Lecture 3C Maxwell's: From the differential form to the integral form

We begin by restating Maxwell's equations in both forms:

	differential form	integral form	
Gauss' law	$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$	(1)
no mag. monopoles	$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{A} = 0$	(2)
Faraday's law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}$	(3)
Ampere's law*	$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$	$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed} + \mu_0\epsilon_0 \frac{d}{dt}\int_A \vec{E} \cdot d\vec{A}$	(4)

Since the calculations are similar for the first two and the second two, we will only show the transition for Gauss' law (1) and Ampere's law (4). Let's also write out explicitly the divergence of $\vec{E}$ and the curl of $\vec{B}$.

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
\nabla \times \vec{B} &= \hat{i}[\partial_y B_z - \partial_z B_y] + \hat{j}[\partial_z B_x - \partial_x B_z] + \hat{k}[\partial_x B_y - \partial_y B_x]
\end{aligned}$$

We will also use extensively the fundamental theorem of calculus,

$$\int_a^b \frac{df(x)}{dx} dx = f(b) - f(a) \qquad (5)$$

Gauss law

We begin with the local, differential form of the law and integrate it over a finite volume, V.

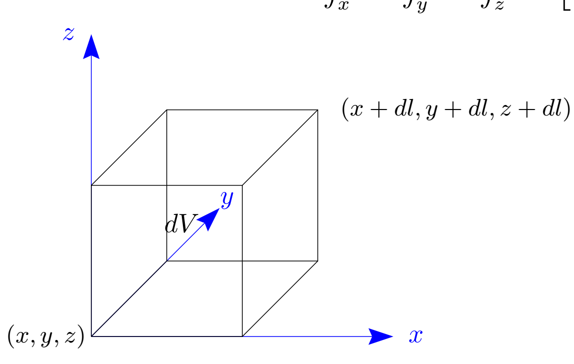
$$\int_V \nabla \cdot \vec{E} dV = \int_V \frac{\rho}{\epsilon_0} dV$$

Since ρ is just the charge density, the right hand side simply gives the charge enclosed in V. We're half way home.

$$\int_V \nabla \cdot \vec{E} dV = \frac{Q_{enc}}{\epsilon_0}$$

For the left hand side, we will chop it up into infinitesimal cubes of sides dl, evaluate the result for one cube, and then extrapolate to the entire volume. So consider an infinitesimal cube with faces orientated along the $x-y$, $x-z$, and $y-z$ planes.

$$\begin{aligned}
\int_{dV} \nabla \cdot \vec{E} dV &= \int_x^{x+dl}\int_y^{y+dy}\int_z^{z+dl} \nabla \cdot \vec{E}\; dxdydz \\
&= \int_x^{x+dl}\int_y^{y+dl}\int_z^{z+dl} \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] dxdydz
\end{aligned}$$



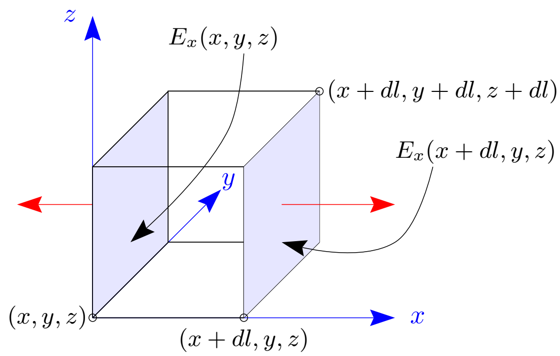
To proceed, we will integrate each term over the dimension of its derivative. And we will use the fundamental theorem of calculus (5) to evaluate that integral. For the first term we have,

$$\begin{aligned}\int_x^{x+dl}\int_y^{y+dl}\int_z^{z+dl}\left[\frac{\partial E_x}{\partial x}\right] dxdydz &= \int_y^{y+dl}\int_z^{z+dl}\left[\int_x^{x+dl}\frac{\partial E_x}{\partial x}dx\right] dydz \\ &= \int_y^{y+dl}\int_z^{z+dl}\left[E_x(x+dl)-E(x)\right] dydz\end{aligned}$$

