This assignment is due in class on Thursday, January 19, 2017.

(1) Consider the topological manifold $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$. For $i = 1, \ldots, n + 1$, define:

$$U_i^+ = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i > 0\},$$

$$U_i^- = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_i < 0\}.$$

For each $i$ let $\phi_i^\pm : U_i^\pm \to \mathbb{R}^n$ be the map defined by

$$\phi_i^\pm(x_1, \ldots, x_{n+1}) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}),$$

in other words $\phi_i^\pm$ is the map that forgets the $i$-th coordinate.

(a) Prove that $(\phi_i^\pm, U_i^\pm)_{0 \leq i \leq n+1}$ is a smooth atlas for $S^n$.

Before proving that $(U_i^\pm, \phi_i^\pm)$ is a smooth atlas, one needs to observe that it is a topological atlas, namely that the maps $\phi_i^\pm$ are homeomorphisms onto open subsets of $\mathbb{R}^n$. To see this, we produce an inverse to $\phi_i^\pm$ for each $i$. The inverse $(\phi_i^\pm)^{-1} : \phi_i^\pm(U_i^\pm) \to U_i^\pm$ is given by the formula

$$(\phi_i^\pm)^{-1}(y_1, \ldots, y_n) = \left(y_1, \ldots, \pm \sqrt{1 - y_1^2 - \cdots - y_{n+1}^2}, \ldots, y_n\right),$$

where the term $\pm \sqrt{1 - y_1^2 - \cdots - y_{n+1}^2}$ is in the $i$-th slot. This proves that $(U_i^\pm, \phi_i^\pm)$ is a topological atlas (or just atlas for short). To show that it is a smooth atlas we need to check that for all $i, j$, the map

$$\phi_j^+ \circ (\phi_i^-)^{-1} : \phi_i^-(U_i^- \cap U_j^+) \to \phi_j^+(U_i^- \cap U_j^+)$$

is diffeomorphism between subsets of $\mathbb{R}^n$. Using (0.1) we see that $\phi_j^+ \circ (\phi_i^-)^{-1}$ is given by the formula

$$(y_1, \ldots, y_n) \mapsto \left(y_1, \ldots, \hat{y}_j, \ldots, -\sqrt{1 - y_1^2 - \cdots - y_{n+1}^2}, \ldots, y_n\right),$$

where $-\sqrt{1 - y_1^2 - \cdots - y_{n+1}^2}$ is in the $i$-th slot and the $j$-th term has been dropped. This is clearly a smooth map. Its inverse is given by a similar formula, and thus is smooth as well. This proves that it is a diffeomorphism. This completes the proof.

(b) With the smooth structure defined above, prove that the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding, where $\mathbb{R}^{n+1}$ has its standard smooth structure.

A smooth embedding is a topological embedding (homeomorphism onto its image) that also happens to be an immersion. Being an inclusion map, the map $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$
by definition is a homeomorphism onto its image. To prove the proposition we need to check that it is an immersion. The property of being an immersion can be verified in charts. It will suffice to check that for all \( i = 1, \ldots, n + 1 \), the map

\[
\iota \circ (\phi_i^\pm)^{-1} : \phi_i^\pm(U_i^\pm) \to \mathbb{R}^{n+1}
\]

is an immersion. By (0.1), it follows that \( \iota \circ (\phi_i^\pm)^{-1} \) is given by the formula

\[
(\phi_i^\pm)^{-1}(y_1, \ldots, y_n) = \left( y_1, \ldots, \pm \sqrt{1 - y_1^2 - \cdots - y_n^2}, \ldots, y_n \right).
\]

To observe that this is an immersion, we just need to compute the Jacobian and see that it has rank \( n \).

(3) Prove that the following are submanifolds of \( \text{Mat}(n \times n) \cong \mathbb{R}^{n^2} \). Compute their dimensions. Must give proof that the dimension is what you say it is!

(a) \( \text{Gl}_n(\mathbb{R}) \).

The space \( \text{Gl}_n(\mathbb{R}) \) is defined to be the complement \( \text{Mat}(n \times n) \setminus \text{det}^{-1}(0) \). Being the preimage of a closed subset under a continuous map, \( \text{det}^{-1}(0) \subset \text{Mat}(n \times n) \) is a closed subset, and thus \( \text{Mat}(n \times n) \setminus \text{det}^{-1}(0) \) is an open subset. Any open subset of a smooth manifold is a submanifold of the same dimension. So \( \text{Gl}_n(\mathbb{R}) \) is a submanifold of dimension \( n^2 \).

