, we establish that $\sup(-E[u(v-p)]) = \sqrt{\sigma_{vv}\sigma_{uu}}/2$ in classes C_1 and C_2. Note that this is precisely the expected loss of the liquidity traders in the canonical one-trader Kyle (1985) model. Thus, it is enough to show that for any other market in C_1 and C_2, the expected loss of the liquidity traders cannot exceed this value. To see this, note first that in such markets, $\sigma_{uv} = 0$. We have $p = E[v|D] = \frac{Cov(v,D)}{Var(D)}D$. The expected loss of the liquidity traders is $E[u(p-v)] = Cov(u, p-v) = Cov(u,p) = \frac{Cov(v,D)Cov(u,D)}{Var(D)}$. Project D on v and u: we then have $D = av + bu + cw$, for some coefficients a, b, and c, where w is normally distributed with mean zero and variance one, and is independent of v and of u. We then get:

$$\begin{aligned} E[u(p-v)] &= \frac{\sigma_{vv}\sigma_{uu}ab}{\sigma_{vv}a^2 + \sigma_{uu}b^2 + c^2} \\ &\leq \frac{\sigma_{vv}\sigma_{uu}ab}{\sigma_{vv}a^2 + \sigma_{uu}b^2} \\ &= \frac{1}{\sigma_{uu}^{-1}a/b + \sigma_{vv}^{-1}b/a} \\ &\leq \frac{1}{2}\sqrt{\sigma_{vv}\sigma_{uu}}, \end{aligned}$$

where the last inequality follows from the fact that for any two real numbers (in this case, $\sigma_{uu}^{-1}a/b$ and $\sigma_{vv}^{-1}b/a$), their arithmetic average is weakly higher than their geometric average.

Second, we establish that $\sup\left(-E[u(v-p)]\right) = \sqrt{\sigma_{vv}\sigma_{uu}}$ in C_3. We have $E[u(p-v)] = Cov(u, p-v) \leq \sqrt{\sigma_{uu}}\sqrt{Var(p-v)} \leq \sqrt{\sigma_{uu}}\sqrt{\sigma_{vv}}$, where the first inequality is just the classical Cauchy-Schwarz inequality, and the second inequality follows from the fact that $E[v-p|p] = 0$, and thus $Var(v) = Var(v-p) + Var(p) \geq Var(v-p)$.

To show that the bound is tight, consider a rescaled version of Example OA.1 in Section 1.1. Given a normal variable w with mean zero and variance one, let the asset value be $v = \sqrt{\sigma_{vv}}w$ and the liquidity demand be $u = -\sqrt{\sigma_{uu}}w$, so that $Var(v) = \sigma_{vv}$, $Var(u) = \sigma_{uu}$, and the asset value is perfectly negatively correlated with liquidity demand. There is one strategic trader who observes signal $\theta_1 = v$ (and thus also knows u). The resulting covariance matrix is

$$\Omega = \begin{pmatrix} \sigma_{vv} & \sigma_{vv} & -\sqrt{\sigma_{vv}\sigma_{uu}} \\ \sigma_{vv} & \sigma_{vv} & -\sqrt{\sigma_{vv}\sigma_{uu}} \\ -\sqrt{\sigma_{vv}\sigma_{uu}} & -\sqrt{\sigma_{vv}\sigma_{uu}} & \sigma_{uu} \end{pmatrix}.$$

As there is only one strategic trader, many matrices become scalars. We have $\Lambda = \Sigma_{diag} + \Sigma_{\theta\theta} = 2\sigma_{vv}$, $\Lambda^{-1} = 1/(2\sigma_{vv})$, $A_v = \Lambda^{-1}\Sigma_{\theta v} = 1/2$, and $A_u = \Lambda^{-1}\Sigma_{\theta u} = -\frac{1}{2}\sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}$.

Using our closed-form characterization, we find $\beta_D = \sqrt{\frac{\sigma_{vv}}{\sigma_{uu}}}$ and $\alpha_1 = \sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}$. Thus, in this market, the demand of the strategic trader is equal to $\sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}v = -u$, and the aggregate demand and the price are always equal to zero. Since the strategic trader demands quantity $-u$ and the price is zero, his expected profit is $E[-uv] = \sqrt{\sigma_{vv}\sigma_{uu}}$, which is also the expected loss of liquidity traders.

Upper bounds on β_D. First, consider any market in class C_1. As $p = \beta_D D$, we have $E[u(p-v)] = Cov(u,p) - Cov(u,v) = Cov(u,p) = \beta_D Cov(u,D)$. And as the signals of the traders are uncorrelated with u, $Cov(u,D) = \sigma_{uu}$. Hence, $\beta_D = E[u(p-v)]/\sigma_{uu}$. Using the upper bound obtained for the expected loss of liquidity traders, we immediately obtain the upper bound on β_D.

Next, consider classes C_2 and C_3. Consider the two-group market used to establish the lower bound on $Var(D)$ in classes C_2 and C_3. Recall that, in this configuration, we obtained $\beta_D = \sqrt{n}\sqrt{\frac{\sigma_{vv}}{\sigma_{uu}}}$, which grows without bound as n grows large.

5 Examples

In this section, we give several additional examples that illustrate the general framework presented in the main body of the paper and also help develop intuition for our information aggregation results. We first present a simple yet seemingly counterintuitive example in which a trader informed about the value of the security trades in the direction opposite to his estimate of that value. Next, we study what happens when one of the strategic traders is informed about the demand of liquidity traders. We conclude by analyzing several examples in which the market maker possesses private information about the value of the security and study how this information gets incorporated into the price of the security and how it affects equilibrium trading strategies and the sensitivity of equilibrium prices to market demand.

5.1 Trading "Against" One's Own Signal

We start with an example of information structure under which a trader who receives a signal about the value of the security trades in the opposite direction: if, based on his information, the expected value of the security is positive, then he shorts the security; if it is negative, then he buys it. Note that since our model is single-period, there cannot be any dynamic incentives to manipulate prices, of the form "I will try to mislead others first, and then take advantage of the mispricing."[2]

Example OA.3 *The value of the security is distributed as* $v \sim N(0,1)$. *There are two strategic traders. Trader 1 observes a noisy estimate of* v: $\theta_1 = v + \rho_1 \xi$, *where* $\xi \sim N(0,1)$ *is a random variable independent of* v, *and* ρ_1 *is a parameter that determines how accurate trader 1's signal is (e.g., if* $\rho_1 = 0$, *then trader 1 observes* v *exactly, and if* ρ_1 *is very large, then trader 1's signal is not very accurate). Trader 2 also observes a noisy estimate of* v: $\theta_2 = v + \rho_2 \xi$, *with the same "driver" of noise,* ξ, *as in trader 1's signal, but with a potentially different magnitude of noise,* ρ_2. *Finally, there is demand from liquidity traders,* $u \sim N(0,1)$, *which is independent of all other random variables. Formally, the resulting covariance matrix is*

$$\Omega = \left(\begin{array}{c:cc:c} 1 & 1 & 1 & 0 \\ \hdashline 1 & 1+\rho_1^2 & 1+\rho_1\rho_2 & 0 \\ 1 & 1+\rho_1\rho_2 & 1+\rho_2^2 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right).$$

From the analysis and closed-form characterization in the main body of the paper, we know that in the unique linear equilibrium the pricing rule is characterized by some $\beta_D > 0$, and the strategies of traders 1 and 2 are characterized by:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\beta_D} \Lambda^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{OA.6}$$

where $\Lambda = \begin{pmatrix} 2+2\rho_1^2 & 1+\rho_1\rho_2 \\ 1+\rho_1\rho_2 & 2+2\rho_2^2 \end{pmatrix}$.

Using the matrix inversion formula and setting $\delta = \frac{1}{\beta_D \det(\Lambda)}$ (which is positive, since Λ is positive definite), we get

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \delta \begin{pmatrix} 2+2\rho_2^2 & -1-\rho_1\rho_2 \\ -1-\rho_1\rho_2 & 2+2\rho_1^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \delta \begin{pmatrix} 1+2\rho_2^2-\rho_1\rho_2 \\ 1+2\rho_1^2-\rho_1\rho_2 \end{pmatrix}. \tag{OA.7}$$

Thus, if $\rho_1 = 2\rho_2 + \frac{1}{\rho_2}$, trader 1 *never trades*, despite θ_1 being informative about the value of the security, and for $\rho_1 > 2\rho_2 + \frac{1}{\rho_2} > 0$, trader 1 *always trades in the direction*

[2]For examples of settings in which such dynamic incentives do arise, see Brunnermeier (2005) and Sadzik and Woolnough (2015).

opposite to his signal θ_1, despite θ_1 being positively correlated with the value of the security, v. Similarly, if ρ_2 is equal to or greater than $2\rho_1 + \frac{1}{\rho_1}$, then trader 2 does not trade or trades in the direction opposite to his signal.

To get the intuition behind this seemingly puzzling behavior, consider a slight variation of Example OA.3.

Example OA.4 *The value of the security is* $v \sim N(0,1)$. *There are two strategic traders. Trader 1 observes a noisy estimate of* v: $\theta_1 = v + \xi$, *where* $\xi \sim N(0,1)$, *independent of* v. *Trader 2 observes* ξ: $\theta_2 = \xi$. *The demand from liquidity traders,* $u \sim N(0,1)$, *is independent of all other random variables. The resulting covariance matrix is*

$$\Omega = \left(\begin{array}{c:cc:c} 1 & 1 & 0 & 0 \\ \hdashline 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right).$$

In this case, $\Lambda = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$ and

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\beta_D}\Lambda^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta\begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{(OA.8)}$$

for some $\delta > 0$, and thus trader 2 trades in the direction opposite to his signal. Note that in this example, trader 2 is not informed about the value of the security: his signal ξ is independent of v. However, he is informed about the bias in trader 1's signal, and thus knows in which direction trader 1 is likely to "err" when submitting his demand. Thus, trader 2, by partly "undoing" this error (i.e., trading against it), can in expectation make a positive profit, despite not having any direct information about the value of the security. In a sense, while trader 1 trades on "fundamental" information, trader 2 trades on "technical" information: trader 1's ability to make money is due to his information about the value of the security, while trader 2's ability to make a profit is due to his information about the "mistakes" of other agents in the economy.[3]

[3]Formally, we say that trader i has "fundamental" information if $Cov(\theta_i, v|\theta_M) \neq 0$, and say that trader i has "technical" information if $Cov(\theta_i, u|\theta_M) \neq 0$ or $Cov(\theta_i, \theta_j|\theta_M) \neq 0$ for some $j \neq i$. If a trader has neither fundamental nor technical information, then in equilibrium he does not trade, and

In Example OA.3, the intuition is similar. If ρ_2 is large relative to $2\rho_1 + \frac{1}{\rho_1}$, then the main "chunk" of trader 2's information is about the mistake that trader 1 makes, and not about the fundamental value of the security. This causes trader 2 to want to "undo" that mistake and trade "against" his signal, while trader 1 continues to trade in a natural direction. When $\rho_2 = 2\rho_1 + \frac{1}{\rho_1}$, the incentives of trader 2 to trade on "fundamental" information (the positive correlation of his signal with the value of the security) and on the "technical" information (the positive correlation of his signal with the mistake of trader 1) cancel out, and trader 2 ends up not trading.

Examples OA.3 and OA.4 illustrate that a strategic trader's behavior in equilibrium is driven not only by the correlation of his information and the value of the asset, but also by the informational content of his signals relative to the information already contained in the signals and the resulting behavior of other agents—potentially even to the point of reversing the direction of his trade. It is this flexibility that allows strategic traders' information to get fully aggregated and incorporated in prices as market size grows, even for very rich information structures. In contrast, the behavior of liquidity traders is exogenous, and is not endogenously affected by what information it contains. As a result, the information contained in liquidity demand is fully incorporated in market prices only under appropriate correlation structures (see Section 5 in the main body of the paper for details).

5.2 Information about Liquidity Demand

In this section, we present an example showing what happens when one of the strategic traders does not know anything about the value of the security, but is informed about the amount of liquidity trading. We then compare the equilibrium to that of the standard model without such a trader.

Example OA.5 *The value of the security is distributed as $v \sim N(0, \sigma_{vv})$, and the demand from liquidity traders is distributed as $u \sim N(0, \sigma_{uu})$, independently of v. There are two strategic traders. Trader 1's signal is equal to v: $\theta_1 = v$. He is fully informed about the value of the security, just like in the standard Kyle model. Trader 2 is uninformed about the value of the security, but has insider information about the*

does not make any profit. In Section 6, we formally state and prove this result (Proposition OA.2), and also explore in more detail the dependence and nondependence of equilibrium trading strategies on various types of information.

demand from liquidity traders: $\theta_2 = u$. *Formally, the covariance matrix is*

$$\Omega = \left(\begin{array}{c:c:c:c} \sigma_{vv} & \sigma_{vv} & 0 & 0 \\ \hdashline \sigma_{vv} & \sigma_{vv} & 0 & 0 \\ 0 & 0 & \sigma_{uu} & \sigma_{uu} \\ \hdashline 0 & 0 & \sigma_{uu} & \sigma_{uu} \end{array}\right).$$

The auxiliary matrices in this example are:

$$\Lambda = \begin{pmatrix} 2\sigma_{vv} & 0 \\ 0 & 2\sigma_{uu} \end{pmatrix}, \; A_u = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \text{ and } A_v = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

Coefficient b in the quadratic equation is equal to zero, and therefore

$$\begin{aligned} \gamma &= \sqrt{-\frac{c}{a}} = \sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}, \\ \alpha_1 &= \frac{1}{2}\gamma = \frac{1}{2}\sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}, \\ \alpha_2 &= -\frac{1}{2}. \end{aligned}$$

For comparison, if the second strategic trader was not present, the model would reduce to the standard model of Kyle (1985), and the equilibrium would be characterized by

$$\begin{aligned} \gamma &= 2\sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}, \\ \alpha_1 &= \sqrt{\frac{\sigma_{uu}}{\sigma_{vv}}}. \end{aligned}$$

In other words, when the second strategic trader (who is informed about the demand from liquidity traders) is present in the market, that trader "takes away" one half of that "liquidity" demand. As a result, the first strategic trader, who knows the value of the security, trades half as much as he would in the absence of that second trader, and the market maker's pricing rule is twice as sensitive. Therefore, for any realization of v and u, the price in the market with the second strategic trader will be exactly the same as that in the market without that trader—and thus the informativeness of prices is not affected in either direction by whether there is a trader in that market who observes the trading flow from liquidity traders. Likewise, the expected loss of liquidity traders is also unaffected by the presence of a trader who observes their demand. Since, by construction, the market maker in expectation breaks even, it has to be the case that the profit of the second strategic trader comes out of the first trader's pocket. In fact, the second trader takes away exactly one half of the first trader's profit.[4] Also, as in Example OA.4, the second trader is trading on "technical" information, and is only able to make a profit because of the "mistakes" of other agents in the economy.

Example OA.5 shows that when liquidity demand is fully observed by some strategic traders, they may have an incentive to trade in the opposite direction, effectively removing part of that demand from the market. If the number of such strategic traders grows large, they may end up removing all liquidity demand from the market, potentially hindering information aggregation (and possibly the existence of limit equilibrium) in large markets (see, e.g., footnote 20 in the main body of the paper). Thus, in Sections 5 and 6 in the main body of the paper we assume not only that the variance of liquidity demand is positive, but also that it remains positive when we condition it on the signals of large groups of strategic traders and the market maker.

5.3 Informed Market Maker

In the preceding examples, the market maker does not receive any information other than the aggregate demand coming from strategic and liquidity traders. In this subsection, we turn to examples in which the market maker does possess some additional information. We show how this information affects the strategies of other traders and illustrate the interplay between the weight the market maker places on this additional information and the weight she places on market demand.

Our first two examples illustrate that the equilibrium obtained when the market maker has private information is generally not the same as when that information is publicly available (i.e., known both to the market maker and to all strategic traders).[5] This difference turns out to be important when we study the informativeness of prices

[4] To see this, note that the prices in the two markets are always the same, realization by realization, while the demand of the first strategic trader, in the presence of the second one, is exactly one half of what it would be in the absence of that trader.

[5] Jain and Mirman (1999) and Luo (2001) study extensions of the Kyle (1985) model with a partially informed market maker and with partially informative public information, respectively. The difference between the two cases can be seen by comparing their results (setting $\sigma_i^2 = 0$ in Luo (2001)). Our Examples OA.6 and OA.7 are similar, though not identical, to the models of Jain and Mirman (1999) and Luo (2001). We present the examples to emphasize the distinction between the two cases within the same setup, as this distinction is important for our hybrid-market information aggregation results.

as the sizes of some (but not all) groups of strategic traders become large (Section 6 in the main body of the paper). In that setting, as the sizes of some of the groups become large, the market behaves as if the signals of those groups were observed directly by the market maker—and not as if the signals of those groups were observed publicly.

Example OA.6 *The value of the security is* $v \sim N(0,1)$. *There is one strategic trader, who observes signal* $\theta_1 = v + \epsilon_1$. *The market maker observes signal* $\theta_M = v + \epsilon_2$. *Variables* ϵ_1 *and* ϵ_2 *are distributed normally with mean zero and variance one, independently of each other and of all other variables. The demand from liquidity traders is also independently distributed as* $u \sim N(0,1)$. *Formally, the covariance matrix that describes this information structure is*

$$\Omega = \left(\begin{array}{c:c:c:c} 1 & 1 & 1 & 0 \\ \hdashline 1 & 2 & 1 & 0 \\ \hdashline 1 & 1 & 2 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right).$$

Applying the formulas derived in Section 3, we get $\Sigma_{diag} = \Sigma_{\theta\theta} = \Sigma_{MM} = 2$ and $\Sigma_{\theta M} = \Sigma_{\theta v} = \Sigma_{Mv} = 1$. Thus, $\Lambda = 2+2-1/2 = 7/2$, $A_u = 0$, and $A_v = 2(1-1/2)/7 = 1/7$. The coefficients in the quadratic equation for γ are $a = -2/49$, $b = 0$, and $c = 1$, and thus

$$\beta_D = \frac{1}{\gamma} = \frac{\sqrt{2}}{7}.$$

Hence, the strategic trader's behavior is given by

$$\alpha_1 = \frac{1}{\beta_D} A_v = \frac{1}{2}\sqrt{2},$$

and the market maker's sensitivity to her own signal is

$$\beta_M = \Sigma_{MM}^{-1}\left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v\right) = \frac{3}{7}.$$

Consider now a variation of Example OA.6, in which the market maker's signal is public information (i.e., known to both the market maker and the strategic trader).