The remaining integral is just an area integral (on each side of the cube). The shaded areas in the figure to the left are the two areas integrated over. The left face is over $E_x(x,y,z)$ and the right over $E_x(x+dl,y,z)$. The negative sign difference between the two is due to the oppositely directed normal vectors (pointing out of the volume), indicated by the red arrows.
Note, that if $E_x(x,y,z)$ is pointing to the left it is negative, this will cancel the negative sign in the above term. Thus if the E filed points out of the box, it gives a positive contribution to the integral. This can be expressed as a dot product between the field and the normal vector,

$$-\hat{i}E_x(x,y,z)\cdot\hat{n}+\hat{i}E_x(x+dl,y,z)\cdot\hat{n}.$$

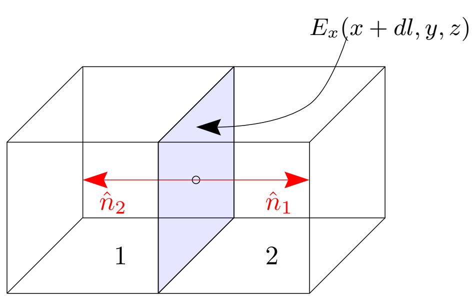
By symmetry, it should be clear that the later two terms yield the same result, over their respective faces.



In this manner it should be clear that we can express the total expression as the integral over the surface of the cube over the dot product of the electric field with the normal vector pointing outwards to the surface. The boundary of the volume dV is a bounded, closed area dA.

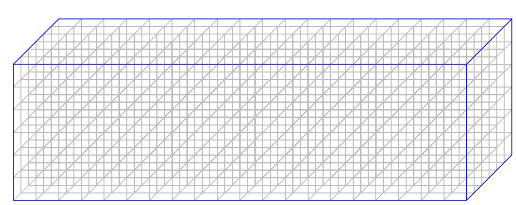
$$\begin{aligned}\int_{dV}\nabla\cdot\vec{E}dV &= \int_x^{x+dl}\int_y^{y+dl}\int_z^{z+dl}\left[\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right] dxdydz \\ &= \oint_{dA}\vec{E}\cdot\hat{n}dA \equiv \oint_{dA}\vec{E}\cdot d\vec{A} = \frac{dQ_{enc}}{\epsilon_0}\end{aligned}$$

We continue to label the volume, area, and enclosed charges as infinitesimals (dV, dA, dQ_{enc}) since we are only considering an infinitesimal volume. However, if you place these infinitesimal cubes next to each other, filling the macroscopic volume V, any neighboring faces which share a face will have canceling contributions. See the figure on the left below. This is because the outward pointing normal vectors will point in opposite directions ($\hat{n}_1$ and $\hat{n}_2$ in the figure).



Many infinitesimal volumes forming a macroscopic volume, V

All sides on the interior having vanishing contribution to the integral.



The figure on the right shows that by building up a macroscopic volume by stacking infinitesimal volumes, all interior sides cancel (flux into one side equals flux out of its neighbor). The remaining integral is only over the boundary of the volume. And we have arrived at the macroscopic (integral) form of Gauss law.

Ampere's law

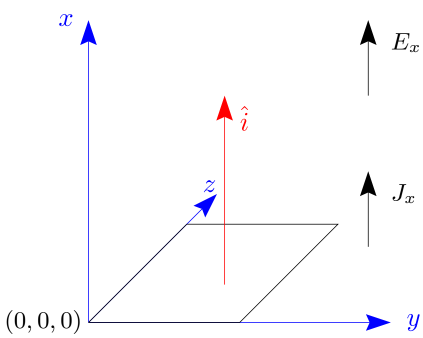
In this case, instead of integrating over a volume element we integrate over an area element. The basic idea is the same, we will see that neighboring plaquettes will have contributions that cancel and only an integral over the boundary remains. Since this relation results in a vector expression, we will have three separate expressions.