(b) \( \text{Sl}_n(\mathbb{R}) \).

The space \( \text{Sl}_n(\mathbb{R}) \) is defined to the pre-image \( \text{det}^{-1}(1) \subset \text{Mat}(n \times n) \). One has to verify that the smooth map \( \text{det} : \text{Mat}(n \times n) \to \mathbb{R} \) has 1 as a regular value. Once verified that 1 is a regular value for \( \text{det} \), it follows from the regular value theorem that \( \text{Sl}_n(\mathbb{R}) \subset \text{Mat}(n \times n) \) is a submanifold of dimension \( n^2 - 1 \). To verify that 1 is a regular value for \( \text{det} \), one needs to compute the derivative of \( \text{det} \). First observe that for \( A \in \text{Mat}(n \times n) \), for any \( i = 1, \ldots, n \) we have

\[
\text{det}(A) = (-1)^{i+1} \sum_{j=1}^{n} (-1)^{j+1} a_{i,j} \text{det}(A(i, j))
\]

where \( a_{i,j} \) is the \((i,j)\)th entry of \( A \), and where \( A(i, j) \) denotes the minor matrix obtained from \( A \) by deleting \( i \)th row and \( j \)th column. We may then compute the partial derivative with respect to the coordinate \( a_{i,j} \):

\[
\frac{\partial}{\partial a_{i,j}} \text{det}(A) = (-1)^{i+1}(-1)^{j+1} \text{det}(A(i, j)).
\]

This calculation used the product rule and the fact that

\[
\frac{\partial}{\partial a_{i,j}} \text{det}(A(i, j)) = 0,
\]

because \( A(i, j) \) has no dependence on the variable \( a_{i,j} \). Now, if \( A \in \text{det}^{-1}(1) \), then \( \text{det}(A(i, j)) \) must be non-zero for some \((i,j)\), for otherwise the determinant \( \text{det}(A) \) would be zero. The Jacobian of \( \text{det} \) (at \( A \in \text{det}^{-1}(A) \)) is a \( 1 \times n \) matrix with at least one non-zero entry and thus it has rank equal to 1. This proves that 1 is a regular value for \( \text{det} \).
SO\(_n\).

The space \(O_n\) can be identified with the subset of Mat\((n \times n)\) consisting of all matrices whose column vectors are orthonormal. \(SO_n \subset O_n\) is the path component consisting of those matrices with determinant equal to 1 (as opposed to \(-1\)). Since \(SO_n \subset O_n\) is a whole path component, it is an open subset and thus is submanifold of the same dimension (assuming that \(O_n\) is indeed a manifold). To solve the problem, at hand it will suffice to prove that \(O_n\) is submanifold and then determine the dimension of \(O_n\).

For \(i \leq j\), consider the map
\[
\sigma_{i,j} : \text{Mat}(n \times n) \rightarrow \mathbb{R}
\]
defined by
\[
\sigma_{i,j}(A) = \langle A_i, A_j \rangle
\]
where \(A_i\) denotes the \(i\)-th column vector and \(\langle A_i, A_j \rangle\) is the standard inner product on \(\mathbb{R}^n\). Using an argument similar to part (b), it can be shown that 1 is a regular value for each of these maps. We then consider
\[
\sigma := \prod_{1 \leq i \leq j \leq n} \sigma_{i,j} : \text{Mat}(n \times n) \rightarrow \mathbb{R} \times \cdots \times \mathbb{R}
\]
where the product on the right-hand side has \(n(n+1)/2\)-many terms. It follows that \((1, \ldots, 1)\) is a regular value for \(\sigma\). The pre-image \(\sigma^{-1}(1, \ldots, 1)\) is precisely the set of matrices with orthonormal column vectors, hence \(O_n\). By the regular value theorem it has dimension
\[
n^2 - \left(\binom{n}{2} + n\right) = n(n-1) - \binom{n}{2} = n(n-1) - n(n-1)/2 = n(n-1).
\]

(4) We give solutions to the following problems from Hirsch, "Differential Topology":

(a) Chapter 1, Section 2, Problem 12.

The problem asserts that there is a diffeomorphism
\[
T(S^n \times \mathbb{R}) \xrightarrow{\simeq} S^n \times \mathbb{R}^{n+1}.
\]

For this problem, there is a basic fact that we will want to exploit.

**Fact:** Let \(M\) be a smooth manifold with an atlas consisting of a single chart
\[
\phi : M \rightarrow \mathbb{R}^n.
\]

Then this single chart \(\phi\), induces a diffeomorphism
\[
T\phi : TM \xrightarrow{\simeq} M \times \mathbb{R}^n, \quad (x, a) \mapsto (x, a)
\]
where \((x, a) \in TM\) is a tangent vector written in local coordinates with respect to \(\phi\) (which in this case is a global coordinate representation).