Example OA.7 *The value of the security is* $v \sim N(0,1)$. *The market maker observes signal* $\theta_M = v + \epsilon_2$. *The strategic trader now observes* two *signals,* $\theta^1 = v + \epsilon_1$ *and* $\theta^2 = v + \epsilon_2$. *Both* ϵ_1 *and* ϵ_2 *are normally distributed with mean zero and variance* *one, independently of each other and of all other variables. The demand from liquidity traders is independently distributed as* $u \sim N(0,1)$. *The covariance matrix that describes this information structure is now*

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now have $\Sigma_{diag} = \Sigma_{\theta\theta} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\Sigma_{MM} = 2$, $\Sigma_{\theta M} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\Sigma_{\theta v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\Sigma_{Mv} = 1$. Thus, $\Lambda = \begin{pmatrix} 7/2 & 1 \\ 1 & 2 \end{pmatrix}$, $\Lambda^{-1} = 1/6 \begin{pmatrix} 2 & -1 \\ -1 & 7/2 \end{pmatrix}$, $A_u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $A_v = \Lambda^{-1} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot 1/2 \right) = 1/6 \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$.

The coefficients of the quadratic equation on γ are now:

$$a = -1/36 \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} = -1/24, \; b = 0, \; \text{and } c = 1,$$

and thus

$$\beta_D = \frac{1}{\gamma} = \frac{\sqrt{6}}{12},$$

the strategic trader's behavior is given by

$$\alpha_1 = \frac{1}{\beta_D} A_v = \begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \end{pmatrix},$$

and the market maker's sensitivity to her own signal is now given by

$$\beta_M = \Sigma_{MM}^{-1} \left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v \right) = \frac{1}{2}.$$

The equilibria in these two examples are substantively different: the sensitivities of the market maker to the aggregate demand and to her own signal are different, and the sensitivity of the strategic trader's demand to signal θ^1 is different as well. We can also compute the expected profits that the strategic trader makes in these two markets (and thus the losses of liquidity traders): in the first example, the expected profit is $\sqrt{2}/7$, while in the second one it is greater: $\sqrt{6}/12$. These differences illustrate the point that having the market maker observe a signal is substantively different from having that signal observed publicly.

Our next example considers the case in which a strategic trader's information is strictly worse than the information available to the market maker.

Example OA.8 *Let ν_1, ν_2, ϵ_1, ϵ_2, and u be independent random variables, each distributed normally with mean zero and variance one. The value of the security is $v = \nu_1 + \nu_2$. The demand from liquidity traders is u. There are two partially informed strategic traders and a partially informed market maker. Trader 1's signal is $\theta_1 = \nu_1 + \epsilon_1$. Trader 2's signal is $\theta_2 = \nu_2 + \epsilon_2$. The market maker's signal is $\theta_M = \nu_2$. Note that while trader 1 possesses some "exclusive" information about the value of the security, trader 2 does not (because ν_2 is observed by the market maker, and ϵ_2 is pure noise). Formally, the covariance matrix is*

$$\Omega = \left(\begin{array}{c:cc:c:c} 2 & 1 & 1 & 1 & 0 \\ \hdashline 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ \hdashline 1 & 0 & 1 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

The auxiliary matrices in this example are:

$$\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}, \ A_u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ and } A_v = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}.$$

Therefore, in this case, we have

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\beta_D} \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix},$$

and so $\alpha_2 = 0$. Thus, trader 2 does not trade in equilibrium. This illustrates a more general phenomenon: in equilibrium, a strategic trader cannot make a positive profit (and does not trade) if his information is the same as or worse than (in the informationtheoretic sense) that of the market maker.[6]

Our final example considers a sequence of markets, indexed by the number of strategic traders, m. All traders receive the same information, which is imperfectly correlated with both the value of the asset and the market maker's information.

Example OA.9 *The value of the security, v, the demand from liquidity traders, u, and two information shocks, ϵ_1 and ϵ_2, are all distributed normally with mean zero and variance one, independently of each other. There are m identically informed strategic traders and a partially informed market maker. Each strategic trader observes a signal $\theta_1 = v + \epsilon_1$. The market maker observes a signal $\theta_M = v + \epsilon_2$. Formally (indexing all matrices by the number of strategic traders in the market, m), the covariance matrix is*

$$\Omega^m = \left(\begin{array}{c:cccc:c:c} 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ \hdashline 1 & 2 & 2 & \cdots & 2 & 1 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 2 & \cdots & 2 & 1 & 0 \\ \hdashline 1 & 1 & 1 & \cdots & 1 & 2 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right).$$

The auxiliary matrices are:

$$\Lambda^m = \begin{pmatrix} 3\frac{1}{2} & 1\frac{1}{2} & \cdots & 1\frac{1}{2} & 1\frac{1}{2} \\ 1\frac{1}{2} & 3\frac{1}{2} & \cdots & 1\frac{1}{2} & 1\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1\frac{1}{2} & 1\frac{1}{2} & \cdots & 3\frac{1}{2} & 1\frac{1}{2} \\ 1\frac{1}{2} & 1\frac{1}{2} & \cdots & 1\frac{1}{2} & 3\frac{1}{2} \end{pmatrix}, \text{ so that } (\Lambda^m)^{-1} = \begin{pmatrix} \frac{3m+1}{6m+8} & \frac{-3}{6m+8} & \cdots & \frac{-3}{6m+8} & \frac{-3}{6m+8} \\ \frac{-3}{6m+8} & \frac{3m+1}{6m+8} & \cdots & \frac{-3}{6m+8} & \frac{-3}{6m+8} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-3}{6m+8} & \frac{-3}{6m+8} & \cdots & \frac{3m+1}{6m+8} & \frac{-3}{6m+8} \\ \frac{-3}{6m+8} & \frac{-3}{6m+8} & \cdots & \frac{-3}{6m+8} & \frac{3m+1}{6m+8} \end{pmatrix},$$

$$A_u^m = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ and } A_v^m = \begin{pmatrix} \frac{1}{3m+4} \\ \vdots \\ \frac{1}{3m+4} \end{pmatrix}.$$

[6]See Proposition OA.2 in Section 6 for a formal statement and proof of this result.

Coefficient b in the quadratic equation is equal to zero, and so

$$\begin{aligned}\gamma^m &= \sqrt{-\frac{c}{a}} = \frac{3m+4}{\sqrt{2m}},\\ \alpha_i^m &= \gamma^m A_{vi}^m = \frac{1}{\sqrt{2m}},\\ \beta_M^m &= \frac{1}{2}(1-\frac{m}{3m+4}) = \frac{2m+4}{6m+8} = \frac{m+2}{3m+4}.\end{aligned}$$

Note that the weight β_M that the market maker places on her own signal is not constant in m. If there were no strategic traders at all, and only noise traders ($m=0$, although strictly speaking that case is not allowed by our general setup), it would be equal to $\frac{1}{2} = \frac{Cov(v,\theta_M)}{Var(\theta_M)}$. As m grows, this weight is monotonically decreasing (converging to $\frac{1}{3}$ in the limit). Intuitively, as m grows, an increasingly large fraction of the market maker's information about the value of the security is also contained in the strategic demand, and can be extracted from it by the market maker—thus leaving a smaller part for the signal θ_M that the market maker observes directly.

The second observation concerns the informativeness of prices. Take any m, and consider a realization of θ_1, θ_M, and u. In this realization, demand D is equal to $m\alpha_i^m\theta_1 + u = \frac{m}{\sqrt{2m}}\theta_1 + u$, and the market price P set by the market maker is equal to $\beta_D^m D + \beta_M^m\theta_M = \frac{m}{3m+4}\theta_1 + \frac{m+2}{3m+4}\theta_M + \frac{\sqrt{2m}}{3m+4}u$. Now, fix the realization of random variables, and let the number of strategic traders, m, grow to infinity. Then price P converges to $\frac{1}{3}\theta_1 + \frac{1}{3}\theta_M$. But notice that this expression is precisely the expected value of the asset, v, conditional on the information available in the market: u is uninformative, because it is independent of all other random variables, and

$$\begin{aligned}E[v|\theta_1,\theta_M] &= Cov\left(v, \begin{pmatrix}\theta_1\\ \theta_M\end{pmatrix}\right)^T Var\left(\begin{pmatrix}\theta_1\\ \theta_M\end{pmatrix}\right)^{-1}\begin{pmatrix}\theta_1\\ \theta_M\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}^{-1}\begin{pmatrix}\theta_1\\ \theta_M\end{pmatrix}\\ &= \frac{1}{3}\theta_1 + \frac{1}{3}\theta_M.\end{aligned}$$

Hence, as the number of strategic traders becomes large, their information and the information of the market maker get incorporated into the market price with precisely the weights that a Bayesian observer with access to *all* information available in the market would assign. In other words, as the number of strategic traders becomes large, all information available in the market is aggregated and revealed by the market price. Of course, this is not a coincidence: as we show in Theorem 2 in the main body of the paper, the information aggregation result holds very generally.

6 The Impact of Informative and Uninformative Signals on Trading Strategies

In this section, we prove the statement made in footnote 3 in Section 5.1 and also following Example OA.8 in Section 5.3 that a trader who possesses neither fundamental nor technical information (and who is thus less informed than the market maker) does not trade in equilibrium and thus does not make a profit. We also show that the presence of such a trader does not affect the trading behavior of other agents or the pricing behavior of the market maker. We then present two examples showing that the interaction of different types of information can in general be quite subtle: a trader who possesses only technical information may be able to make a positive profit even if he is the only strategic trader in the market, and a trader who has some signals less informative than those of the market maker may nevertheless use those signals for trading if he also has some other, more informative signals. We conclude by showing that if a subvector of strategic traders' signals is uninformative (rather than simply being less informative than the signal of the market maker), then the traders do not use this subvector of signals in equilibrium, and the presence of these signals has no impact on the agents' trading behavior or on the pricing behavior of the market maker.

Formally, we say that trader i is (weakly) *less informed* than the market maker if conditional on the market maker's signal θ_M, trader i's signal θ_i does not contain any additional information about other random variables in the model: v, u, and θ_j for $j \neq i$:

$$Cov(\theta_i, v|\theta_M) = Cov(\theta_i, u|\theta_M) = Cov(\theta_i, \theta_j|\theta_M) = 0.$$

Note that this is equivalent to the trader having neither fundamental nor technical information, as defined in footnote 3 in Section 5.1 in the main body of the paper.

Proposition OA.2 *Suppose trader i is less informed than the market maker. Then in equilibrium, $\alpha_i = 0$, and the profit of trader i is zero. The equilibrium strategies of all other traders and the pricing behavior of the market maker are the same as in the economy in which trader i is not present.*

Proof. For notational simplicity, suppose $i = 1$, and suppose the dimensionality of trader 1's signal is k (i.e., $\theta_1 \in \mathbb{R}^k$). Note that by the maintained Assumption 1 of Section 3, at least one strategic trader receives at least some information about the value of the security that is not contained in the market maker's signal, and thus there are at least two strategic traders in this market: $n \geq 2$.

From the closed-form solution given in Sections 4.1 and 4.2 of the main body of the paper, the vector α that "stacks" all trading strategy vectors α_j on top of each other is given by

$$\begin{aligned}\alpha &= \frac{1}{\beta_D} A_v - A_u \\ &= \Lambda^{-1} \left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right).\end{aligned}$$

Note first that the assumption that trader 1 is less informed than the market maker implies that the top k elements of vector $\left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right)$ are all zero.

Next, consider matrix $\Lambda = \Sigma_{diag} + Var(\theta|\theta_M)$.

$$\Lambda = \begin{pmatrix} \Sigma_{11} & 0 & \cdots & 0 \\ 0 & \Sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{nn} \end{pmatrix} + \begin{pmatrix} Var(\theta_1|\theta_M) & 0 & \cdots & 0 \\ 0 & Var(\theta_2|\theta_M) & \cdots & Cov(\theta_2, \theta_n|\theta_M) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Cov(\theta_n, \theta_2|\theta_M) & \cdots & Var(\theta_n|\theta_M) \end{pmatrix} \tag{OA.9}$$

$$= \begin{pmatrix} \Sigma_{11} + Var(\theta_1|\theta_M) & 0 \\ 0 & M \end{pmatrix}, \tag{OA.10}$$

where M is an invertible matrix.

Thus,

$$\Lambda^{-1} = \begin{pmatrix} (\Sigma_{11} + Var(\theta_1|\theta_M))^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}, \tag{OA.11}$$

and therefore in vector

$$\alpha = \Lambda^{-1} \left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right),$$

the first k elements are all zero—and these are precisely the elements that describe the equilibrium trading strategy of trader 1, α_1. The result about the zero profit of trader 1 is then immediate.

Next, the first k elements of vector $A_v = \Lambda^{-1}Cov(\theta, v|\theta_M)$ and vector $A_u = \Lambda^{-1}Cov(\theta, u|\theta_M)$ are all zero, and so it is immediate that the coefficients a, b, and c of the quadratic equation that pins down market depth $\gamma = 1/\beta_D$ are the same as in the corresponding quadratic equation for the market in which trader 1 is not present (see Section 3.2 of the main body of the paper for the closed-form solution formulas). Thus, the presence of trader 1 does not affect β_D.

Finally, from the block-diagonal formulas in equations (OA.9), (OA.10), and (OA.11), it follows that the elements of vectors A_v and A_u *after* the first k are exactly the same as in the corresponding vectors for the market in which trader 1 is not present. This observation, combined with the observations that the first k elements of A_v and A_u are all zero, and that β_D is not affected by the presence of trader 1, imply that β_M is not affected by the presence of trader 1 and that the elements of vector α *after* the first k are exactly the same as in the corresponding vector for the market in which trader 1 is not present. ■

The statement of Proposition OA.2 is intuitive, but it is important to note that the interaction of information and trading can in general be quite subtle. To illustrate the subtlety, we present two examples. The first example shows that a trader who possesses only technical information may still be able to make a positive profit even if he is the only strategic trader in the market.[7] In the example, the reason for the ability of the technical trader to make a profit is that liquidity demand is informative about the value of the security, and so the market maker's pricing rule is sensitive to aggregate demand. The technical strategic trader, in turn, is informed about the "bias" in liquidity demand.

Example OA.10 *The value of the security is distributed as $v \sim N(0,1)$. Liquidity demand is distributed as $u = v + \epsilon$, where $\epsilon \sim N(0,1)$. Random variables v and ϵ are independent. There is a single strategic trader in the market, whose signal is given by $\theta_1 = \epsilon$. The market maker does not directly observe any signals ($k_m = 0$).*

To compute a linear equilibrium, suppose the sensitivity of the market maker's pricing rule is $\beta_D > 0$ and suppose the strategic trader's trading rule is given by

[7] Strictly speaking, the setting of the example violates Assumption 1 of our model, which states that at least one strategic trader in the market must possess some fundamental information. Thus, our general results are not applicable to this setting. Nevertheless, as we show, there still exists a unique linear equilibrium in this example.

multiplier α_1. The aggregate demand D is then equal to $v+(1+\alpha_1)\epsilon$, and the expected value of the security conditional on aggregate demand is then given by

$$E[v|D] = \frac{1}{1+(1+\alpha_1)^2} D.$$

Thus, $\beta_D = \frac{1}{1+(1+\alpha_1)^2}$.

On the other hand, suppose the strategic trader observes a realization $\tilde{\theta}$ of his signal. If he submits demand d, his expected profit is equal to

$$E\left[d\left(v - \beta_D(v+\tilde{\theta}+d)\right)\right] = -\beta_D\left(\tilde{\theta}d + d^2\right).$$

Thus, the optimal demand is $d = -\tilde{\theta}/2$, and so $\alpha_1 = -1/2$, which in turn implies $\beta_D = 4/5$ and a positive expected profit of trader 1 (conditional on observing realization $\tilde{\theta}$ of his signal, trader 1's expected profit is equal to $\tilde{\theta}^2/5$).

Our next example shows that for Proposition OA.2 to hold, it is important that a trader's entire vector of signals is less informative than the signal of the market maker. If only some of trader i's signals are less informative, while other ones contain useful information that the market maker does not observe, then the trader may end up putting nonzero weights on the "less informative" signals in equilibrium.

Example OA.11 *The value of the security is distributed as $v \sim N(0,1)$. There is a single strategic trader in the market, who observes a vector of signals $(\theta^1;\theta^2) \in \mathbb{R}^2$. The first component of the vector is given by $\theta^1 = v + \delta$, where $\delta \sim N(0,1)$ is a random variable independent of v. The second component is $\theta^2 = \delta$. The market maker's signal is given by $\theta_M = \delta$. Finally, the demand from liquidity traders is $u \sim N(0,1)$, distributed independently of v and δ.*

Note that the second component of the strategic trader's vector of signals is (weakly) less informative than the market maker's signal: it contains no additional information. By Proposition OA.2, if that were the only information that the trader observed, he would not trade based on it. In Example OA.11, however, that is not the case. In the unique equilibrium, the strategic trader will put negative weight on component θ^2: his demand is given by $d = (\theta^1 - \theta^2) = v$, and so $\alpha_1 = (1;-1)$. The market maker ignores her signal ($\beta_M = 0$), and puts weight $\beta_D = 1/2$ on aggregate demand. (Note that this equilibrium is essentially the same as the equilibrium of the standard Kyle (1985) model, and that trader 1 having access to signal θ^2 is essential for this equivalence: if trader 1 had access only to signal θ^1, the equilibrium would be substantively different, with, e.g., $\beta_M \neq 0$.)