We begin by integrating Ampere's law over an area A, which we will break into infinitesimal pieces of area dA.[1]

$$\begin{aligned}\int_{dA} \nabla \times \vec{B} dA &= \int_{dA} \left\{ \hat{i}[\partial_y B_z - \partial_z B_y] + \hat{j}[\partial_z B_x - \partial_x B_z] + \hat{k}[\partial_x B_y - \partial_y B_x] \right\} dA \\ &= \int_{dA} \left[\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] dA\end{aligned}$$

Let's attack this component by component, we begin with the x component.

$$\hat{i} \int_{dA} [\partial_y B_z - \partial_z B_y] dydz = \hat{i} \int_{dA} \left[\mu_0 J_x + \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \right] dydz$$



As before, the right hand side is easy to simplify. Upon integrating, the area current density J_x becomes the current in the x direction that passes through the plaquette, dI_{enc}. The integral over the x component of the field does not simplify any further.

$$\int_{dA} \left[\mu_0 J_x + \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \right] dydz = \mu_0 dI_{enc} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int_{dA} \vec{E} \cdot d\vec{A}$$

Integrating over a macroscopic area A, the right hand side takes the integral form of the generalized Ampere law.

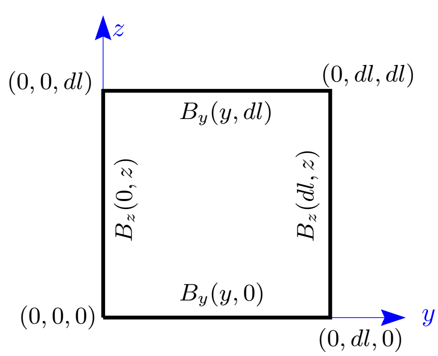
We attack the left side in the same manner as for Gauss' law. The corner is placed at the origin ($x = 0, y = 0, z = 0$).

$$\int_{dA} [\partial_y B_z - \partial_z B_y] dydz = \int_0^{dl} \int_0^{dl} \frac{\partial B_z}{\partial y} dydz - \int_0^{dl} \int_0^{dl} \frac{\partial B_y}{\partial z} dydz$$

Applying (5) to the two terms we get,

$$\int_0^{dl} \int_0^{dl} \frac{\partial B_z}{\partial y} dydz - \int_0^{dl} \int_0^{dl} \frac{\partial B_y}{\partial z} dydz = \int_0^{dl} [B_z(dl, z) - B_z(0, z)] dz - \int_0^{dl} [B_y(y, dl) - B_y(y, 0)] dy$$

Notice that each term is a line integral over an edge of the plaquette.



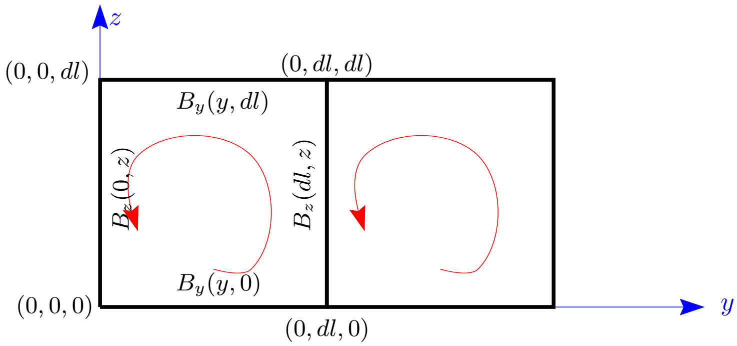
To finish the connection we flip the end points of the integral for the negative terms. Thus we get,

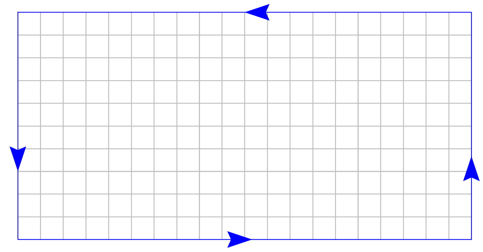
$$\begin{aligned}&\int_0^{dl} B_z(dl, z) + \int_{dl}^0 B_z(0, z) dz + \int_{dl}^0 B_y(y, dl) + \int_0^{dl} B_y(y, 0) dy \\ &= \int_0^{dl} B_y(y, 0) dy \int_0^{dl} B_z(dl, z) + \int_{dl}^0 B_y(y, dl) + \int_{dl}^0 B_z(0, z) dz\end{aligned}$$

In the second line we rearranged the terms to highlight the fact that we are integrating around the boundary $dC = \partial dA$ in a counterclockwise manner.

