Consider the topological manifold $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1\}$. For $i = 1, \ldots, n + 1$, define:

$U_i^+ = \{(x_1, \ldots, x_{n+1}) \in S^n | x_i > 0\},$

$U_i^- = \{(x_1, \ldots, x_{n+1}) \in S^n | x_i < 0\}.$

For each $i$ let \( \phi_i^\pm : U_i^\pm \to \mathbb{R}^n \) be the map defined by

\[ \phi_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}), \]

in other words $\phi_i^\pm$ is the map that forgets the $i$-th coordinate.

(a) Prove that $(\phi_i^\pm, U_i^\pm)_{0 \leq i \leq n+1}$ is a smooth atlas for $S^n$.

(b) With the smooth structure defined above, prove that the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding, where $\mathbb{R}^{n+1}$ has its standard smooth structure.

(2) Let $\pi: \tilde{X} \to X$ be a covering space. Let $\tilde{\Phi}$ be a smooth structure on $\tilde{X}$. Prove that there exists smooth structure $\Phi$ on $X$ such that $\pi: (\tilde{X}, \tilde{\Phi}) \to (X, \Phi)$ is an immersion.

(3) Prove that the following are submanifolds of $\text{Mat}(n \times n) \cong \mathbb{R}^{n^2}$. Compute their dimensions. Must give proof that the dimension is what you say it is!

(a) $\text{Gl}_n(\mathbb{R})$,

(b) $\text{Sl}_n(\mathbb{R})$,

(c) $\text{SO}_n$.

(4) The following problems from Hirsch, "Differential Topology":

(a) Chapter 1, Section 1: 1, 2, 3.

(b) Chapter 1 Section 2: 10, 12, 13.