The last result of this section shows that if some elements of traders' vectors of signals are uninformative about *all* other relevant random variables (including the information of the market maker), then traders will indeed not trade on that information—it will have no impact on their trading. We use this result in the proof of the general case of Theorem 3 in Section 10, where in some cases we add auxiliary uninformative signals to avoid having to deal with zero covariance matrices.

Formally, suppose each strategic trader i's signal θ_i can be represented as a pair of signals $(\theta_i^{'}; \theta_i^{''})$ in such a way that the combined vectors $\theta' = (\theta_1^{'}; \ldots; \theta_n^{'})$ and $\theta'' = (\theta_1^{''}; \ldots; \theta_n^{''})$ have the property that vector θ'' is *uninformative*:

$$Cov(\theta'', v) = Cov(\theta'', u) = Cov(\theta'', \theta') = Cov(\theta'', \theta_M) = 0.$$

(Note that for some traders i, $\theta_i^{'}$ or $\theta_i^{''}$ may be empty, but we assume that both θ' and θ'' have at least one element.) Let α' and α'' be the vectors of components of equilibrium demand multiplier α corresponding to θ' and θ''.

Proposition OA.3 *Suppose θ'' is uninformative. Then $\alpha'' = 0$. Moreover, the equilibrium is not affected by the presence of signals in θ'': pricing multipliers β_D and β_M are the same as those in the market where only signals in θ' are observed, and vector of strategy multipliers α' is the same as the vector of strategy multipliers in the market where only signals in θ' are observed.*

Proof. The proof is similar to that of Proposition OA.2. From the closed-form solution given in Sections 4.1 and 4.2 of the main body of the paper, the vector α is given by

$$\begin{aligned} \alpha &= \frac{1}{\beta_D} A_v - A_u \\ &= \Lambda^{-1} \left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right). \end{aligned}$$

Since by assumption, θ'' is uninformative, it is independent of u, v, and θ_M, and so the rows corresponding to elements of θ'' in the matrix $\left(\frac{1}{\beta_D} Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M) \right)$ are all zero. Call this "Observation 1".

Next, consider matrix $\Lambda = \Sigma_{diag} + Var(\theta|\theta_M) = \Sigma_{diag} + \Sigma_{\theta\theta} - \Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T$. Take any element i from vector θ' and any element j from vector θ'' (these elements can be parts of the same trader's signal or be observed by different traders). Since by assumption, θ'' is independent of both θ' and θ_M, the entries in matrix Λ in cells (i, j) and (j, i) are both zero (because the corresponding entries are all zero in matrices Σ_{diag}, $\Sigma_{\theta\theta}$, and $\Sigma_{\theta M}\Sigma_{MM}^{-1}\Sigma_{\theta M}^T$. This, in turn, implies that the entries in matrix Λ^{-1} in cells (i, j) and (j, i) are also both zero.[8] Call this "Observation 2".

Combining Observation 1 and Observation 2, we find that the entries in vector $\alpha = \Lambda^{-1}\left(\frac{1}{\beta_D}Cov(\theta, v|\theta_M) - Cov(\theta, u|\theta_M)\right)$ corresponding to the elements of θ'' are all equal to zero.

The proof of the statement that the pricing multipliers β_D and β_M are the same as those in the market where only signals in θ' are observed, and that the vector of strategy multipliers α' is the same as the vector of strategy multipliers in the market where only signals in θ' are observed, is completely analogous to the proof of the statement in Proposition OA.2 that the equilibrium strategies of traders $j \neq i$ and the pricing behavior of the market maker are the same as in the economy in which trader i (less informed than the market maker) is not present. ■

7 Rescaling Liquidity Demand

In this section, we formally show that equilibrium prices are not affected by the scale of liquidity demand.

Take an "original" market as defined in Section 2 of the main body of the paper, and consider an "alternative" market, in which the demand from liquidity traders, $u^{(alt)}$, is equal to the demand from liquidity traders in the original market multiplied

[8] To see this, note that by rearranging the rows and columns of matrix Λ to first list the entries corresponding to θ' and then the entries corresponding to θ'', one gets a block-diagonal matrix

$$\begin{pmatrix} \Sigma'_{diag} + Var(\theta'|\theta_M) & 0 \\ 0 & \Sigma''_{diag} + Var(\theta'') \end{pmatrix},$$

the inverse of which is also block-diagonal:

$$\begin{pmatrix} \left(\Sigma'_{diag} + Var(\theta'|\theta_M)\right)^{-1} & 0 \\ 0 & \left(\Sigma''_{diag} + Var(\theta'')\right)^{-1} \end{pmatrix}.$$

by a constant factor $\rho > 0$: $u^{(alt)} = \rho u$. For notational convenience, we append the superscript (alt) to variables associated with the alternative market, while variables associated with the original markets have the same notation as before.

The total demand observed by the market maker in the equilibrium of the alternative market is thus

$$D^{(alt)} = \sum_i d_i^{(alt)} + u^{(alt)},$$

where $d_i^{(alt)} = (\alpha^{(alt)})^T \theta_i$ is the demand of trader i. The price in the equilibrium of this alternative market is then a random variable given by

$$p^{(alt)} = \beta_D^{(alt)} \left(\sum_i (\alpha^{(alt)})^T \theta_i + \rho u \right) + (\beta_M^{(alt)})^T \theta_M.$$

Proposition OA.4 *In the unique linear equilibrium of the alternative market,*

$$\beta_D^{(alt)} = \frac{\beta_D}{\rho},$$
$$\beta_M^{(alt)} = \beta_M,$$
$$\alpha^{(alt)} = \rho\alpha.$$

In particular, equilibrium price does not depend on the scale of liquidity demand:

$$p^{(alt)} = p.$$

Proof. The proof follows directly from the equilibrium characterization of Section 3.2. First, note that $\Sigma_{\theta\theta}^{(alt)} = Cov(\theta, \theta) = \Sigma_{\theta\theta}$ and similarly $\Sigma_{diag}^{(alt)} = \Sigma_{diag}$, $\Sigma_{MM}^{(alt)} = Cov(\theta_M, \theta_M) = \Sigma_{MM}$, $\Sigma_{\theta v}^{(alt)} = Cov(\theta, v) = \Sigma_{\theta v}$, and $\Sigma_{Mv}^{(alt)} = Cov(\theta_M, v) = \Sigma_{Mv}$.

Next, note that $\Sigma_{\theta u}^{(alt)} = Cov(\theta, \rho u) = \rho \Sigma_{\theta u}$, $\Sigma_{Mu}^{(alt)} = Cov(\theta_M, \rho u) = \rho \Sigma_{Mu}$, $\sigma_{uu}^{(alt)} = Var(\rho u) = \rho^2 \sigma_{uu}$, and $\sigma_{vu}^{(alt)} = Cov(v, \rho u) = \rho \sigma_{vu}$.

This immediately yields $A_v^{(alt)} = A_v$, $A_u^{(alt)} = \rho A_u^{(alt)}$, $a^{(alt)} = a$, $b^{(alt)} = \rho b$, and $c^{(alt)} = \rho^2 c$. Hence, $\gamma^{(alt)} = \rho\gamma$, $\beta_D^{(alt)} = \beta_D/\rho$, $\beta_M^{(alt)} = \beta_M$, and $\alpha^{(alt)} = \rho\alpha$, and therefore $p^{(alt)} = p$. ■

8 Proof of Theorem 2 (General Case)

The proof of Theorem 2 in the main body of the paper applies to the special case in which the covariance matrix of random vector $(\theta; \theta_M; u)$ is full rank. In this section, we prove Theorem 2 for the general case, imposing only Assumptions 1 and 2L: $Cov(v, \theta|\theta_M) \neq 0$ and $Var(u|\theta, \theta_M) > 0$.

Before proceeding to the proof, we first prove the following lemma.

Lemma OA.2 *Let ϕ be a random vector normally distributed with mean zero and covariance matrix Σ. Let α be a vector of the same dimensionality, and let $\{\alpha_k\}$ be a sequence of vectors such that*

$$\lim_{k\to\infty} (\alpha_k)^T \widetilde{\phi} = \alpha^T \widetilde{\phi}$$

for every realization $\widetilde{\phi}$ of random vector ϕ except possibly on a set of probability zero. Then the limit also holds in the L^2 sense, that is,

$$\lim_{k\to\infty} E\left[\left((\alpha_k)^T \phi - \alpha^T \phi\right)^2\right] = 0.$$

Proof. Let N be the rank of matrix $Var(\phi)$, and take an orthogonal matrix Φ such that

$$\Phi^T Var(\phi) \Phi = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},$$

where M is a symmetric positive definite matrix of size N (if matrix $Var(\phi)$ is itself positive definite, and thus full rank, then we can simply take Φ to be the identity matrix and thus $M = Var(\phi)$.)

Let $\psi = \Phi^T \phi$ (and thus, since Φ is orthogonal, $\phi = \Phi\psi$). If, for almost all realizations $\widetilde{\phi}$ of ϕ, we have $(\alpha_k)^T \widetilde{\phi} \to \alpha^T \widetilde{\phi}$, then for almost all realizations $\widetilde{\psi}$ of ψ, we also have $(\alpha_k)^T \Phi \widetilde{\psi} \to \alpha^T \Phi \widetilde{\psi}$, which can be rewritten as

$$(\Phi^T \alpha_k)^T \widetilde{\psi} \to (\Phi^T \alpha)^T \widetilde{\psi}. \tag{OA.12}$$

Since ψ is distributed normally with mean zero and covariance matrix $\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$, where M is a positive definite matrix of rank N, the fact that the convergence in Equation (OA.12) holds for almost all realizations $\widetilde{\psi}$ of random vector ψ implies that the first N components of vector $\Phi^T \alpha_k$ converge to the first N components of vector

$\Phi^T\alpha$. Then,

$$
\begin{aligned}
E\left[\left((\alpha_k)^T\phi - \alpha^T\phi\right)^2\right] &= E\left[\left((\alpha_k)^T\Phi\psi - \alpha^T\Phi\psi\right)^2\right] \\
&= E\left[(\Phi^T\alpha_k - \Phi^T\alpha)^T\psi\psi^T(\Phi^T\alpha_k - \Phi^T\alpha)\right] \\
&= (\Phi^T\alpha_k - \Phi^T\alpha)^T Var(\psi)(\Phi^T\alpha_k - \Phi^T\alpha) \\
&= (\Phi^T\alpha_k - \Phi^T\alpha)^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} (\Phi^T\alpha_k - \Phi^T\alpha).
\end{aligned}
$$

Since the first N components of $\Phi^T\alpha_k$ converge to the first N components of $\Phi^T\alpha$, we get that the first N components of $(\Phi^T\alpha_k - \Phi^T\alpha)$ converge to 0, which in turn implies by the last equation that $E\left[\left((\alpha_k)^T\phi - \alpha^T\phi\right)^2\right] \to 0$. ■

We are now ready to prove Theorem 2. The proof proceeds in five steps.

Step 1. Step 1 is identical to Step 1 in the proof of the special case in Appendix B in the main body of the paper, and is therefore omitted. The remaining steps are different from those in the proof of the special case.

Step 2. Let us now consider the entire sequence of markets, and restore superscript (m) for the variables. We introduce, for each market m, a random vector $\widehat{\theta}^{(m)}$, which is independent of the other random variables of the model, and is distributed normally with mean zero and covariance matrix $\widehat{\Sigma}^{(m)}_{diag}$.

Applying the usual projection formulas for jointly normal random variables, we note that the expressions for $\widehat{\Lambda}^{(m)}$, $\widehat{A}^{(m)}_u$, and $\widehat{A}^{(m)}_v$ that were obtained in Step 1 can be written as

$$
\begin{aligned}
\widehat{\Lambda}^{(m)} &= Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)} | \theta_M), \\
\widehat{A}^{(m)}_u &= [Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)} | \theta_M)]^{-1} Cov(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, u | \theta_M), \\
\widehat{A}^{(m)}_v &= [Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)} | \theta_M)]^{-1} Cov(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, v | \theta_M).
\end{aligned}
$$

Note that, unlike the special case considered in Appendix B, the limit of $\widehat{\Lambda}^{(m)}$ is not guaranteed to be invertible, and so we cannot directly consider the limits of $\widehat{A}^{(m)}_u$ and $\widehat{A}^{(m)}_v$. Instead, we (in essence) will perform a change of basis, and will work in the new system of coordinates.

Formally, let Φ be an orthogonal matrix such that

$$\Phi^T Var(\theta|\theta_M)\Phi = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},$$

where M is a symmetric positive definite matrix whose size is the rank of $Var(\theta|\theta_M)$ (and, if $Var(\theta|\theta_M)$ is itself positive definite, we can simply take Φ to be the identity matrix and $M = Var(\theta|\theta_M)$). Note that Assumption 1 ($Cov(v, \theta|\theta_M) \neq 0$) implies that $Var(\theta|\theta_M) \neq 0$, and thus the size of matrix M is greater than zero.

Let $(\theta'; \theta'')$ be the random vector defined as $\Phi^T\theta$, where the dimensionality of θ' is equal to the rank of M. (Note that $Var(\theta'|\theta_M) = M$ and that $Var(\theta''|\theta_M) = 0$.) In a similar fashion, we let $((\widehat{\theta}^{(m)})'; (\widehat{\theta}^{(m)})'') = \Phi^T\widehat{\theta}^{(m)}$ and $((\xi^{(m)})'; (\xi^{(m)})'') = \Phi^T\xi^{(m)}$.

We will first show that the following limits hold.

$$\begin{aligned}
Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') &\rightarrow 0,\\
Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) &\rightarrow 0,\\
Var((\widehat{A}_u^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) &\rightarrow 0,\\
Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) &\rightarrow 0,\\
Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) &\rightarrow 0,\\
Var((\widehat{A}_v^{(m)})^T\xi^{(m)}) &\rightarrow 0,\\
Var((\widehat{A}_u^{(m)})^T\xi^{(m)}) &\rightarrow 0,\\
Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) &\rightarrow 0,\\
Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) &\rightarrow 0,\\
Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) &\rightarrow 0,\\
Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) &\rightarrow 0,\\
Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) &\rightarrow 0.
\end{aligned}$$

Step 2(a). We show that $Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') \rightarrow 0$. Note that

$$Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') \leq Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i),$$

where $(\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i$ denotes the i-th element of random vector $(\xi^{(m)})' + (\widehat{\theta}^{(m)})'$. We then observe that, using the variance-covariance inequality,

$$\begin{aligned}\big|Cov((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i, (\xi^{(m)})'_j + (\widehat{\theta}^{(m)})'_j|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')\big|^2 \\ \leq Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') \\ \times Var((\xi^{(m)})'_j + (\widehat{\theta}^{(m)})'_j|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'') \\ \leq Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i) Var((\xi^{(m)})'_j + (\widehat{\theta}^{(m)})'_j).\end{aligned}$$

Since $Var((\xi^{(m)})'_i + (\widehat{\theta}^{(m)})'_i) = Var((\xi^{(m)})'_i) + Var((\widehat{\theta}^{(m)})'_i) \to 0$ for all i, we conclude that every element of matrix $Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')$ converges to 0.

Step 2(b). We now show that $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)}+\widehat{\theta}^{(m)}) \to 0$ and $Var((\widehat{A}_u^{(m)})^T(\xi^{(m)}+\widehat{\theta}^{(m)})) \to 0$. We focus on the first limit; the second one can be obtained via a similar derivation. For any vector $\widetilde{\theta}^{(m)}$ of the same dimensionality as $\widehat{\theta}^{(m)}$ and $\xi^{(m)}$, we can rewrite $(\widehat{A}_v^{(m)})^T\widetilde{\theta}^{(m)}$ as

$$\begin{aligned}&E[v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)} = \widetilde{\theta}^{(m)}, \theta_M = 0] \\ &\quad = E[v|\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)} = \Phi^T\widetilde{\theta}^{(m)}, \theta_M = 0] \\ &\quad = E[v|\theta' + (\xi^{(m)})' + (\widehat{\theta}^{(m)})' = (\widetilde{\theta}^{(m)})', (\xi^{(m)})'' + (\widehat{\theta}^{(m)})'' = (\widetilde{\theta}^{(m)})'', \theta_M = 0],\end{aligned}$$

where we define $((\widetilde{\theta}^{(m)})'; (\widetilde{\theta}^{(m)})'') = \Phi^T\widetilde{\theta}^{(m)}$ and the second equality makes use of the fact that, conditionally on $\theta_M = 0$, it is the case that $\theta'' = 0$ almost surely.