Since the normal vector to this area is out of the page, we see that this integral is positive if traverse the boundary in accordance with the right hand rule (thumb in direction of normal vector, fingers curl in direction of integration). As before, examine two neighboring plaquettes and the direction of integration over each.

[1] Note well the notation. We are considering an infinitesimal area element dA that is not necessarily closed. However, when we use the notation ∂A this means the boundary of A.





Notice on the figure on the left the shared boundary is integrated in opposite directions for each plaquette. Thus, these two contributions cancel. On the right figure we place many plaquettes together to form a macroscopic area A. Thus the remaining integral is over the boundary of the area $C = \partial A$. We have arrived at the macroscopic form of Ampere's law for the x component. A similar analysis yields the other two components.

The Big Take Away

These two results are two of the more significant in the study of vector calculus. They are stated below in their mathematical form,

$$\text{Gauss' theorem:} \qquad \oint_{A=\partial V} \vec{V} \cdot d\vec{A} = \int_V \nabla \cdot \vec{V} dV$$
$$\text{Stokes's theorem:} \qquad \oint_{C=\partial A} \vec{V} \cdot d\vec{l} = \int_A \nabla \times \vec{V} \cdot d\vec{A}$$

In the above the relation between the volumes, areas and line elements are related as follows (where ∂ means boundary):

Volume V has boundary	∂V	which is a closed bounded area A,	$A = \partial V$
Area A has a boundary	∂A	which is a closed bounded line C,	$C = \partial A$
note well though,		the boundary of a boundary is 0	
$\partial\partial V = \partial^2 V = 0$			

Maxwell's wave equation

From Maxwell's equation it is a simple calculation to arrive at the following form of the wave equation (for the E field, there is another equation just substituting B for E),

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Here we want to show that this is equal to the usual wave equation if there is no charge density around (in vacuum). Thus, we want to simplify the expression,

$$\begin{aligned}
\nabla \times \nabla \times \vec{E} &= \nabla \times \Big[\hat{i}[\partial_y E_z - \partial_z E_y] + \hat{j}[\partial_z E_x - \partial_x E_z] + \hat{k}[\partial_x E_y - \partial_y E_x]\Big] \\
&= \Big[\hat{i}\partial_x + \hat{j}\partial_y + \hat{z}\partial_z\Big] \times \Big[\hat{i}[\partial_y E_z - \partial_z E_y] + \hat{j}[\partial_z E_x - \partial_x E_z] + \hat{k}[\partial_x E_y - \partial_y E_x]\Big] \\
&= \hat{i}\partial_x \times \Big[\hat{j}[\partial_z E_x - \partial_x E_z] + \hat{k}[\partial_x E_y - \partial_y E_x]\Big] \\
&\quad + \hat{j}\partial_y \times \Big[\hat{i}[\partial_y E_z - \partial_z E_y] + \hat{k}[\partial_x E_y - \partial_y E_x]\Big] \\
&\quad + \hat{k}\partial_z \times \Big[\hat{i}[\partial_y E_z - \partial_z E_y] + \hat{j}[\partial_z E_x - \partial_x E_z]\Big] \\
&= \hat{k}[\partial_x\partial_z E_x - \partial_x^2 E_z] - \hat{j}[\partial_x^2 E_y - \partial_x\partial_y E_x] \\
&\quad - \hat{k}[\partial_y^2 E_z - \partial_y\partial_z E_y] + \hat{i}[\partial_y\partial_x E_y - \partial_y^2 E_x] \\
&\quad + \hat{j}[\partial_z\partial_y E_z - \partial_z^2 E_y] - \hat{i}[\partial_z^2 E_x - \partial_z\partial_x E_z]
\end{aligned}$$

