This assignment is due in class on Thursday, January 26, 2017.

(1) Let $M$ be a smooth manifold of dimension $k$. If $n > k$, prove that all smooth maps $f : M \to S^n$ are null-homotopic. You must use material that was covered in the class.

(2) (a) Let $f : M \to N$ be an injective immersion. Prove that if $M$ is compact, then $f$ is an embedding. You may have to recall some basic point set topology. Recall that all manifolds (as far as we are concerned) are Hausdorff.

(b) Produce an example of an injective immersion (from a non-compact manifold) that is not an embedding. You may give a “proof by picture”, but you need to clearly explain your picture.

(3) Consider the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Denote open subsets

$V_+ = \{(x, y, z) \in S^2 \mid z \neq 1\}$,

$V_- = \{(x, y, z) \in S^2 \mid z \neq -1\}$.

Consider the maps $h_\pm : V_\pm \to \mathbb{R}^2$ defined by the formulas

$h_+(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right)$,

$h_-(x, y, z) = \left(\frac{x}{1 + z}, \frac{-y}{1 + z}\right)$.

(a) Prove that $(h_\pm, V_\pm)$ is a smooth atlas for the sphere $S^2$. Prove that this smooth atlas is compatible with the smooth structure obtained by considering $S^2$ to be a submanifold of $\mathbb{R}^3$.

(b) In Problem 2 on page 14 of Hirsch, a smooth structure on $\mathbb{C}P^n$ is given. Prove that there is a diffeomorphism $\mathbb{C}P^1 \cong S^2$.

(4) For a smooth manifold $M$, we define the diagonal to be the space given by

$\Delta_M = \{(x, y) \in M \times M \mid x = y\}$.

The smooth structure on $M$ induces a smooth structure on $M$ via the homeomorphism,

$M \cong \Delta_M$,

$x \mapsto (x, x)$.

(a) With the smooth structure on $\Delta_M$ given above, prove that the inclusion $\Delta_M \hookrightarrow M \times M$ is a smooth embedding.

(b) Prove that there is a diffeomorphism $\Delta_{TM} \cong T(\Delta_M)$, i.e. the tangent bundle of the diagonal is diffeomorphic to the diagonal of the tangent bundle.
(c) Let $A \subset M$ be a submanifold and let $f : N \to M$ be a smooth map. Prove that $f$ is transverse to $A$ if and only if the product map

$$f \times i_A : N \times A \to M \times M$$

is transverse to $\Delta_M$, where $i_A$ denotes the inclusion of $A$.

(5) For a smooth function $f : M \to N$, let the graph of $f$ be defined by

$$\Gamma(f) = \{(x, y) \in M \times N \mid y = f(x)\}.$$

(a) Prove that the graph $\Gamma(f) \subset M \times N$ is a submanifold. Use the diagonal $\Delta_N \subset N \times N$.

(b) Verify that the map

$$M \to \Gamma(f), \quad x \mapsto (x, f(x))$$

is a diffeomorphism.

(c) Consider the graph $\Gamma(D(f)) \subset TM \times TN$ of the differential $Df : TM \to TN$ (which recall is a smooth map). Prove that there is a diffeomorphism $T(\Gamma(f)) \cong \Gamma(D(f))$.

(6) Construct a smooth function $f : \mathbb{R} \to \mathbb{R}$ whose set of critical values is the rational numbers.