Let $\chi^{(m)} = (\xi^{(m)})' + (\widehat{\theta}^{(m)})' - E[(\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'']$, i.e., $\chi^{(m)}$ is the residual of the projection of $(\xi^{(m)})' + (\widehat{\theta}^{(m)})'$ on $(\xi^{(m)})'' + (\widehat{\theta}^{(m)})''$. As $(\xi^{(m)})$ and $(\widehat{\theta}^{(m)})$ are independent of v, $(\xi^{(m)})'' + (\widehat{\theta}^{(m)})''$ is also independent of v, and noting that $(\xi^{(m)})'' + (\widehat{\theta}^{(m)})''$ is independent of θ', $\chi^{(m)}$, and θ_M, we get

$$\begin{aligned}(\widehat{A}_v^{(m)})^T\widetilde{\theta}^{(m)} &= E[v|\theta' + (\xi^{(m)})' + (\widehat{\theta}^{(m)})' = (\widetilde{\theta}^{(m)})', (\xi^{(m)})'' + (\widehat{\theta}^{(m)})'' = (\widetilde{\theta}^{(m)})'', \theta_M = 0] \\ &= E[v|\theta' + \chi^{(m)} = f(\widetilde{\theta}^{(m)}), \theta_M = 0],\end{aligned}$$

where we define $f(\widetilde{\theta}^{(m)}) = (\widetilde{\theta}^{(m)})' - E[(\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'' = (\widetilde{\theta}^{(m)})'']$. Next,

$$E[v|\theta' + (\chi^{(m)})' = f(\widetilde{\theta}^{(m)}), \theta_M = 0] = Cov(v, \theta' + \chi^{(m)}|\theta_M)[Var(\theta' + \chi^{(m)}|\theta_M)]^{-1} f(\widetilde{\theta}^{(m)}).$$

Hence, since $f(\xi^{(m)} + \widehat{\theta}^{(m)}) = \chi^{(m)}$,

$$Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) = Var(Cov(v, \theta' + \chi^{(m)}|\theta_M)[Var(\theta' + \chi^{(m)}|\theta_M)]^{-1}\chi^{(m)}).$$

Therefore, to show that $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)}) \to 0$, it is enough to show that

$$Var(\chi^{(m)}) \to 0,$$

and that

$$Cov(v, \theta' + \chi^{(m)}|\theta_M)[Var(\theta' + \chi^{(m)}|\theta_M)]^{-1} \to Cov(v, \theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}.$$

To get the first limit, note that $Var(\chi^{(m)}) = Var((\xi^{(m)})' + (\widehat{\theta}^{(m)})'|(\xi^{(m)})'' + (\widehat{\theta}^{(m)})'')$, which was shown to converge to 0 in Step 2(a). The second limit follows directly from the first one, along with the observation that $Var(\theta'|\theta_M)$ is, by construction, positive definite, and thus invertible.

Thus, $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) \to 0$.

Step 2(c). Since $\xi^{(m)}$ and $\widehat{\theta}^{(m)}$ are independent, we have that $Var((\widehat{A}_v^{(m)})^T(\xi^{(m)} + \widehat{\theta}^{(m)})) = Var((\widehat{A}_v^{(m)})^T\xi^{(m)}) + Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)})$, and as the left-hand side converges to 0 as $m \to \infty$, and the two terms of the right-hand side are nonnegative, we get that $Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) \to 0$ and $Var((\widehat{A}_v^{(m)})^T\xi^{(m)}) \to 0$. We also get $Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) \to 0$ and $Var((\widehat{A}_u^{(m)})^T\xi^{(m)}) \to 0$ by the same argument.

Step 2(d). We now show that $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0$; the proof of the convergence for $Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0$, $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) \to 0$, and $Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \xi^{(m)}) \to 0$ is again completely analogous.

By the covariance inequality,

$$|Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}_i^{(m)})|^2 \leq Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)})Var(\widehat{\theta}_i^{(m)})$$

and we know from Step 2(c) that $Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) \to 0$. Besides we also have that $Var(\widehat{\theta}_i^{(m)}) \to 0$. Hence $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}_i^{(m)}) \to 0$ for all i, and thus $Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0$.

Step 2(e). Again, by the covariance inequality,

$$|Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)})|^2 \leq Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)})Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}),$$

and we have just shown that both variances converge to 0. Thus,

$$Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) \to 0.$$

Step 3. In this step we will show the following:

$$\begin{aligned} a^{(m)} &\to 0, \\ b^{(m)} &\to -Cov(u, v|\theta, \theta_M), \\ c^{(m)} &\to Var(u|\theta, \theta_M). \end{aligned}$$

By the projection formulas for jointly normal random variables, we observe that the expressions for the coefficients $a^{(m)}$, $b^{(m)}$ and $c^{(m)}$, obtained in Step 1, can be written as follows:

$$\begin{aligned} a^{(m)} &= Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}), \\ b^{(m)} &= 2Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) - Cov(u, v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, \theta_M), \\ c^{(m)} &= Var((\widehat{A}_u^{(m)})^T(\theta + \xi^{(m)}) - u|\theta_M). \end{aligned}$$

Step 3(a). We showed in Step 2 that $Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) \to 0$. It implies $a^{(m)} \to 0$.

Step 3(b). Let us show that $b^{(m)} \to -Cov(u, v|\theta, \theta_M)$.

In Step 2, we showed that

$$Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) \to 0$$

so it remains to show $Cov(u, v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, \theta_M) \to Cov(u, v|\theta, \theta_M)$.

Using the projection formula, we write $Cov(u, v|\theta + \xi^{(m)} + \widehat{\theta}^{(m)}, \theta_M)$ as

$$Cov(u, v|\theta_M) - Cov(u, \theta|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\theta, v|\theta_M),$$

where we used the fact that $\xi^{(m)} + \widehat{\theta}^{(m)}$ is independent of v and θ. Then, using the orthogonality of Φ, i.e., that $\Phi^T = \Phi^{-1}$, we observe that

$$
\begin{aligned}
&Cov(u,\theta|\theta_M)[Var(\theta+\xi^{(m)}+\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\theta,v|\theta_M)\\
&\qquad = Cov(u,\Phi^T\theta|\theta_M)[Var(\Phi^T\theta+\Phi^T\xi^{(m)}+\Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\Phi^T\theta,v|\theta_M).
\end{aligned}
$$

Also observe that $Cov(u,\Phi^T\theta|\theta_M) = (Cov(u,\theta'|\theta_M),0)$ and $Cov(\Phi^T\theta,v|\theta_M) = (Cov(\theta',v|\theta_M);0)$. From the block matrix inversion formula,[9] and using the facts that $Var(\theta''|\theta_M) = 0$ and that random vectors θ and $\xi^{(m)}+\widehat{\theta}^{(m)}$ are independent, we get

$$
\begin{aligned}
&Cov(u,\Phi^T\theta|\theta_M)[Var(\Phi^T\theta+\Phi^T\xi^{(m)}+\Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\Phi^T\theta,v|\theta_M)\\
&= Cov(u,\theta'|\theta_M)\\
&\ \times[Var(\theta'+(\xi^{(m)})'+(\widehat{\theta}^{(m)})'|\theta_M)\\
&\ - Cov((\xi^{(m)})'+(\widehat{\theta}^{(m)})',(\xi^{(m)})''+(\widehat{\theta}^{(m)})'')Var((\xi^{(m)})''+(\widehat{\theta}^{(m)})'')Cov((\xi^{(m)})''+(\widehat{\theta}^{(m)})'',(\xi^{(m)})'+(\widehat{\theta}^{(m)})')]^{-1}\\
&\ \times Cov(\theta',v|\theta_M)\\
&= Cov(u,\theta'|\theta_M)[Var(\theta'+(\xi^{(m)})'+(\widehat{\theta}^{(m)})'|\theta_M,(\xi^{(m)})''+(\widehat{\theta}^{(m)})'')]^{-1}Cov(\theta',v|\theta_M).
\end{aligned}
$$

Now, as

$$
Var(\theta'+(\xi^{(m)})'+(\widehat{\theta}^{(m)})'|\theta_M,(\xi^{(m)})''+(\widehat{\theta}^{(m)})'') = Var(\theta'|\theta_M)+Var((\xi^{(m)})'+(\widehat{\theta}^{(m)})'|(\xi^{(m)})''+(\widehat{\theta}^{(m)})''),
$$

and since we have already proven in Step 2 that $Var((\xi^{(m)})'+(\widehat{\theta}^{(m)})'|(\xi^{(m)})''+(\widehat{\theta}^{(m)})'') \to 0$, we have

$$
\begin{aligned}
&Cov(u,\Phi^T\theta|\theta_M)[Var(\Phi^T\theta+\Phi^T\xi^{(m)}+\Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\Phi^T\theta,v|\theta_M) \to\\
&\qquad\qquad Cov(u,\theta')[Var(\theta'|\theta_M)]^{-1}Cov(\theta',v|\theta_M).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&Cov(u,v|\theta_M) - Cov(u,\theta|\theta_M)[Var(\theta+\xi^{(m)}+\widehat{\theta}^{(m)}|\theta_M)]^{-1}Cov(\theta,v|\theta_M) \to\\
&\qquad\qquad Cov(u,v|\theta_M) - Cov(u,\theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}Cov(\theta',v|\theta_M),
\end{aligned}
$$

and the latter expression is equal to $Cov(u,v|\theta',\theta_M)$, which in turn is equal to $Cov(u,v|\theta,\theta_M)$. Thus, $b^{(m)} \to -Cov(u,v|\theta,\theta_M)$.

[9]See, for example, `http://en.wikipedia.org/wiki/Block_matrix_pseudoinverse`.

Step 3(c). Finally, let us show that $c^{(m)} \to Var(u|\theta, \theta_M)$. We have already shown that

$$\begin{aligned}c^{(m)} &= Var(Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}(\theta + \xi^{(m)}) - u|\theta_M) \\ &= Var(Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}\theta - u|\theta_M) + Var((\widehat{A}_u^{(m)})^T \xi^{(m)}).\end{aligned}$$

We have shown in Step 2 that $Var((\widehat{A}_u^{(m)})^T \xi^{(m)}) \to 0$. Now, again using $\Phi^T = \Phi^{-1}$, we note that

$$\begin{aligned}&Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}\theta \\ &\qquad = Cov(u, \Phi^T\theta|\theta_M)[Var(\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}\Phi^T\theta.\end{aligned}$$

Next, using the same block matrix inversion formula as in Step 3(b), and the same facts about vectors θ'', θ, and $\widehat{\theta}^{(m)}$, we get

$$\begin{aligned}&Var(Cov(u, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)[Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M)]^{-1}\theta - u|\theta_M) = \\ &\quad Var(Cov(u, \Phi^T\theta|\theta_M)[Var(\Phi^T\theta + \Phi^T\xi^{(m)} + \Phi^T\widehat{\theta}^{(m)}|\theta_M)]^{-1}\Phi^T\theta - u|\theta_M) \to \\ &\qquad Var(Cov(u, \theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}\theta' - u|\theta_M).\end{aligned}$$

Finally,

$$\begin{aligned}Var(Cov(u, \theta'|\theta_M)[Var(\theta'|\theta_M)]^{-1}\theta' - u|\theta_M) &= Var(u|\theta', \theta_M) \\ &= Var(u|\theta, \theta_M),\end{aligned}$$

and thus

$$c^{(m)} \to Var(u|\theta, \theta_M).$$

For the last observation in Step 3, note that by Assumption 2L, $Var(u|\theta, \theta_M) > 0$, and thus the results above on the limits of $a^{(m)}$, $b^{(m)}$, and $c^{(m)}$ imply that $\beta_D^{(m)}$ also converges to a finite value.

Step 4. By equilibrium condition (i), $\beta_D^{(m)}$ and $\beta_M^{(m)}$ are coefficients of the projection of v on the total demand $(\alpha^{(m)})^T(\theta + \xi^{(m)}) + u$ and the market maker's signal θ_M:

$$v = \beta_D^{(m)}((\alpha^{(m)})^T(\theta + \xi^{(m)}) + u) + (\beta_M^{(m)})^T\theta_M + \epsilon_{v,(\alpha^{(m)})^T(\theta+\xi^{(m)})+u,\theta_M}, \qquad \text{(OA.13)}$$

where $\epsilon_{v,(\alpha^{(m)})^T(\theta+\xi^{(m)})+u,\theta_M}$ is independent of $(\alpha^{(m)})^T(\theta+\xi^{(m)})+u$ and θ_M.

Also, condition (ii), expressed via Equation (17) in the proof of the special case of Theorem 2 in Appendix B of the main body of the paper,[10] can also be rewritten as a projection equation, of $v-\beta_D^{(m)}u-(\beta_M^{(m)})^T\theta_M$ (corresponding to the right-hand side of Equation (17)) on $\theta+\xi^{(m)}+\widehat{\theta}^{(m)}$ (corresponding to the left-hand side of Equation (17)):

$$v-\beta_D^{(m)}u-(\beta_M^{(m)})^T\theta_M=\beta_D^{(m)}(\alpha^{(m)})^T(\theta+\xi^{(m)}+\widehat{\theta}^{(m)})+\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}. \qquad \text{(OA.14)}$$

We will now show that θ_M is independent of $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$. Because of joint normality, it is enough to show that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta_M)=0$. From Equation (OA.14), and using the independence of $\widehat{\theta}^{(m)}$ and θ_M,

$$\begin{aligned}&Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta_M)=\\&\quad Cov(v,\theta_M)-Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta,\theta_M)-Cov(\beta_D^{(m)}u,\theta_M)-Cov((\beta_M^{(m)})^T\theta_M,\theta_M).\end{aligned}$$

From Equation (OA.13), we get that

$$Cov(v,\theta_M)=Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta,\theta_M)+Cov(\beta_D^{(m)}u,\theta_M)+Cov((\beta_M^{(m)})^T\theta_M,\theta_M),$$

hence establishing the equality $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta_M)=0$.

In the remainder of the proof, $p^{(m)}$ is the random variable that corresponds to the equilibrium price of the asset in market m, i.e.,

$$p^{(m)}=\beta_D^{(m)}(\alpha^{(m)})^T(\theta+\xi^{(m)})+\beta_D^{(m)}u+(\beta_M^{(m)})^T\theta_M. \qquad \text{(OA.15)}$$

Step 5. As $m\to\infty$, the behavior of the market depends on the sign of $Cov(u,v|\theta,\theta_M)$.

Step 5(a). Let us first consider the case in which $Cov(u,v|\theta,\theta_M)>0$.

In this case, $\beta_D^{(m)}\to Cov(u,v|\theta,\theta_M)/Var(u|\theta,\theta_M)$, which is immediately seen from the limits of the coefficients of the quadratic equation on $\beta_D^{(m)}$ in Step 3. We will show that $p^{(m)}\xrightarrow{L^2}E[v|\theta_M,\theta,u]$, i.e., $p^{(m)}$ converges to $E[v|\theta_M,\theta,u]$ in the L^2 sense.

[10] It is straightforward to check that the derivation of Equation (17) does not depend on the additional assumptions imposed in the special case, and remains valid in the general case that we consider in the current proof.

We observe that

$$E[v|\theta_M, \theta, u] - p^{(m)} = E[v - p^{(m)}|\theta_M, \theta, u]$$

and, from equations (OA.14) and (OA.15),

$$v - p^{(m)} = \beta_D^{(m)} (\alpha^{(m)})^T \widehat{\theta}^{(m)} + \epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},$$

and since $\widehat{\theta}^{(m)}$ is independent of θ, θ_M, u, we get that

$$E[v|\theta_M, \theta, u] - p^{(m)} = E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta_M, \theta, u] = E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta, u],$$

where the second equality comes from the fact that $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$ and θ_M are independent.

By Lemma OA.2, to prove

$$E\left[E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta, u]^2\right] \to 0,$$

it is enough to show that, for almost every realization $(\widetilde{\theta}; \widetilde{u})$ of random vector $(\theta; u)$,

$$E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta = \widetilde{\theta}, u = \widetilde{u}] \to 0.$$

To show the latter convergence, it is in turn sufficient to show two convergences:

$$Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \to 0$$

and $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, u) \to 0$.

We first show that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \to 0$. From Equation (OA.14), and using the fact that $\widehat{\theta}^{(m)}$ and θ are independent, we have

$$\begin{aligned} &Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \\ &\qquad = Cov(v, \theta) - Cov(\beta_D^{(m)} u, \theta) - Cov((\beta_M^{(m)})^T \theta_M, \theta) - Cov(\beta_D^{(m)} (\alpha^{(m)})^T \theta, \theta). \end{aligned}$$

Recalling that $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$ and $\theta + \xi^{(m)} + \widehat{\theta}^{(m)}$ are independent and that $\widehat{\theta}^{(m)}$ and $\xi^{(m)}$ are independent of each other and of v, u, θ, and θ_M, we also have

$$
\begin{aligned}
Cov(v,\theta) = Cov(v, \theta + \xi^{(m)} + \widehat{\theta}^{(m)}) \\
&= Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, \theta) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \\
&+ Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, \xi^{(m)}) + Cov(\beta_D^{(m)}u, \theta) + Cov((\beta_M^{(m)})^T\theta_M, \theta).
\end{aligned}
$$

Thus,

$$
Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) = Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, \xi^{(m)}).
$$

Now, since $\beta_D^{(m)}$ converges to a finite value, by Step 2 we have

$$
Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) = Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) - \beta_D^{(m)} Cov((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}, \widehat{\theta}^{(m)}) \to 0.
$$

Similarly, we have

$$
Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, \xi^{(m)}) = Cov((\widehat{A}_v^{(m)})^T\xi^{(m)}, \xi^{(m)}) - \beta_D^{(m)} Cov((\widehat{A}_u^{(m)})^T\xi^{(m)}, \xi^{(m)}) \to 0.
$$

And thus we get that

$$
Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, \theta) \to 0.
$$

Observe that to establish this convergence, we did not rely on the fact that $\beta_D^{(m)}$ converges to a positive value; we only used the fact that it converges to a finite value. We will use this observation in Step 5(b).