Now collect terms that have pure second derivatives and those that have mixed derivatives,

$$\begin{aligned}
\nabla \times \nabla \times \vec{E} &= \Big[-\hat{k}\partial_x^2 E_z - \hat{j}\partial_x^2 E_y - \hat{k}\partial_y^2 E_z - \hat{i}\partial_y^2 E_x - \hat{j}\partial_z^2 E_y - \hat{i}\partial_z^2 E_x\Big] \\
&\quad + \Big[\hat{i}[\partial_x\partial_y E_y + \partial_x\partial_z E_x] + \hat{j}[\partial_y\partial_x E_x + \partial_y\partial_z E_z] + \hat{k}[\partial_z\partial_x E_x + \partial_z\partial_y E_y]\Big] \\
&= -[\hat{i}(\partial_y^2 E_x + \partial_z^2 E_x) + \hat{j}(\partial_x^2 E_y + \partial_z^2 E_y) + \hat{k}(\partial_x^2 E_z + \partial_y^2 E_z)] \\
&\quad + \Big[\hat{i}[\partial_x\partial_y E_y + \partial_x\partial_z E_x] + \hat{j}[\partial_y\partial_x E_x + \partial_y\partial_z E_z] + \hat{k}[\partial_z\partial_x E_x + \partial_z\partial_y E_y]\Big]
\end{aligned}$$

At this point we recognize that if we add and subtract pure second differentials, we can express the relation in a separate form. So, we add the following terms $\hat{i}(\partial^2 E_x - \partial^2 E_x)$, $\hat{j}(\partial^2 E_y - \partial^2 E_y)$, and $\hat{k}(\partial^2 E_z - \partial^2 E_z)$.

$$\begin{aligned}
\nabla \times \nabla \times \vec{E} &= -[\hat{i}(\partial_x^2 E_x + \partial_y^2 E_x + \partial_z^2 E_x) + \hat{j}(\partial_x^2 E_y + \partial_y^2 E_y + \partial_z^2 E_y) + \hat{k}(\partial_x^2 E_z + \partial_y^2 E_z + +\partial_z^2 E_z)] \\
&\quad + \Big[\hat{i}[\partial_x\partial_x E_x + \partial_x\partial_y E_y + \partial_x\partial_z E_x] + \hat{j}[\partial_y\partial_x E_x + \partial_y\partial_y E_y + \partial_y\partial_z E_z] + \hat{k}[\partial_z\partial_x E_x + \partial_z\partial_y E_y + \partial_z\partial_z E_z]\Big].
\end{aligned}$$

We recognize the first line as the Laplacian and the second we can rearrange as follows,

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} + \hat{i}\partial_x[\partial_x E_x + \partial_y E_y + \partial_z E_z] + \hat{j}\partial_y[\partial_x E_x + \partial_y E_y + \partial_z E_z] + \hat{k}\partial_z[\partial_x E_x + \partial_y E_y + \partial_z E_z]$$

Each term in the square bracket is the divergence of the E field.

$$\begin{aligned}
\nabla \times \nabla \times \vec{E} &= -\nabla^2 \vec{E} + \hat{i}\partial_x \nabla \cdot \vec{E} + \hat{j}\partial_y \nabla \cdot \vec{E} + \hat{k}\partial_z \nabla \cdot \vec{E} \\
\nabla \times \nabla \times \vec{E} &= -\nabla^2 \vec{E} + \nabla(\nabla \cdot \vec{E})
\end{aligned}$$

This is the result we desire. In empty space there is no charge density and Gauss law ($\nabla \cdot \vec{E} = 0$) so we obtain the desired result,

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Or, using the relation $c^2 = \frac{1}{\mu_0 \epsilon_0}$, the final version is,

$$\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} = 0$$

The reason this form is important is that the metric for 4-dimensional spacetime is $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$, and this second derivative is really the natural second derivative in spacetime (i.e. $\Box = \frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$). Thus Maxwell's wave equation takes the Lorentz covariant form $\Box \vec{E} = 0$. [2]

[2] The operator $\Box$ as shown is called the d'Alembertian.