Now, let us consider the smooth embedding,
\[
\phi : S^n \times \mathbb{R} \hookrightarrow \mathbb{R}^{n+1}, \quad (x_1, \ldots, x_{n+1}, t) \mapsto (\alpha(t)x_1, \ldots, \alpha(t)x_{n+1}),
\]
where \(\alpha : \mathbb{R} \rightarrow (-\frac{1}{2}, \frac{1}{2})\) is some diffeomorphism (it doesn’t matter which one). By the “basic fact” above, it follows that \(\phi\) induces a diffeomorphism
\[
T\phi : T(S^n \times \mathbb{R}) \xrightarrow{\simeq} (S^n \times \mathbb{R}) \times \mathbb{R}^{n+1}.
\]
Now, in the next problem we construct an explicit diffeomorphism

\[ e : T(S^n \times \mathbb{R}) \xrightarrow{\cong} TS^n \times T\mathbb{R}. \]

The atlas on \( \mathbb{R} \) given by the identity map induces a diffeomorphism

\[ T(\text{Id}) : T\mathbb{R} \xrightarrow{\cong} \mathbb{R} \times \mathbb{R}. \]

Consider the composite of diffeomorphisms

\[ TS^n \times (\mathbb{R} \times \mathbb{R}) \xrightarrow{(\text{Id})^{-1}} TS^n \times T\mathbb{R} \xrightarrow{e^{-1}} T(S^n \times \mathbb{R}) \xrightarrow{T\phi} S^n \times \mathbb{R} \times \mathbb{R}^{n+1}. \]

The proof follows by observing that (0.4) maps \( TS^n \times \{0\} \times \mathbb{R} \) to \( S^n \times \{0\} \times \mathbb{R}^{n+1} \). One has to trace through the definitions of the maps given above. The definition of the map \( e \) is given in the solution of the next problem.

(b) Chapter 1, Section 2, Problem 12.

Let \( M \) and \( N \) be smooth manifolds of dimension \( m \) and \( n \) respectively. We will prove that for any two manifolds \( M \) and \( N \), there is a diffeomorphism \( T(M \times N) \cong TM \times TN \).

Let \( (\phi_i, U_i)_{i \in I} \) and \( (\psi_j, V_j)_{j \in J} \) be smooth atlases for \( M \) and \( N \) respectively. Consider the product atlas \( (\phi_i \times \psi_j, U_i \times U_j)_{(i,j) \in I \times J} \) on \( M \times N \). We may express the tangent bundles \( TM \) and \( TN \) as the quotient spaces

\[ TM = \left( \coprod_{i \in I} U_i \times \mathbb{R}^m \right) / \sim, \quad TN = \left( \coprod_{j \in J} V_j \times \mathbb{R}^n \right) / \sim. \]

Elements of this quotient space (the one on the right specifically) can written as \((x, a)_i\), where \((x, a) \in U_i \times \mathbb{R}^m\). The equivalence relation is defined by declaring

\[ (x, a)_s = (y, b)_r \]

if and only if

\[ x = y \in U_s \cap U_r \quad \text{and} \quad D_{\phi_i}(x)(\phi_r \circ \phi_s^{-1})a = b. \]

Using the product atlas \((\phi_i \times \psi_j, U_i \times U_j)_{(i,j) \in I \times J}\) we may express \( T(M \times N) \) as a quotient space similar to (0.5), with a generic element given by \((x, y, a)_{i,j}\), with \((x, y) \in U_i \times V_j\) and \(a \in \mathbb{R}^{m+n}\). We then define a map

\[ F : T(M \times N) \rightarrow TM \times TN, \]

\[ F((x, y, a)_{i,j}) = ((x, \pi_m(a))_i, (y, \pi_n(a))_j), \]

where \(\pi_m : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m\) and \(\pi_n : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n\) are the standard projections onto the first \(m\) coordinates and last \(n\) coordinates respectively. This is easily seen to be well defined. Its inverse is given by the map

\[ G : TM \times TN \rightarrow T(M \times N), \quad ((x, a), (y, b)) \mapsto ((x, y), (a, b)). \]

Using the charts \(T(\phi_i), T(\psi_j)\) and \(T(\phi_i \times \psi_j)\) on \(TM, TN\), and \(TM \times N\); it is easy to see that these maps \(F\) and \(G\) are smooth. This completes the proof.