Next, we show that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, u) \to 0$. From Equation (OA.13), we get that

$$
\begin{aligned}
& Cov(v, (\alpha^{(m)})^T\theta) + Cov(v, u) \\
&= Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, (\alpha^{(m)})^T\theta) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, (\alpha^{(m)})^T\xi^{(m)}) + Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, u) \\
&+ Cov(\beta_D^{(m)}u, (\alpha^{(m)})^T\theta) + Cov(\beta_D^{(m)}u, u) + Cov((\beta_M^{(m)})^T\theta_M, (\alpha^{(m)})^T\theta) + Cov((\beta_M^{(m)})^T\theta_M, u).
\end{aligned}
$$

From Equation (OA.14), it follows that

$$
\begin{aligned}
& Cov(v, (\alpha^{(m)})^T\theta) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, (\alpha^{(m)})^T\theta) \\
&\quad - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, (\alpha^{(m)})^T\xi^{(m)}) - Cov(\beta_D^{(m)}u, (\alpha^{(m)})^T\theta) \\
&\quad\quad - Cov((\beta_M^{(m)})^T\theta_M, (\alpha^{(m)})^T\theta) = Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, (\alpha^{(m)})^T\widehat{\theta}^{(m)}),
\end{aligned}
$$

where the last term can be rewritten as

$$
\begin{aligned}
&Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, (\alpha^{(m)})^T\widehat{\theta}^{(m)}) \\
&= (\beta_D^{(m)})^{-1}Var((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}) - 2Cov((\widehat{A}_v^{(m)})^T\widehat{\theta}^{(m)}, (\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}) + \beta_D^{(m)}Var((\widehat{A}_u^{(m)})^T\widehat{\theta}^{(m)}).
\end{aligned}
$$

As $\beta_D^{(m)}$ converges to a positive value, the limits established in Step 2 imply that all the terms on the right-hand side of the last equation converge to zero, and thus

$$
Cov(\beta_D^{(m)}(\alpha^{(m)})^T\widehat{\theta}^{(m)}, (\alpha^{(m)})^T\widehat{\theta}^{(m)}) \to 0,
$$

and so

$$
\begin{aligned}
&Cov(v, (\alpha^{(m)})^T\theta) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, (\alpha^{(m)})^T\theta) \\
&\qquad - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\xi^{(m)}, (\alpha^{(m)})^T\xi^{(m)}) - Cov(\beta_D^{(m)}u, (\alpha^{(m)})^T\theta) \\
&\qquad\qquad - Cov((\beta_M^{(m)})^T\theta_M, (\alpha^{(m)})^T\theta) \to 0.
\end{aligned}
$$

Thus

$$
Cov(v, u) - Cov(\beta_D^{(m)}(\alpha^{(m)})^T\theta, u) - Cov(\beta_D^{(m)}u, u) - Cov((\beta_M^{(m)})^T\theta_M, u) \to 0
$$

and so $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}, u) \to 0$.

Step 5(b). We now examine the remaining case: $Cov(u, v|\theta, \theta_M) \leq 0$. In this case, $\beta_D^{(m)} \to 0$, which again follows from the limits of the coefficients $a^{(m)}$, $b^{(m)}$, and $c^{(m)}$ in Step 3. We will show that $p^{(m)} \xrightarrow{L^2} E[v|\theta, \theta_M]$.

Expressing v using Equation (OA.14), and taking expectations on both sides conditional on θ and θ_M, we have

$$
\begin{aligned}
E[v|\theta, \theta_M] &= \beta_D^{(m)}E[u|\theta, \theta_M] + \left[(\beta_M^{(m)})^T\theta_M + \beta_D^{(m)}(\alpha^{(m)})^T(\theta + \xi^{(m)})\right] + E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta, \theta_M] \\
&= \beta_D^{(m)}E[u|\theta, \theta_M] + \left[p^{(m)} - \beta_D^{(m)}u\right] + E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta, \theta_M],
\end{aligned}
$$

and so we have

$$
p^{(m)} - E[v|\theta, \theta_M] = \beta_D^{(m)}(u - E[u|\theta, \theta_M]) - E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta, \theta_M].
$$

As $\beta_D^{(m)} \to 0$, the first term on the right-hand side, $\beta_D^{(m)}(u - E[u|\theta, \theta_M])$, converges to 0 (in L^2 sense). Also, in Step 4 we showed that $\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}$ is independent of

θ_M, and therefore $E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta,\theta_M] = E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta]$. In Step 5(a), we showed that $Cov(\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}},\theta) \to 0$, implying that $E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta = \widetilde{\theta},\theta_M = \widetilde{\theta}_M]$ converges to 0 for almost every realization $(\widetilde{\theta};\widetilde{\theta}_M)$ of random vector $(\theta;\theta_M)$. Therefore, by Lemma OA.2, $E[\epsilon_{v,\theta+\xi^{(m)}+\widehat{\theta}^{(m)}}|\theta,\theta_M] \xrightarrow{L^2} 0$. Thus, $p^{(m)} \xrightarrow{L^2} E[v|\theta,\theta_M]$.

9 Proof of Theorem 3 (Special Case)

In this Section, we prove Theorem 3 in the special case in which the covariance matrix of random vector $(\theta_S;\theta_L;\theta_M;u)$ is full rank. In Section 10 below, we provide the full proof of Theorem 3, without making this simplifying assumption.

Step 1. In addition to the markets indexed $m = 1, 2, \ldots$, we consider the alternative market which includes s groups of traders $i = 1, \ldots, s$. The size of group i is ℓ_i. Each trader j of group i receives signal θ_i. In this alternative market, the market maker receives signal $(\theta_L;\theta_M)$. We use superscript (m) when we refer to the variables of the market m, and we use superscript (alt) when we refer to the variables in the alternative market. By Theorem 1 a unique linear equilibrium exists for each market m and for the alternative market. Recall that $\ell_i^{(m)}$ is constant in m for $i \le s$.

For $i = s+1, \ldots, n$, we define $\xi_i^{(m)} = (\sum_j \xi_{i,j})/\ell_i^{(m)}$ and $\xi_L^{(m)} = (\xi_{s+1}^{(m)};\ldots;\xi_n^{(m)})$. For $i \le s$ we let by convention $\Sigma_i^\xi = 0$, $\xi_i^{(m)} = 0$, and $\xi_S^{(m)} = (\xi_1^{(m)};\ldots;\xi_s^{(m)})$. Let $\xi^{(m)} = (\xi_S^{(m)};\xi_L^{(m)})$.

Also, as before, we define

$$\widehat{\Sigma}_{diag}^{(alt)} = \begin{pmatrix} \frac{1}{\ell_1}\Sigma_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\ell_s}\Sigma_{ss} \end{pmatrix}$$

and

$$\widehat{\Sigma}_{diag}^{(m)} = \begin{pmatrix} \frac{1}{\ell_1^{(m)}}(\Sigma_{11}+\Sigma_1^\xi) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\ell_n^{(m)}}(\Sigma_{nn}+\Sigma_n^\xi) \end{pmatrix}.$$

We also define

$$\Sigma^{\xi,(m)} = \begin{pmatrix} \frac{1}{\ell_1^{(m)}}\Sigma_1^\xi & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\ell_n^{(m)}}\Sigma_n^\xi \end{pmatrix}.$$

Note that $\Sigma^{\xi,(m)}$ is the covariance matrix of $\xi^{(m)}$. Also note that as $m \to \infty$, $\Sigma^{\xi,(m)} \to 0$ and $\widehat{\Sigma}_{diag}^{(m)} \to \widehat{\Sigma}_{diag}^{(\infty)}$, where we define

$$\widehat{\Sigma}_{diag}^{(\infty)} = \begin{pmatrix} \widehat{\Sigma}_{diag}^{(alt)} & 0 \\ 0 & 0 \end{pmatrix}.$$

We could proceed by showing various convergence results directly, by matrix manipulation, as in the proof of the special case of Theorem 2 in Appendix B in the main body of the paper. However, it turns out that the proof becomes simpler and more intuitive if instead we follow the methodology of the proof of the general case of Theorem 2, introduce auxiliary random variables, and interpret various matrices in the proof as covariance matrices of various combinations of these auxiliary random variables and the random variables in the model.

Specifically, for each market m, we introduce a random vector $\widehat{\theta}^{(m)}$, which is independent of the other random variables in the model, and is distributed normally with mean zero and covariance matrix $\widehat{\Sigma}_{diag}^{(m)}$. We also introduce a random vector $\widehat{\theta}_S$, which is independent of the other random variables in the model, and is distributed normally with mean zero and covariance matrix $\widehat{\Sigma}_{diag}^{(alt)}$. Finally, we introduce a random vector $\widehat{\theta}^{(\infty)}$, which is defined as $\widehat{\theta}^{(\infty)} = (\widehat{\theta}_S; 0)$, and is therefore distributed normally with mean zero and covariance matrix $\widehat{\Sigma}_{diag}^{(\infty)}$.

First let us focus on the linear equilibrium in market m. We have

$$\begin{aligned} \widehat{\Lambda}^{(m)} &= \widehat{\Sigma}_{diag}^{(m)} + \Sigma^{\xi,(m)} + Var(\theta|\theta_M) = Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M), \\ \widehat{A}_u^{(m)} &= (\widehat{\Lambda}^{(m)})^{-1} Cov(\theta, u|\theta_M), \\ \widehat{A}_v^{(m)} &= (\widehat{\Lambda}^{(m)})^{-1} Cov(\theta, v|\theta_M). \end{aligned}$$

Finding the linear equilibrium is equivalent to solving the quadratic equation

$$c^{(m)}(\beta_D^{(m)})^2 + b^{(m)}\beta_D^{(m)} + a^{(m)} = 0,$$

where

$$
\begin{aligned}
a^{(m)} &= -(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_v^{(m)}, \\
b^{(m)} &= (\widehat{A}_v^{(m)})^T \left(2\widehat{\Sigma}_{diag}^{(m)} + \widehat{\Lambda}^{(m)}\right) \widehat{A}_u^{(m)} - Cov(u, v|\theta_M), \\
c^{(m)} &= Var((\widehat{A}_u^{(m)})^T(\theta + \xi^{(m)}) - u|\theta_M).
\end{aligned}
$$

Similarly, there exists a unique linear equilibrium of the alternative market. Let

$$
\begin{aligned}
\widehat{\Lambda}^{(alt)} &= \widehat{\Sigma}_{diag}^{(alt)} + Var(\theta_S|\theta_M, \theta_L) = Var(\theta_S + \widehat{\theta}_S|\theta_M, \theta_L), \\
\widehat{A}_u^{(alt)} &= (\widehat{\Lambda}^{(alt)})^{-1} Cov(\theta_S, u|\theta_M, \theta_L), \\
\widehat{A}_v^{(alt)} &= (\widehat{\Lambda}^{(alt)})^{-1} Cov(\theta_S, v|\theta_M, \theta_L).
\end{aligned}
$$

Finding the linear equilibrium is equivalent to solving the quadratic equation

$$
c^{(alt)}(\beta_D^{(alt)})^2 + b^{(alt)}\beta_D^{(alt)} + a^{(alt)} = 0,
$$

where

$$
\begin{aligned}
a^{(alt)} &= -(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_v^{(alt)}, \\
b^{(alt)} &= (\widehat{A}_v^{(alt)})^T \left(2\widehat{\Sigma}_{diag}^{(alt)} + \widehat{\Lambda}^{(alt)}\right) \widehat{A}_u^{(alt)} - Cov(u, v|\theta_M, \theta_L), \\
c^{(alt)} &= Var((\widehat{A}_u^{(alt)})^T \theta_S - u|\theta_M, \theta_L).
\end{aligned}
$$

The equilibrium price in market m is

$$
\begin{aligned}
p^{(m)} &= (\beta_M^{(m)})^T \theta_M + \beta_D^{(m)} \left((\alpha^{(m)})^T \theta^{(m)} + u\right) \\
&= (\beta_M^{(m)})^T \theta_M + \beta_D^{(m)} \left((\alpha_S^{(m)})^T \theta_S + (\alpha_L^{(m)})^T (\theta_L^{(m)} + \xi_L^{(m)}) + u\right),
\end{aligned}
$$

where we "decompose" the vector $\alpha^{(m)}$ as $\alpha^{(m)} = \left(\alpha_S^{(m)}; \alpha_L^{(m)}\right)$.

The equilibrium price in the alternative market is

$$
\begin{aligned}
p^{(alt)} &= (\beta_M^{(alt)})^T \theta_M^{(alt)} + \beta_D^{(alt)} \left((\alpha^{(alt)})^T \theta_S + u\right) \\
&= (\beta_{M,M}^{(alt)})^T \theta_M + (\beta_{M,L}^{(alt)})^T \theta_L + \beta_D^{(alt)} \left((\alpha^{(alt)})^T \theta_S + u\right),
\end{aligned}
$$

where $\theta_M^{(alt)} = (\theta_M; \theta_L)$ and $\beta_M^{(alt)}$ is "decomposed" as $\beta_M^{(alt)} = (\beta_{M,M}^{(alt)}; \beta_{M,L}^{(alt)})$.

We will show in Step 2 that $\beta_D^{(m)} \to \beta_D^{(alt)}$, and then in Step 3 we will show that $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$, $\beta_D^{(m)}\alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$, and $\beta_D^{(m)}\alpha_S^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$. By the same argument as in Step 3 of the proof of the special case of Theorem 2, showing these four convergence results is sufficient to prove the statement of Theorem 3.

Step 2. First, we show that the coefficients of the quadratic equation that $\beta_D^{(m)}$ satisfies converge to those of the quadratic equation that $\beta_D^{(alt)}$ satisfies. As the coefficient on $(\beta_D^{(alt)})^2$ in the latter equation is positive (as shown in Step 5 on the proof of Theorem 1 in Appendix A in the main body of the paper), this convergence implies that $\beta_D^{(m)}$ converges to $\beta_D^{(alt)}$.

Step 2(a). We first show that $a^{(m)} \to a^{(alt)}$. We have

$$\widehat{\Sigma}_{diag}^{(m)} \to \widehat{\Sigma}_{diag}^{(\infty)} := Var((\widehat{\theta}_S; 0)),$$

thus

$$\widehat{\Lambda}^{(m)} \to \widehat{\Lambda}^{(\infty)} := Var((\theta_S + \widehat{\theta}_S; \theta_L)|\theta_M),$$

and, as $\widehat{\Lambda}^{(\infty)}$ is positive definite (which follows from Assumption 2H),

$$\begin{aligned}
\widehat{A}_v^{(m)} \to \widehat{A}_v^{(\infty)} &:= (\widehat{\Lambda}^{(\infty)})^{-1} Cov(\theta, v|\theta_M) \\
&= Var((\theta_S + \widehat{\theta}_S; \theta_L)|\theta_M)^{-1} Cov((\theta_S + \widehat{\theta}_S; \theta_L), v|\theta_M).
\end{aligned}$$

This identity implies that for any (fixed) vectors $\widetilde{\theta}_S$ (of the same dimension as random vector θ_S) and $\widetilde{\theta}_L$ (of the same dimension as random vector θ_L), we have

$$(\widehat{A}_v^{(\infty)})^T (\widetilde{\theta}_S; \widetilde{\theta}_L) = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = \widetilde{\theta}_L]. \tag{OA.16}$$

Now, note that

$$\begin{aligned}
a^{(m)} \to a^{(\infty)} &:= -(\widehat{A}_v^{(\infty)})^T \widehat{\Sigma}_{diag}^{(\infty)} \widehat{A}_v^{(\infty)} \\
&= -(\widehat{A}_v^{(\infty)})^T Var((\widehat{\theta}_S; 0)) \widehat{A}_v^{(\infty)} \\
&= -Var\left((\widehat{A}_v^{(\infty)})^T (\widehat{\theta}_S; 0)\right).
\end{aligned}$$

Likewise, for any (fixed) vector $\widetilde{\theta}_S$ (of the same dimension as θ_S), we have

$$(\widehat{A}_v^{(alt)})^T\widetilde{\theta}_S = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0]. \tag{OA.17}$$

Also,

$$\begin{aligned} a^{(alt)} &= -(\widehat{A}_v^{(alt)})^T\widehat{\Sigma}_{diag}^{(alt)}\widehat{A}_v^{(alt)} \\ &= -(\widehat{A}_v^{(alt)})^T Var(\widehat{\theta}_S)\widehat{A}_v^{(alt)} \\ &= -Var\left((\widehat{A}_v^{(alt)})^T\widehat{\theta}_S\right). \end{aligned}$$

Equations (OA.16) and (OA.17) imply that for every realization $\widetilde{\theta}_S$ of random vector $\widehat{\theta}_S$,

$$\begin{aligned} (\widehat{A}_v^{(\infty)})^T(\widetilde{\theta}_S; 0) &= E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0] \\ &= (\widehat{A}_v^{(alt)})^T\widetilde{\theta}_S, \end{aligned}$$

and thus

$$Var\left((\widehat{A}_v^{(\infty)})^T(\widehat{\theta}_S; 0)\right) = Var\left((\widehat{A}_v^{(alt)})^T\widehat{\theta}_S\right)$$

and so $a^{(m)} \to a^{(\infty)} = a^{(alt)}$.

