This is assignment is due in class on March 2nd.

(1) A lie group is defined to be a smooth manifold $G$ that also has the structure of a group with
that satisfies the following conditions:

- the product $- \cdot - : G \times G \rightarrow G$ is a smooth map;
- the map $G \rightarrow G, g \mapsto g^{-1}$ is smooth.

A lie group homomorphism is a group homomorphism that is also a smooth map. Lie groups include many of your good friends: $GL(n, \mathbb{R})$, $SO(n)$, $\mathbb{R}$, $S^1$, etc... (you should verify that these are in fact lie groups). Prove that every lie group $G$ is paralellizable, i.e. the tangent bundle $TG$ is trivial.

(2) Let $M$ be a smooth manifold. Recall that two closed submanifolds $N, N' \subset M$ are said to be cobordant in $M$ if there exists a compact submanifold $X \subset [0,1] \times M$ for which

$$\partial X = X \cap ([0,1] \times M)$$

and

$$X \cap ([0,\varepsilon) \times M) = N \times [0,\varepsilon) \quad \text{and} \quad X \cap ((1-\varepsilon,1] \times M) = N' \times (1-\varepsilon,1]$$

for some $\varepsilon > 0$.

(i) Prove that cobordant in $M$ is an equivalence relation on the set of all closed submanifolds of $M$. Prove that framed cobordant in $M$ is an equivalence relation as well.

(ii) Two submanifolds $N, N' \subset M$ are said to be isotopic if there exists a continuous family of diffeomorphisms,

$$h_t : M \rightarrow M, \quad t \in [0,1],$$

for which $h_0 = \text{Id}_M$ and $h_1(N) = N'$. Prove that if $N$ and $N'$ are isotopic then they are cobordant.

(iii) Similarly, let $(N, \bar{v})$ and $(N', \bar{v}')$ be framed manifolds in $M$. Suppose that $N$ and $N'$ are isotopic. Does this imply that $(N, \bar{v})$ and $(N', \bar{v}')$ are framed cobordant? Prove your answer.

(3) Fix an orientation on the sphere $S^n$.

(a) Determine the (integral) degree of the antipodal map,

$$S^n \rightarrow S^n, \quad x \mapsto -x.$$

(b) Suppose that $n$ is odd. Show that the tangent bundle $T S^n \rightarrow S^n$ has a non-vanishing section (or in other words that $S^n$ has a non-vanishing vector field).

(c) Use the your calculation in part (a) to show that if $n$ is even, then $S^n$ cannot have a non-vanishing vector field. (This is outlined on page 125 of Hirsch, fill in the details).

(4) The following exercises from Hirsch, Section 5.1 (pages 130-131):
(5) Use the techniques developed in the previous exercise (specifically 7 parts (a) and (b) and 9 part (b)) to construct a non-trivial element in $[S^3, S^2]$. Prove that there is a surjection $\pi_3(S^2) \rightarrow \mathbb{Z}$. 