Step 2(b). Next, we show that $b^{(m)} \to b^{(alt)}$. In the limit,

$$b^{(m)} \to b^{(\infty)} := (\widehat{A}_v^{(\infty)})^T\left(2\widehat{\Sigma}_{diag}^{(\infty)} + \widehat{\Lambda}^{(\infty)}\right)\widehat{A}_u^{(\infty)} - Cov(u, v|\theta_M),$$

where

$$\widehat{A}_u^{(\infty)} := \lim_{m\to\infty}\widehat{A}_u^{(m)} = (\widehat{\Lambda}^{(\infty)})^{-1}Cov(\theta, u|\theta_M).$$

Similarly to equations (OA.16) and (OA.17) above, for any fixed vectors $\widetilde{\theta}_S$ and $\widetilde{\theta}_L$, we have

$$(\widehat{A}_u^{(\infty)})^T(\widetilde{\theta}_S; \widetilde{\theta}_L) = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = \widetilde{\theta}_L]; \tag{OA.18}$$

$$(\widehat{A}_u^{(alt)})^T\widetilde{\theta}_S = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0]. \tag{OA.19}$$

Note that

$$
\begin{aligned}
(\widehat{A}_v^{(\infty)})^T \widehat{\Sigma}_{diag}^{(\infty)} \widehat{A}_u^{(\infty)} &= (\widehat{A}_v^{(\infty)})^T Var((\widehat{\theta}_S;0)) \widehat{A}_u^{(\infty)} \\
&= Cov\left((\widehat{A}_v^{(\infty)})^T (\widehat{\theta}_S;0), (\widehat{A}_u^{(\infty)})^T (\widehat{\theta}_S;0) \right)
\end{aligned}
$$

and

$$
\begin{aligned}
(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)} &= (\widehat{A}_v^{(alt)})^T Var(\widehat{\theta}_S) \widehat{A}_u^{(alt)} \\
&= Cov\left((\widehat{A}_v^{(alt)})^T \widehat{\theta}_S, (\widehat{A}_u^{(alt)})^T \widehat{\theta}_S \right).
\end{aligned}
$$

By equations (OA.16)–(OA.19), for any realization $\widetilde{\theta}_S$ of random vector $\widehat{\theta}_S$,

$$
\begin{aligned}
&(\widehat{A}_v^{(\infty)})^T (\widetilde{\theta}_S;0) = (\widehat{A}_v^{(alt)})^T \widetilde{\theta}_S \quad \text{and} \\
&(\widehat{A}_u^{(\infty)})^T (\widetilde{\theta}_S;0) = (\widehat{A}_u^{(alt)})^T \widetilde{\theta}_S,
\end{aligned}
$$

and so

$$
(\widehat{A}_v^{(\infty)})^T \widehat{\Sigma}_{diag}^{(\infty)} \widehat{A}_u^{(\infty)} = (\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)}.
$$

Next, note that

$$
(\widehat{A}_v^{(\infty)})^T \widehat{\Lambda}^{(\infty)} \widehat{A}_u^{(\infty)} = Cov(v, (\theta_S+\widehat{\theta}_S;\theta_L)|\theta_M)[Var((\theta_S+\widehat{\theta}_S;\theta_L)|\theta_M)]^{-1} Cov((\theta_S+\widehat{\theta}_S;\theta_L), u|\theta_M)
$$

and so

$$
(\widehat{A}_v^{(\infty)})^T \widehat{\Lambda}^{(\infty)} \widehat{A}_u^{(\infty)} - Cov(u,v|\theta_M) = -Cov(u,v|\theta_M,\theta_L,\theta_S+\widehat{\theta}_S).
$$

Similarly,

$$
(\widehat{A}_v^{(alt)})^T \widehat{\Lambda}^{(alt)} \widehat{A}_u^{(alt)} = Cov(v, \theta_S+\widehat{\theta}_S|\theta_M,\theta_L)[Var(\theta_S+\widehat{\theta}_S|\theta_M,\theta_L)]^{-1} Cov(\theta_S+\widehat{\theta}_S, u|\theta_M,\theta_L),
$$

and so

$$
(\widehat{A}_v^{(alt)})^T \widehat{\Lambda}^{(alt)} \widehat{A}_u^{(alt)} - Cov(u,v|\theta_M,\theta_L) = -Cov(u,v|\theta_M,\theta_L,\theta_S+\widehat{\theta}_S).
$$

Therefore, we have

$$
b^{(m)} \to b^{(\infty)} = 2(\widehat{A}_v^{(\infty)})^T \widehat{\Sigma}_{diag}^{(\infty)} \widehat{A}_u^{(\infty)} + \left((\widehat{A}_v^{(\infty)})^T \widehat{\Lambda}^{(\infty)} \widehat{A}_u^{(\infty)} - Cov(u,v|\theta_M) \right)
$$

$$
\begin{aligned}
&= 2(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)} - Cov(u, v|\theta_M, \theta_L, \theta_S + \widehat{\theta}_S) \\
&= b^{(alt)}.
\end{aligned}
$$

Step 2(c). Finally, we show that $c^{(m)} \to c^{(alt)}$. We have

$$
c^{(m)} \to c^{(\infty)} := Var((\widehat{A}_u^{(\infty)})^T \theta - u|\theta_M)
$$

and

$$
c^{(alt)} = Var((\widehat{A}_u^{(alt)})^T \theta_S - u|\theta_M, \theta_L).
$$

Let random variable χ be the residual from the projection of u on $(\theta_S + \widehat{\theta}_S; \theta_L; \theta_M)$. By construction, χ is orthogonal to θ_L and θ_M and thus, by the properties of the normal distribution, is independent of those two random variables. Recall that $\widehat{\theta}_S$ was also chosen to be independent of θ_L and θ_M.

Next,

$$
\begin{aligned}
Var((\widehat{A}_u^{(\infty)})^T \theta - u|\theta_M) &= Var\left(u - (\widehat{A}_u^{(\infty)})^T (\theta_S + \widehat{\theta}_S; \theta_L) + (\widehat{A}_u^{(\infty)})^T (\widehat{\theta}_S; 0)|\theta_M\right) \\
&= Var\left(\chi + (\widehat{A}_u^{(\infty)})^T (\widehat{\theta}_S; 0)|\theta_M\right)
\end{aligned}
$$

and

$$
\begin{aligned}
Var((\widehat{A}_u^{(alt)})^T \theta_S - u|\theta_M, \theta_L) &= Var\left(u - (\widehat{A}_u^{(alt)})^T (\theta_S + \widehat{\theta}_S) + (\widehat{A}_u^{(alt)})^T \widehat{\theta}_S|\theta_M, \theta_L\right) \\
&= Var\left(\chi + (\widehat{A}_u^{(alt)})^T \widehat{\theta}_S|\theta_M, \theta_L\right).
\end{aligned}
$$

Since χ and $\widehat{\theta}_S$ are both independent of θ_M and θ_L, we have

$$
Var\left(\chi + (\widehat{A}_u^{(\infty)})^T (\widehat{\theta}_S; 0)|\theta_M\right) = Var\left(\chi + (\widehat{A}_u^{(\infty)})^T (\widehat{\theta}_S; 0)\right)
$$

and

$$
Var\left(\chi + (\widehat{A}_u^{(alt)})^T \widehat{\theta}_S|\theta_M, \theta_L\right) = Var\left(\chi + (\widehat{A}_u^{(alt)})^T \widehat{\theta}_S\right).
$$

Take any realizations $\widetilde{\chi}$ and $\widetilde{\theta}_S$ of random variables χ and $\widehat{\theta}_S$. From equations (OA.18) and (OA.19) in Step 2(b), we have

$$\begin{aligned}\widetilde{\chi} + (\widehat{A}_u^{(\infty)})^T(\widetilde{\theta}_S; 0) &= \widetilde{\chi} + E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0] \\ &= \widetilde{\chi} + (\widehat{A}_u^{(alt)})^T\widetilde{\theta}_S,\end{aligned}$$

and so

$$Var\left(\chi + (\widehat{A}_u^{(\infty)})^T(\widehat{\theta}_S; 0)\right) = Var\left(\chi + (\widehat{A}_u^{(alt)})^T\widehat{\theta}_S\right)$$

and thus

$$c^{(m)} \to c^{(\infty)} = c^{(alt)}.$$

Step 3. We now show that $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$, $\beta_D^{(m)}\alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$, and $\beta_D^{(m)}\alpha_S^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$. The arguments below rely on Assumption 2H, which implies that various conditional expectations that we compute below are guaranteed to be well defined. They also rely on the result we showed in the previous step, $\beta_D^{(\infty)} = \beta_D^{(alt)}$.

First, note that for any $\widetilde{\theta}_S$, $(\alpha_S^{(m)})^T\widetilde{\theta}_S = (\alpha^{(m)})^T(\widetilde{\theta}_S; 0)$, and so

$$\begin{aligned}\lim_{m\to\infty} \beta_D^{(m)}(\alpha^{(m)})^T(\widetilde{\theta}_S; 0) &= \beta_D^{(\infty)}\left((\widehat{A}_v^{(\infty)})^T/\beta_D^{(\infty)} - (\widehat{A}_u^{(\infty)})^T\right)(\widetilde{\theta}_S; 0) \\ &= E[v - \beta_D^{(\infty)}u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0] \\ &= \beta_D^{(alt)}(\alpha_S^{(alt)})^T\widetilde{\theta}_S.\end{aligned}$$

Thus, $\beta_D^{(m)}\alpha_S^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$.

Next, we have

$$\beta_M^{(m)} \to \beta_M^{(\infty)} := \Sigma_{MM}^{-1}\left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v^{(\infty)}\right) - \beta_D^{(\infty)}\Sigma_{MM}^{-1}\left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u^{(\infty)}\right),$$

and so for any $\widetilde{\theta}_M$, we have

$$(\beta_M^{(\infty)})^T\widetilde{\theta}_M = E[v - \beta_D^{(\infty)}u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0].$$

Also, similarly to the above expression for $\beta_D^{(\infty)}\alpha_S^{(\infty)}$, for any $\widetilde{\theta}_L$, we have

$$\beta_D^{(\infty)}(\alpha_L^{(\infty)})^T\widetilde{\theta}_L = E[v - \beta_D^{(\infty)}u|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L = \widetilde{\theta}_L].$$

Thus,

$$(\beta_M^{(\infty)}; \beta_D^{(\infty)} \alpha_L^{(\infty)})^T (\widetilde{\theta}_M; \widetilde{\theta}_L) = E[v - \beta_D^{(\infty)} u | \theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = \widetilde{\theta}_L].$$

Analogously to the expression for $(\beta_M^{(\infty)})^T \widetilde{\theta}_M$, we also have

$$\begin{aligned}(\beta_M^{(alt)})^T (\widetilde{\theta}_M; \widetilde{\theta}_L) &= E[v - \beta_D^{(alt)} u | \theta_S + \widehat{\theta}_S = 0, (\theta_M; \theta_L) = (\widetilde{\theta}_M; \widetilde{\theta}_L)] \\ &= (\beta_M^{(\infty)}; \beta_D^{(\infty)} \alpha_L^{(\infty)})^T (\widetilde{\theta}_M; \widetilde{\theta}_L).\end{aligned}$$

Thus, $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$ and $\beta_D^{(m)} \alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$, and combining all the convergence results above and using the same argument as in Step 3 of the proof of the special case of Theorem 2 in Appendix B in the main body of the paper, we conclude the proof of Theorem 3.

10 Proof of Theorem 3 (General Case)

The proof in Section 9 applies to the special case in which the covariance matrix of random vector $(\theta_S; \theta_L; \theta_M; u)$ is full rank. In this section, we prove Theorem 3 for the general case, imposing only Assumptions 1H and 2H: $Cov(v, \theta_S | \theta_L, \theta_M) \neq 0$ and $Var(u | \theta_L, \theta_M) > 0$.

In addition to the markets indexed $m = 1, 2, \ldots$, we consider the alternative market which includes s groups of traders $i = 1, \ldots, s$. The size of group i is ℓ_i. Each trader j of group i receives the same signal θ_i. In this alternative market, the market maker receives signal $(\theta_L; \theta_M)$. We use superscript (m) when we refer to the variables of the market m, and we use superscript (alt) when we refer to the variables in the alternative market. By Theorem 1 a unique linear equilibrium exists for each market m and for the alternative market.

For $i = s+1, \ldots, n$, we define $\xi_i^{(m)} = (\sum_j \xi_{i,j}) / \ell_i^{(m)}$ and $\xi_L^{(m)} = (\xi_{s+1}^{(m)}; \ldots; \xi_n^{(m)})$. For $i \leq s$, we let $\Sigma_i^\xi = 0$, $\xi_i^{(m)} = 0$, and $\xi_S^{(m)} = (\xi_1^{(m)}; \ldots; \xi_s^{(m)})$. Let $\xi^{(m)} = (\xi_1^{(m)}; \ldots; \xi_n^{(m)})$.

We also define

$$\widehat{\Sigma}_{diag}^{(alt)} = \begin{pmatrix} \frac{1}{\ell_1} \Sigma_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\ell_s} \Sigma_{ss} \end{pmatrix}$$

and

$$\widehat{\Sigma}^{(m)}_{diag} = \begin{pmatrix} \frac{1}{\ell_1^{(m)}}(\Sigma_{11}+\Sigma_1^{\xi}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\ell_n^{(m)}}(\Sigma_{nn}+\Sigma_n^{\xi}) \end{pmatrix}$$

where $\ell_i^{(m)} = \ell_i$ if $i \leq s$. Note that

$$\widehat{\Sigma}^{(m)}_{diag} \to \widehat{\Sigma}^{(\infty)}_{diag} := \begin{pmatrix} \widehat{\Sigma}^{(alt)}_{diag} & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, we define

$$\Sigma^{\xi,(m)} = \begin{pmatrix} \frac{1}{\ell_1^{(m)}}\Sigma_1^{\xi} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{\ell_n^{(m)}}\Sigma_n^{\xi} \end{pmatrix}.$$

Note that $\Sigma^{\xi,(m)}$ is the covariance matrix of the random vector $\xi^{(m)}$, and that

$$\lim_{m\to\infty} \Sigma^{\xi,(m)} = 0.$$

We follow the methodology of the proof of the general case of Theorem 2 and the special case of Theorem 3. We introduce auxiliary random variables, and interpret various matrices in the proof as covariance matrices of combinations of these auxiliary random variables and the original random variables in the model.

Specifically, for each market m, we introduce a random vector $\widehat{\theta}^{(m)}$, which is independent of the other random variables in the model, and is distributed normally with mean zero and covariance matrix $\widehat{\Sigma}^{(m)}_{diag}$. We also introduce a random vector $\widehat{\theta}_S$, which is independent of the other random variables in the model, and is distributed normally with mean zero and covariance matrix $\widehat{\Sigma}^{(alt)}_{diag}$. Finally, we introduce a random vector $\widehat{\theta}^{(\infty)} = (\widehat{\theta}_S; 0)$, which is distributed normally with mean zero and covariance matrix $\widehat{\Sigma}^{(\infty)}_{diag}$.

The remainder of the proof assumes that $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S) \neq 0$. This assumption is without loss of generality: if the signals of the large groups are such that $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S) = 0$, we can append a randomly and independently distributed signal to the common signals component of one large group, so that the conditional variance becomes nonzero. By Proposition OA.3 in Section 6, traders never trade based on this additional, uninformative signal, and the equilibrium outcome is not impacted by its presence.

Finally, we define the following matrices and vectors, similarly as in the proof of Theorem 1. For market m, we let

$$\begin{aligned}\widehat{\Lambda}^{(m)} &= Var(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}|\theta_M),\\ \widehat{A}_u^{(m)} &= (\widehat{\Lambda}^{(m)})^{-1}Cov(\theta, u|\theta_M),\\ \widehat{A}_v^{(m)} &= (\widehat{\Lambda}^{(m)})^{-1}Cov(\theta, v|\theta_M).\end{aligned}$$

For the alternative market, we let

$$\begin{aligned}\widehat{\Lambda}^{(alt)} &= Var(\theta_S + \widehat{\theta}_S|\theta_M, \theta_L),\\ \widehat{A}_u^{(alt)} &= (\widehat{\Lambda}^{(alt)})^{-1}Cov(\theta_S, u|\theta_M, \theta_L),\\ \widehat{A}_v^{(alt)} &= (\widehat{\Lambda}^{(alt)})^{-1}Cov(\theta_S, v|\theta_M, \theta_L).\end{aligned}$$

We remark that, for a vector $\widetilde{\theta}$ of the same dimensionality as θ,

$$(\widehat{A}_v^{(m)})^T\widetilde{\theta} = E[v \mid \theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = \widetilde{\theta}_L]$$

where we decompose $\widetilde{\theta}$ as $(\widetilde{\theta}_S; \widetilde{\theta}_L)$. We get an analogous expression for $\widehat{A}_u^{(m)}$. Similarly, we have

$$(\widehat{A}_v^{(alt)})^T\widetilde{\theta}_S = E[v \mid \theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S]$$

and we get an analogous expression for $\widehat{A}_u^{(alt)}$.

Change of Basis

As in the proof of the general case of Theorem 2 in Section 8, to handle the problem of covariance matrices that are not positive definite, we perform a change of basis.

Let Φ be an orthogonal matrix such that

$$\Phi^T Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S)\Phi = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix},$$

where M is a symmetric positive definite matrix whose size is the rank of $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S)$. Since, by assumption, $Var(\theta_L|\theta_M, \theta_S + \widehat{\theta}_S) \neq 0$, the size of matrix M is greater than zero.

Let $(\theta_L'; \theta_L'')$ be the random vector defined as $\Phi^T \theta_L$, where the dimensionality of θ_L' is equal to the rank of M. Note that $Var(\theta_L' | \theta_M, \theta_S + \widehat{\theta}_S) = M$ and that $\theta_L'' = 0$. In a similar fashion, we let $((\widehat{\theta}_L^{(m)})'; (\widehat{\theta}_L^{(m)})'') = \Phi^T \widehat{\theta}_L^{(m)}$ and $((\xi_L^{(m)})'; (\xi_L^{(m)})'') = \Phi^T \xi_L^{(m)}$. For a vector $\widetilde{\theta}_L$ of the same dimensionality as θ_L, define $(\widetilde{\theta}_L'; \widetilde{\theta}_L'') = \Phi^T \widetilde{\theta}_L$.

We let $\chi_L^{(m)}$ be the residual of the projection of $(\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})'$ on $(\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})''$, i.e., $\chi_L^{(m)} = (\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})' - E[(\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})' | (\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'']$. We have already shown in the general proof of Theorem 2 in Section 8 that $Var(\chi_L^{(m)}) \to 0$.

For a vector $\widetilde{\theta}_L$ of the same dimensionality as θ_L, we define

$$f(\widetilde{\theta}_L) = (\widetilde{\theta}_L)' - E[(\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})' | (\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'' = (\widetilde{\theta}_L)''].$$

In addition, let

$$\widehat{B}_v^{(m)} = [Var((\theta_S + \widehat{\theta}_S; \theta_L' + \chi_L^{(m)}) | \theta_M)]^{-1} Cov((\theta_S; \theta_L'), v \mid \theta_M)$$

and note that since $(\theta_S + \widehat{\theta}_S; \theta_L'; \theta_M)$ is not degenerate, and that $Var(\chi_L^{(m)}) \to 0$ and $Var(\widehat{\theta}_L^{(m)}) \to 0$, we have the following limit:

$$\lim_{m \to \infty} \widehat{B}_v^{(m)} = \widehat{B}_v^{(\infty)} := [Var((\theta_S + \widehat{\theta}_S; \theta_L') | \theta_M]^{-1} Cov((\theta_S; \theta_L'), v | \theta_M).$$

Similarly, define

$$\widehat{B}_u^{(m)} = [Var((\theta_S + \widehat{\theta}_S; \theta_L' + \chi_L^{(m)}) | \theta_M)]^{-1} Cov((\theta_S; \theta_L'), u \mid \theta_M)$$

and note that we have the following limit:

$$\lim_{m \to \infty} \widehat{B}_u^{(m)} = \widehat{B}_u^{(\infty)} := [Var((\theta_S + \widehat{\theta}_S; \theta_L') | \theta_M)]^{-1} Cov((\theta_S; \theta_L'), u \mid \theta_M).$$

Finally, let

$$\widehat{C}_v = [Var((\theta_S + \widehat{\theta}_S; \theta_L) | \theta_M)]^{-1} Cov((\theta_S; \theta_L), v \mid \theta_M)$$

and

$$\widehat{C}_u = [Var((\theta_S + \widehat{\theta}_S; \theta_L) | \theta_M)]^{-1} Cov((\theta_S; \theta_L), u \mid \theta_M),$$

where the matrix inverse in the last two equations denotes a Moore-Penrose pseudoinverse, because the random vector $(\theta_L; \theta_M)$ may be degenerate. Note that, for $\widetilde{\theta}_L$ a realization of θ_L, and any vector $\widetilde{\theta}_S$ of the same dimension as θ_S, we still have the equality

$$(\widehat{C}_v)^T(\widetilde{\theta}_S; \widetilde{\theta}_L) = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = \widetilde{\theta}_L],$$

and similarly

$$(\widehat{C}_u)^T(\widetilde{\theta}_S; \widetilde{\theta}_L) = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = \widetilde{\theta}_L].$$

Auxiliary Results

Next, we first state and prove several auxiliary results.

Auxiliary Result (a). First, we prove that

$$\lim_{m\to\infty} Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)}) = Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L).$$

We have, noting that $\theta_L'' = 0$,

$$\begin{aligned} &Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)}) \\ &\quad = Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L' + (\widehat{\theta}_L^{(m)})' + (\xi_L^{(m)})', (\widehat{\theta}_L^{(m)})'' + (\xi_L^{(m)})'') \\ &\quad = Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L' + \chi_L^{(m)}). \end{aligned}$$

The covariance matrix of $(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L')$ is positive definite. As the covariance matrix of $\chi_L^{(m)}$ converges to zero, we get

$$\begin{aligned} \lim_{m\to\infty} Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L' + \chi_L^{(m)}) &= Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L') \\ &= Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L), \end{aligned}$$

which yields the desired result.

Auxiliary Result (b). Next we prove that, for every vector $\widetilde{\theta}_S$ of the same dimensionality as θ_S, we have the limit

$$\lim_{m\to\infty} E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)} = 0] = E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0].$$

Indeed,

$$
\begin{aligned}
&E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)} = 0\right] \\
&\quad = E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L + (\widehat{\theta}_L^{(m)})' + (\xi_L^{(m)})' = 0, (\widehat{\theta}_L^{(m)})'' + (\xi_L^{(m)})'' = 0\right] \\
&\quad = E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L + \chi_L^{(m)} = 0, (\widehat{\theta}_L^{(m)})'' + (\xi_L^{(m)})'' = 0\right] \\
&\quad = E\left[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L + \chi_L^{(m)} = 0\right].
\end{aligned}
$$

Observing that the covariance matrix of $(\theta_M, \theta_S + \widehat{\theta}_S, \theta'_L)$ is positive definite and the covariance matrix of $\chi_L^{(m)}$ converges to zero yields the desired result.

Auxiliary Result (c). We get, in the same way, that for every vector $\widetilde{\theta}_S$ of the same dimensionality as θ_S, we have the limit

$$
\lim_{m\to\infty} E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] = E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = 0]
$$

and that for every vector $\widetilde{\theta}_M$ of the same dimensionality as θ_M, we have the limit

$$
\begin{aligned}
\lim_{m\to\infty} E[v|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] &= E[v|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0], \\
\lim_{m\to\infty} E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = 0] &= E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0].
\end{aligned}
$$

Auxiliary Result (d). Next, we prove that

$$
\lim_{m\to\infty} Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; \widehat{\theta}_L^{(m)})) = Var((\widehat{A}_v^{alt})^T \widehat{\theta}_S).
$$

We have, using the independence of the two random vectors $\widehat{\theta}_S$ and $\widehat{\theta}_L^{(m)}$, the equality

$$
Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; \widehat{\theta}_L^{(m)})) = Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; 0)) + Var((\widehat{A}_v^m)^T(0; \widehat{\theta}_L^{(m)})).
$$

By Auxiliary Result (b),

$$
\lim_{m\to\infty} Var((\widehat{A}_v^m)^T(\widehat{\theta}_S; 0)) = Var((\widehat{A}_v^{alt})^T \widehat{\theta}_S),
$$

so it remains to prove that

$$
\lim_{m\to\infty} Var((\widehat{A}_v^m)^T(0; \widehat{\theta}_L^{(m)})) = 0. \tag{OA.20}
$$

We have

$$
\begin{aligned}
Var((\widehat{A}_v^m)^T(0;\widehat{\theta}_L^{(m)})) &\le Var((\widehat{A}_v^m)^T(0;\widehat{\theta}_L^{(m)})) + Var((\widehat{A}_v^m)^T(0;\xi_L^{(m)})) \\
&= Var((\widehat{A}_v^m)^T(0;\widehat{\theta}_L^{(m)}+\xi_L^{(m)})).
\end{aligned}
$$

We have, for any vector $\widetilde{\theta}_L$ of the same dimension as θ_L,

$$
\begin{aligned}
&E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = \widetilde{\theta}_L] \\
&= E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L' + (\xi_L^{(m)})' + (\widehat{\theta}_L^{(m)})' = (\widetilde{\theta}_L)', (\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'' = (\widetilde{\theta}_L)''] \\
&= E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L' + \chi_L^{(m)} = f(\widetilde{\theta}_L), (\xi_L^{(m)})'' + (\widehat{\theta}_L^{(m)})'' = (\widetilde{\theta}_L)''] \\
&= E[v|\theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L' + \chi_L^{(m)} = f(\widetilde{\theta}_L)].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Var((\widehat{A}_v^{(m)})^T(0;\widehat{\theta}_L^{(m)}+\xi_L^{(m)})) &= Var((\widehat{B}_v^{(m)})^T(0;f(\widehat{\theta}_L^{(m)}+\xi_L^{(m)}))) \\
&= Var((\widehat{B}_v^{(m)})^T(0;f(\chi_L^{(m)}))) \\
&\to 0
\end{aligned}
$$

since $\widehat{B}_v^{(m)} \to \widehat{B}_v^{(\infty)}$ and $Var(\chi_L^{(m)}) \to 0$. Limit (OA.20) follows.

Auxiliary Result (e). Next, we show that

$$
(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} \to (\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)}. \qquad \text{(OA.21)}
$$

We note that

$$
\begin{aligned}
(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} &= Cov((\widehat{A}_v^{(m)})^T(\widehat{\theta}_S;\widehat{\theta}_L^{(m)}), (\widehat{A}_u^{(m)})^T(\widehat{\theta}_S;\widehat{\theta}_L^{(m)})) \\
&= Cov((\widehat{B}_v^{(m)})^T(\widehat{\theta}_S;f(\widehat{\theta}_L^{(m)})), (\widehat{B}_u^{(m)})^T(\widehat{\theta}_S;f(\widehat{\theta}_L^{(m)}))), \\
&\to Cov((\widehat{B}_v^{(\infty)})^T(\widehat{\theta}_S;0), (\widehat{B}_u^{(\infty)})^T(\widehat{\theta}_S;0)),
\end{aligned}
$$

using that, by linearity of f and independence of $\widehat{\theta}_L^{(m)}$ and $\xi_L^{(m)}$, $Var(f(\widehat{\theta}_L^{(m)})) \le Var(f(\widehat{\theta}_L^{(m)} + \xi_L^{(m)})) = Var(\chi_L^m) \to 0$, and also, $\widehat{B}_u^{(m)} \to \widehat{B}_u^{(\infty)}$ and $\widehat{B}_v^{(m)} \to \widehat{B}_v^{(\infty)}$.

In addition, for any vector $\widetilde{\theta}_S$ of the same dimensionality as θ_S, we can write

$$
(\widehat{B}_v^{(\infty)})^T(\widetilde{\theta}_S;0) = E[v|\theta_M = 0, \theta_L = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S] = (\widehat{A}_v^{(alt)})^T \widetilde{\theta}_S
$$

and

$$(\widehat{B}_u^{(\infty)})^T(\widetilde{\theta}_S;0) = E[u|\theta_M = 0, \theta_L = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S] = (\widehat{A}_u^{(alt)})^T\widetilde{\theta}_S.$$

Thus,

$$Cov((\widehat{B}_v^{(\infty)})^T(\widehat{\theta}_S;0), (\widehat{B}_u^{(\infty)})^T(\widehat{\theta}_S;0)) = Cov((\widehat{A}_v^{(alt)})^T\widehat{\theta}_S, (\widehat{A}_u^{(\infty)})^T\widehat{\theta}_S)) = (\widehat{A}_v^{(alt)})^T\widehat{\Sigma}_{diag}^{(alt)}\widehat{A}_u^{(alt)},$$

which allows to get Limit (OA.21).

Auxiliary Result (f). Next, we show that

$$\lim_{m\to\infty} Var((\widehat{A}_u^{(m)})^T(\theta_S;\theta_L + \xi_L^{(m)}) - u|\theta_M) = Var((\widehat{C}_u)^T(\theta_S;\theta_L) - u|\theta_M). \quad \text{(OA.22)}$$

Since $\xi_L^{(m)}$ is independent of $(\theta_S;\theta_L)$, u, and θ_M, we have

$$Var((\widehat{A}_u^{(m)})^T(\theta_S;\theta_L+\xi_L^{(m)})-u|\theta_M) = Var((\widehat{A}_u^{(m)})^T(\theta_S;\theta_L)-u|\theta_M)+Var((\widehat{A}_u^{(m)})^T(0;\xi_L^{(m)})).$$

In Auxiliary Result (d) we have shown in Limit (OA.20) that $Var((\widehat{A}_v^{(m)})^T(0;\xi_L^{(m)})) \to 0$, and similarly we have $Var((\widehat{A}_u^{(m)})^T(0;\xi_L^{(m)})) \to 0$. It remains to show that

$$\lim_{m\to\infty} Var((\widehat{A}_u^{(m)})^T(\theta_S;\theta_L) - u|\theta_M) = Var((\widehat{C}_u)^T(\theta_S;\theta_L) - u|\theta_M). \quad \text{(OA.23)}$$

By the same argument as in Auxiliary Result (d), for any vector $\widetilde{\theta}_S$ of the same dimension as θ_S, and any vector $\widetilde{\theta}_L$ of the same dimension as θ_L,

$$\begin{aligned}(\widehat{B}_u^{(m)})^T(\widetilde{\theta}_S;\widetilde{\theta}_L) &= E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L + \xi_L^{(m)} + \widehat{\theta}_L^{(m)} = \widetilde{\theta}_L] \\ &= E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L' + \chi_L^{(m)} = f(\widetilde{\theta}_L)],\end{aligned}$$

and since the random vector $(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L')$ is not degenerate, and $Var(\chi_L^{(m)}) \to 0$, we have the limit

$$\begin{aligned}\lim_{m\to\infty} & E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L' + \chi_L^{(m)} = f(\widetilde{\theta}_L)] \\ &= E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L' = f(\widetilde{\theta}_L)].\end{aligned}$$

Note that, since $\theta_L'' = 0$, $f(\theta_L) = \theta_L'$. Thus, for any realization $\widetilde{\theta}_L$ of θ_L,

$$E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta'_L = f(\widetilde{\theta}_L)]$$
$$= E[u|\theta_M = 0, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \theta_L = \widetilde{\theta}_L] = (\widehat{C}_u)^T(\widetilde{\theta}_S; \widetilde{\theta}_L).$$

Therefore Limit (OA.23) obtains.

Auxiliary Result (g). This result is an equality. We show that

$$Var((\widehat{C}_u)^T(\theta_S; \theta_L) - u|\theta_M) = Var((\widehat{C}_u)^T(\theta_S; 0) - u|\theta_M, \theta_L).$$

For any vectors $\widetilde{\theta}_M$, $\widetilde{\theta}_S$, $\widetilde{\theta}_L$, let

$$\eta(\widetilde{\theta}_M, \widetilde{\theta}_S, \widetilde{\theta}_L) = u - E[u|\theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = \widetilde{\theta}_S, \widetilde{\theta}_L = \widetilde{\theta}_L].$$

Note that η is linear in its arguments. Also, since $\eta(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L)$ is the residual of the projection of u on $(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L)$, it is independent of the latter random vector. Therefore,

$$\begin{aligned}
Var(u - (\widehat{C}_u)^T(\theta_S; \theta_L)|\theta_M) &= Var(\eta(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L) + (\widehat{C}_u)^T(\widehat{\theta}_S; 0)|\theta_M) \\
&= Var(\eta(\theta_M, \theta_S + \widehat{\theta}_S, \theta_L) + (\widehat{C}_u)^T(\widehat{\theta}_S; 0)|\theta_M, \theta_L) \\
&= Var(\eta(0, \theta_S + \widehat{\theta}_S, 0) + (\widehat{C}_u)^T(\widehat{\theta}_S; 0)|\theta_M, \theta_L) \\
&= Var(u - (\widehat{C}_u)^T(\theta_S + \widehat{\theta}_S; 0) + (\widehat{C}_u)^T(\widehat{\theta}_S; 0)|\theta_M, \theta_L) \\
&= Var(u - (\widehat{C}_u)^T(\theta_S; 0)|\theta_M, \theta_L).
\end{aligned}$$

Main Body of the Proof

We are now ready to prove Theorem 3. The proof proceeds in three steps.

Step 1. First let us focus on the linear equilibrium in market (m). Following Theorem 1, finding the linear equilibrium is equivalent to solving the quadratic equation

$$c^{(m)}(\beta_D^{(m)})^2 + b^{(m)}\beta_D^{(m)} + a^{(m)} = 0,$$

where

$$a^{(m)} = -(\widehat{A}_v^{(m)})^T\widehat{\Sigma}_{diag}^{(m)}\widehat{A}_v^{(m)},$$

$$
\begin{aligned}
b^{(m)} &= (\widehat{A}_v^{(m)})^T \left(2\widehat{\Sigma}_{diag}^{(m)} + \widehat{\Lambda}^{(m)}\right) \widehat{A}_u^{(m)} - Cov(u, v|\theta_M),\\
c^{(m)} &= Var((\widehat{A}^{(m)})_u^T(\theta + \xi^{(m)}) - u|\theta_M).
\end{aligned}
$$

Similarly, there exists a unique linear equilibrium of the alternative market, and inding the linear equilibrium is equivalent to solving the quadratic equation

$$
c^{(alt)}(\beta_D^{(alt)})^2 + b^{(alt)}\beta_D^{(alt)} + a^{(alt)} = 0,
$$

where

$$
\begin{aligned}
a^{(alt)} &= -(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_v^{(alt)},\\
b^{(alt)} &= (\widehat{A}_v^{(alt)})^T \left(2\widehat{\Sigma}_{diag}^{(alt)} + \widehat{\Lambda}^{(alt)}\right) \widehat{A}_u^{(alt)} - Cov(u, v|\theta_M, \theta_L),\\
c^{(alt)} &= Var((\widehat{A}^{(alt)})_u^T \theta_S - u|\theta_M, \theta_L).
\end{aligned}
$$

The equilibrium price in market (m) is

$$
\begin{aligned}
p^{(m)} &= (\beta_M^{(m)})^T \theta_M + \beta_D^{(m)} \left((\alpha^{(m)})^T \theta^{(m)} + u\right)\\
&= (\beta_M^{(m)})^T \theta_M + \beta_D^{(m)} \left((\alpha_S^{(m)})^T \theta_S + (\alpha_L^{(m)})^T (\theta_L^{(m)} + \xi_L^{(m)}) + u\right),
\end{aligned}
$$

where we "decompose" the vector of coefficients $\alpha^{(m)}$ as $\alpha^{(m)} = \left(\alpha_S^{(m)}; \alpha_L^{(m)}\right)$.

The equilibrium price in the alternative market is

$$
\begin{aligned}
p^{(alt)} &= (\beta_M^{(alt)})^T \theta_M^{(alt)} + \beta_D^{(alt)} \left((\alpha^{(alt)})^T \theta_S + u\right)\\
&= (\beta_{M,M}^{(alt)})^T \theta_M + (\beta_{M,L}^{(alt)})^T \theta_L + \beta_D^{(alt)} \left((\alpha^{(alt)})^T \theta_S + u\right),
\end{aligned}
$$

where $\theta_M^{(alt)} = (\theta_M; \theta_L)$ and $\beta_M^{(alt)}$ is "decomposed" as $\beta_M^{(alt)} = (\beta_{M,M}^{(alt)}; \beta_{M,L}^{(alt)})$.

We will show in Step 2 that $\beta_D^{(m)} \to \beta_D^{(alt)}$, and then in Step 3 we will show that $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$, $\beta_D^{(m)}\alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$, and $\beta_D^{(m)}\alpha_S^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$. By the same argument as in Step 3 of the proof of the special case in Section 9, showing these four convergence results is sufficient to prove the statement of the general case of Theorem 3.

Step 2. First, we show that the coefficients of the quadratic equation that $\beta_D^{(m)}$ satisfies converge to those of the quadratic equation that $\beta_D^{(alt)}$ satisfies. As the coefficient on $(\beta_D^{(alt)})^2$ in the latter equation is positive (as shown in Step 5 on the proof of Theorem 1 in Appendix A of the main body of the paper), this convergence implies that $\beta_D^{(m)}$ converges to $\beta_D^{(alt)}$.

Step 2(a). We first show that $a^{(m)} \to a^{(alt)}$, which is a direct consequence of Auxiliary Result (d) when observing that

$$a^{(m)} = -(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_v^{(m)} = -Var((\widehat{A}_v^{(m)})^T (\widehat{\theta}_S; \widehat{\theta}_L^{(m)}))$$

and

$$a^{(alt)} = -(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_v^{(alt)} = -Var((\widehat{A}_v^{(alt)})^T \widehat{\theta}_S).$$

Step 2(b). Next, we show that $b^{(m)} \to b^{(alt)}$. We have

$$\begin{aligned}
b^{(m)} &= (\widehat{A}_v^{(m)})^T \left(2\widehat{\Sigma}_{diag}^{(m)} + \widehat{\Lambda}^{(m)}\right) \widehat{A}_u^{(m)} - Cov(u, v|\theta_M) \\
&= 2(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} + (\widehat{A}_v^{(m)})^T \widehat{\Lambda}^{(m)} \widehat{A}_u^{(m)} - Cov(u, v|\theta_M) \\
&= 2(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} \\
&\quad + Cov((\widehat{A}_v^{(m)})^T(\theta + \xi^{(m)} + \widehat{\theta}^{(m)}), (\widehat{A}_u^{(m)})^T(\theta + \xi^{(m)} + \widehat{\theta}^{(m)})) - Cov(u, v|\theta_M) \\
&= 2(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} - Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)}).
\end{aligned}$$

Similarly, we have

$$b^{(alt)} = 2(\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)} - Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L).$$

That $b^{(m)} \to b^{(alt)}$, is implied by

$$(\widehat{A}_v^{(m)})^T \widehat{\Sigma}_{diag}^{(m)} \widehat{A}_u^{(m)} \to (\widehat{A}_v^{(alt)})^T \widehat{\Sigma}_{diag}^{(alt)} \widehat{A}_u^{(alt)},$$

and

$$\lim_{m\to\infty} Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L + \widehat{\theta}_L^{(m)} + \xi_L^{(m)}) = Cov(u, v|\theta_M, \theta_S + \widehat{\theta}_S, \theta_L),$$

owing to Auxiliary Result (e) and (a), respectively.

Step 2(c). Finally, that $c^{(m)} \to c^{(alt)}$ comes from a successive application of Auxiliary Result (f) then (g).

Step 3. Step 2 of the current proof showed that $\beta_D^{(m)} \to \beta_D^{(alt)}$.

We now show that $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$, $\beta_D^{(m)}\alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$, and $\beta_D^{(m)}\alpha_S^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$.

First, for any vector $\widetilde{\theta}_S$ of the same dimension as θ_S,

$$\begin{aligned}\beta_D^{(m)}(\alpha^{(m)})^T(\widetilde{\theta}_S;0) &= \beta_D^{(m)}\left((\widehat{A}_v^{(m)})^T/\beta_D^{(m)} - (\widehat{A}_u^{(m)})^T\right)(\widetilde{\theta}_S;0)\\ &= E[v-\beta_D^{(m)}u|\theta_M=0,\theta_S+\widehat{\theta}_S=\widetilde{\theta}_S,\theta_L+\xi_L^{(m)}+\widehat{\theta}_L^{(m)}=0],\end{aligned}$$

and

$$\begin{aligned}\beta_D^{(alt)}(\alpha^{(alt)})^T\widetilde{\theta}_S &= \beta_D^{(alt)}\left((\widehat{A}_v^{(alt)})^T/\beta_D^{(alt)} - (\widehat{A}_u^{(alt)})^T\right)\widetilde{\theta}_S\\ &= E[v-\beta_D^{(alt)}u|\theta_M=0,\theta_S+\widehat{\theta}_S=\widetilde{\theta}_S,\theta_L=0]\end{aligned}$$

We have $\beta_D^{(m)} \to \beta_D^{(alt)}$, and by Auxiliary Result (b) and (c) respectively,

$$\lim_{m\to\infty} E[v|\theta_M=0,\theta_S+\widehat{\theta}_S=\widetilde{\theta}_S,\theta_L+\xi_L^{(m)}+\widehat{\theta}_L^{(m)}=0] = E[v|\theta_M=0,\theta_S+\widehat{\theta}_S=\widetilde{\theta}_S,\theta_L=0]$$

and

$$\lim_{m\to\infty} E[u|\theta_M=0,\theta_S+\widehat{\theta}_S=\widetilde{\theta}_S,\theta_L+\xi_L^{(m)}+\widehat{\theta}_L^{(m)}=0] = E[u|\theta_M=0,\theta_S+\widehat{\theta}_S=\widetilde{\theta}_S,\theta_L=0].$$

It follows that $\beta_D^{(m)}\alpha^{(m)} \to \beta_D^{(alt)}\alpha^{(alt)}$.

Next, we have the equality

$$\beta_M^{(m)} = \Sigma_{MM}^{-1}\left(\Sigma_{Mv} - \Sigma_{\theta M}^T A_v^{(m)}\right) - \beta_D^{(m)}\Sigma_{MM}^{-1}\left(\Sigma_{Mu} - \Sigma_{\theta M}^T A_u^{(m)}\right).$$

hence, for any $\widetilde{\theta}_M$,

$$(\beta_M^{(m)})^T\widetilde{\theta}_M = E[v-\beta_D^{(m)}u|\theta_M=\widetilde{\theta}_M,\theta_S+\widehat{\theta}_S=0,\theta_L+\xi_L^{(m)}+\widehat{\theta}_L^{(m)}=0]. \qquad \text{(OA.24)}$$

Similarly, as $\beta_D^{(m)} \to \beta_D^{(alt)}$, and by Auxiliary Result (b) and (c),

$$\lim_{m\to\infty} E[v|\theta_M=\widetilde{\theta}_M,\theta_S+\widehat{\theta}_S=0,\theta_L+\xi_L^{(m)}+\widehat{\theta}_L^{(m)}=0] = E[v|\theta_M=\widetilde{\theta}_M,\theta_S+\widehat{\theta}_S=0,\theta_L=0]$$

and

$$\lim_{m\to\infty} E[u|\theta_M=\widetilde{\theta}_M,\theta_S+\widehat{\theta}_S=0,\theta_L+\xi_L^{(m)}+\widehat{\theta}_L^{(m)}=0] = E[u|\theta_M=\widetilde{\theta}_M,\theta_S+\widehat{\theta}_S=0,\theta_L=0],$$

we get

$$(\beta_M^{(m)})^T \widetilde{\theta}_M \to E[v - \beta_D^{(alt)} u | \theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = 0]. \quad \text{(OA.25)}$$

Finally, for any $\widetilde{\theta}_L$ realization of θ_L, we have

$$\beta_D^{(m)} (\alpha_L^{(m)})^T \widetilde{\theta}_L \to E[v - \beta_D^{(m)} u | \theta_M = 0, \theta_S + \widehat{\theta}_S = 0, \theta_L = \widetilde{\theta}_L].$$

Thus,

$$(\beta_M^{(m)}; \beta_D^{(m)} \alpha_L^{(m)})^T (\widetilde{\theta}_M; \widetilde{\theta}_L) \to E[v - \beta_D^{(alt)} u | \theta_M = \widetilde{\theta}_M, \theta_S + \widehat{\theta}_S = 0, \theta_L = \widetilde{\theta}_L],$$

and we observe that analogously to the expression for $(\beta_M^{(\infty)})^T \widetilde{\theta}_M$, we also have

$$(\beta_M^{(alt)})^T (\widetilde{\theta}_M; \widetilde{\theta}_L) = E[v - \beta_D^{(alt)} u | \theta_S + \widehat{\theta}_S = 0, (\theta_M; \theta_L) = (\widetilde{\theta}_M; \widetilde{\theta}_L)].$$

Thus $\beta_M^{(m)} \to \beta_{M,M}^{(alt)}$ and $\beta_D^{(m)} \alpha_L^{(m)} \to \beta_{M,L}^{(alt)}$. Combining those convergence results, and applying the same argument as in Step 3 of the proof of the special case of Theorem 3 in Section 9, we conclude the proof.

11 Beauty Contest Games

In this section, we show that our information aggregation results do not extend to Beauty Contest games, even though such games share many of the features of the Kyle model and Cournot competition (normally distributed signals, linear best responses, and the uniqueness of linear equilibrium). We first derive a closed-form linear equilibrium for a family of Beauty Contest games, and then use that characterization to illustrate the non-aggregation of information in a specific example.

Our setup is in the general Beauty Contest framework of Morris and Shin (2002a), but with some key differences. First, we consider a game with a finite number of players (instead of a continuum), and then take a limit of such (still finite) games as the number of players becomes large. Second, we allow for an arbitrary, potentially asymmetric covariance matrix of players' multidimensional signals (instead of assuming that players' signals are single-dimensional and symmetrically distributed).[11] These gener-

[11] See also Section 3 of Morris and Shin (2002b) for a two-player version of the Beauty Contest setting in which the signals have a general correlation structure, as well as the discussions in Bergemann and

alizations allow us to consider versions of Beauty Contest games with the same types of complex information structures as in the other models of our paper. In particular, in Example OA.12, we assume that some players are better informed than others.

Our formal model is as follows. There are $n \geq 2$ players, $i = 1, \ldots, n$. A "state of the world" $v \in \mathbb{R}$ is drawn at random. Each player i privately observes a signal $\theta_i \in \mathbb{R}^{k_i}$. Vector $(\theta_1; \ldots; \theta_n; v)$ is drawn from the normal distribution with mean $(0; \ldots; 0; \overline{v})$ and covariance matrix Ω. As before, without loss of generality we assume that for each i, $Var(\theta_i)$ is full rank. We do not impose any other restrictions on matrix Ω.

After observing the signal, each player i independently chooses action $a_i \in \mathbb{R}$. Subsequently, the state of the world v is revealed, and each player i receives the payoff

$$\pi_i = -w(a_i - v)^2 - (1 - w)\left(a_i - \overline{a}_{-i}\right)^2,$$

where

$$\overline{a}_{-i} = \frac{\sum_{j \neq i} a_j}{n - 1}$$

and $w \in (0, 1)$.

As before, we focus on linear equilibria of the form $a_i(\theta_i) = \alpha_i^T \theta_i + \delta_i$.

11.1 Equilibrium Existence and Uniqueness

Proposition OA.5 *In the Beauty Contest game, there exists a unique linear equilibrium.*

Proof. Fix some player i, and suppose every player $j \neq i$ plays according to the strategy $a_j(\theta_j) = \alpha_j^T \theta_j + \delta_j$, with $\alpha_j \in \mathbb{R}^{k_j}$ and $\delta_j \in \mathbb{R}$.

The expected payoff of player i is concave in a_i, and the best response is pinned down by the first-order condition:

$$-2w(a_i - E[v|\theta_i]) - 2(1 - w)(a_i - E[\overline{a}_{-i}|\theta_i]) = 0.$$

Thus,

$$a_i(\theta_i) = wE[v|\theta_i] + (1 - w)E[\overline{a}_{-i}|\theta_i],$$

Morris (2013) on finite-player analogues of continuum-player quadratic economies (of which the Beauty Contest setting is a particular case).

with

$$E[\overline{a}_{-i}|\theta_i] = \frac{\sum_{j\neq i}\delta_j}{n-1} + \frac{\sum_{j\neq i}\alpha_j^T E[\theta_j|\theta_i]}{n-1}.$$

Next,

$$E[\theta_j|\theta_i] = \Sigma_{ji}\Sigma_{ii}^{-1}\theta_i,$$
$$E[v|\theta_i] = \overline{v} + \Sigma_{vi}\Sigma_{ii}^{-1}\theta_i,$$

and so

$$E[\overline{a}_{-i}|\theta_i] = \frac{\sum_{j\neq i}\delta_j}{n-1} + \frac{\sum_{j\neq i}\alpha_j^T\Sigma_{ji}\Sigma_{ii}^{-1}}{n-1}\theta_i.$$

Therefore, we have

$$a_i(\theta_i) = w\overline{v} + w\Sigma_{vi}\Sigma_{ii}^{-1}\theta_i + (1-w)\sum_{j\neq i}\frac{\delta_j}{n-1} + (1-w)\left(\sum_{j\neq i}\frac{1}{n-1}\alpha_j^T\Sigma_{ji}\Sigma_{ii}^{-1}\right)\theta_i,$$

and so the best response of player i is linear: $a_i(\theta) = \alpha_i^T\theta_i + \delta_i$, with

$$\delta_i = w\overline{v} + (1-w)\sum_{j\neq i}\frac{\delta_j}{n-1}, \tag{OA.26}$$

$$\alpha_i^T = w\Sigma_{vi}\Sigma_{ii}^{-1} + (1-w)\sum_{j\neq i}\frac{1}{n-1}\alpha_j^T\Sigma_{ji}\Sigma_{ii}^{-1}. \tag{OA.27}$$

Therefore, all linear equilibria are characterized by vectors $\alpha_1,\ldots,\alpha_n$ and numbers $\delta_1,\ldots,\delta_n$, such that for all i, equations (OA.26) and (OA.27) are satisfied.

We immediately get $\delta_i = \overline{v}$ for all i. For vector $\alpha = (\alpha_1;\ldots;\alpha_n)$, multiplying Equation (OA.27) by Σ_{ii} on the right, and combining the resulting equations for all i, we get

$$\alpha^T\Sigma_{diag} = w\Sigma_{v\theta} + \frac{1-w}{n-1}\alpha^T\left(\Sigma_{\theta\theta} - \Sigma_{diag}\right),$$

where Σ_{diag} is defined as before. Solving for α, we get[12]

$$\alpha = (n-1)w\Big((n-w)\Sigma_{diag} - (1-w)\Sigma_{\theta\theta}\Big)^{-1}\Sigma_{\theta v}.$$

■

11.2 Information Non-aggregation

We can now present a simple example of information non-aggregation in a large Beauty Contest game. In fact, the example has an even more striking feature: half of the agents directly observe the state of the world, without any noise. Yet in equilibrium, there is no information aggregation, and even the perfectly informed agents distort their actions relative to their signals.

Example OA.12 *There are two groups of players, A and B. There are m players in each group. The state of the world v is distributed normally with mean zero and variance one. Every player in group A observes the true state of the world v. Every player j in group B observes signal $\theta_j^B = v + \epsilon_j$, where for each j, ϵ_j is distributed normally with mean zero and variance one, independently of v and of the signals of other players.*

By symmetry, in the unique linear equilibrium in Example OA.12, every player in group A chooses the same action $a^{(m)} = \alpha_A^{(m)} v$, and every player j in group B chooses action $b_j = \alpha_B^{(m)}(v + \epsilon_j)$ (since $\overline{v} = 0$, we also have $\delta = 0$.) From Equation (OA.27) in the proof of Proposition OA.5, we have two equations that determine $\alpha_A^{(m)}$ and $\alpha_B^{(m)}$:

$$\alpha_A^{(m)} = w + \frac{1-w}{2m-1}\left((m-1)\alpha_A^{(m)} + m\alpha_B^{(m)}\right), \tag{OA.28}$$

[12] To see that matrix $(n-w)\Sigma_{diag} - (1-w)\Sigma_{\theta\theta}$ is positive definite (and thus invertible), note first that it is equal to $((n-w) - n(1-w))\Sigma_{diag} + (1-w)(n\Sigma_{diag} - \Sigma_{\theta\theta})$. The first part of the sum, $((n-w)-n(1-w))\Sigma_{diag}$, is positive definite, because Σ_{diag} is positive definite and $(n-w)-n(1-w) = (n-1)w > 0$. The second part, $(1-w)(n\Sigma_{diag} - \Sigma_{\theta\theta})$, is positive semidefinite. To see that, note first that $(1-w) > 0$. Next, to see that matrix $(n\Sigma_{diag} - \Sigma_{\theta\theta})$ is positive semidefinite, take any vector x of the same dimensionality as θ. We need to show that $x^T(n\Sigma_{diag} - \Sigma_{\theta\theta})x \geq 0$, or equivalently, that $nx^T\Sigma_{diag}x \geq x^T\Sigma_{\theta\theta}x$. Let ψ_1 be the scalar random variable equal to the dot product of θ_1 and the vector consisting of the first k_1 components of vector x, ψ_2 be the scalar random variable equal to the dot product of θ_2 and the vector consisting of the next k_2 components of vector x, and so on. Note that $\sum_{i=1}^n \psi_i = x^T\theta$ and that the inequality $nx^T\Sigma_{diag}x \geq x^T\Sigma_{\theta\theta}x$ can be equivalently rewritten as $n\sum_{i=1}^n Var(\psi_i) \geq Var(\sum_{i=1}^n \psi_i)$. The last inequality follows directly from the fact that for every $i, j \leq n$, we have $Cov(\psi_i, \psi_j) \leq \sqrt{Var(\psi_i)Var(\psi_j)} \leq (Var(\psi_i) + Var(\psi_j))/2$.

$$\alpha_B^{(m)} = \frac{w}{2} + \frac{1-w}{2m-1}\left(m\frac{\alpha_A^{(m)}}{2} + (m-1)\frac{\alpha_B^{(m)}}{2} \right), \qquad \text{(OA.29)}$$

solving which gives us

$$\alpha_A^{(m)} = \frac{w(4m-w-1)}{3mw+m-w(w+1)},$$
$$\alpha_B^{(m)} = \frac{w(2m-w)}{3mw+m-w(w+1)}.$$

In the limit, as $m \to \infty$, we get

$$\alpha_A^{(m)} \to \frac{4w}{3w+1},$$
$$\alpha_B^{(m)} \to \frac{2w}{3w+1}.$$

Thus, in the limit, even the players fully informed about the state of the world v (those in group A) end up choosing an action different from v, and the less informed players (those in group B) end up with actions that are, on average, even further away from v. Thus, in equilibrium, information dispersed among the agents does not get aggregated in any meaningful way.

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