Day 9 Lecture - Additional
Falling into a black hole – details

Let’s examine the case of dropping Bob into a black hole more closely. To explore this we will drop various twins (Bob, Bruce, Bryce, Bill, etc.) into various black holes of different masses. What we will want to explore is at what distances from the singularity does Bob (and the twins) feel pain and eventually decapitation due to the onset of spaghettification.

First of all note that, depending upon the size of the black hole, an infalling observer may not experience any unpleasant side effects. You may wonder how does Bob head and feet communicate once inside of the horizon. Note that although a nerve signal may not climb any further distance from the singularity once within the horizon, Bob’s head continues plunging and meets the signal at a lower radius. Thus there is no problem with the functioning of his body. More importantly you must view this from Bob’s inertial reference frame. As long as his entire body can be considered to be within one inertial reference frame ten what he experiences is no different than any other person in an inertial reference frame. The problems arise once the gradient of curvature (the rate at which gravity is changing) becomes significant within his local frame. That is, when tidal forces become non-negligible.

A third way to state the onset of problems above is by examining the geodesics of Bob’s head and feet. As he approaches the singularity they begin to deviate, At some point (well beyond the RIP point) Bob’s head lies outside of his feet’s lightcone – his head and feet our spacelike separated. At this point, if you could survive that long, you would not feel your feet at all – no discomfort. Unfortunately, tidal forces have already ripped your head off of your body.

We want to now try to quantify where this occurs. It will happen before the two points we will consider; the onset of pain (ouch-point) and the point of decapitation (the RIP point). We define the ouch-point as the point where serious discomfort will begin. As a random criterion we will choose 1000 N. (How did I determine this? Well I figure I can hold up a 100 kg weight with my head if my feet are anchored to the ground. On the surface of the Earth this corresponds to roughly 1000 N. (Of course over time a much less force would cause headaches etc). For the decapitation point, I assume I can not support a 200 kg weight if strung over a pulley and attached to my head with my feet anchored. This gives us the rough range of 1000 to 2000 N as the discomfort zone.)

So our goal is to find the tidal forces due to the difference in force between our feet and head (assuming they are each 1 kg in mass) around various black holes. Further assumptions are that Bob is 1 m in height and 50 kg in mass (this corresponds to a small person – Bob is a 10 year old boy say).

We need to calculate the difference in force between our feet and head, this is not hard to due. In what follows we will use the approximation that the feet-head distance (1 m) is much smaller than the distance to the singularity.

\[ dF = F_{\text{feet}} - F_{\text{head}} = GMm \left( \frac{1}{r^2} - \frac{1}{(r + dr)^2} \right) = \frac{GMm}{r^2} \left( 1 - \frac{1}{1 + \frac{dr}{r}} \right), \]

now use the binomial expansion to simplify this expression (recall \((1+x)^n \approx 1 + nx + \ldots \text{ if } x \ll 1\)),

\[ dF = \frac{GMm}{r^2} \left( 1 - \left( 1 + \frac{dr}{r} \right)^{-2} \right) = \frac{GMm}{r^2} \left( 1 - \left( 1 - 2 \frac{dr}{r} + \ldots \right) \right) = \frac{2GMm}{r^3} \cdot dr \]

This is a somewhat general result, tidal forces are often proportional to the inverse cube of the distance. And to further simply our analysis, we express this tidal force in terms of the Schwarzschild radius, \( R_s = \frac{2GM}{c^2} \), and rescale the distance in terms of it by defining a new coordinate, \( x = \frac{r}{R_s} \).

\[ dF = \frac{2GMm}{r^3} \cdot dr = \frac{c^2 R_s m}{r^3} (dr = 1m) \rightarrow \frac{c^2 R_s m}{R_s^3 x^3} = \frac{c^2 m}{R_s^3 x^3}, \text{ where } x = \frac{r}{R_s} \]

Next we will find evaluate this function for our Sun. Later we will describe our black holes in terms of solar masses. This will make the calculations easier.

Sun, \( M = 2 \times 10^{30} \text{ kg} = 1 \text{ M}_{\text{Sun}} \).

\[ R_s = \frac{2(6.67 \times 10^{-11} \text{ Nm}^2 / \text{kg}^2)(2\times10^{30} \text{ kg})}{9 \times 10^{16} \text{ m} / \text{s}^2} = 3000 \text{ m} \]

\[ dF = \frac{c^2}{R_s x^3} = \frac{(3 \times 10^3 \text{ m/s} / 3000 \text{ m})^2(50 \text{ kg})}{(5 \times 10^3 \text{ N}) \cdot x^3} \]

Since \( R_s \) is directly proportional to the mass of the object we can easily find the Schwarzschild radii of other objects if measured in terms of solar masses.

\[ R_s = M_{\text{in solar masses}} \times 3000 \text{ m} \].
Now, since the tidal force is proportional to the inverse square of the Schwarzschild radius, to find the tidal force for another object, we simply divide the result for the Sun by the square of the object’s mass (measured in solar masses).

\[
dF = \frac{c^2}{R_s^3 x^3} = \frac{c^2}{M_{\text{in solar masses}}^2 R_s^2 x^3} = \frac{(5 \times 10^{13} \text{ N})}{M_{\text{in solar masses}}^2 x^3}
\]

It will also be of interest to find the value of the acceleration due to gravity at various radii (especially at the Schwarzschild radius). We use the familiar formula to find this (where we use \( \kappa \) to represent this value, recall \( \kappa = 9.8 \text{ m/s}^2 \) at the surface of the Earth),

\[
\kappa = \frac{GM}{r^2} = \frac{c^2 R_s}{2 \sqrt{R_s}} \rightarrow \frac{c^2}{2 R_s \sqrt{x^3} x^3} \rightarrow \kappa_{\text{Sun}} = \left( \frac{9 \times 10^{16} \frac{m^3}{s^2}}{(6000 \text{m})^2} \right) = 1.5 \times 10^{13} \frac{m}{x^2}.
\]

Setting \( x = 1 \) in the above expression gives us the acceleration due to gravity at the horizon of a 1 solar mass black hole. And just to compare we find the results for the Earth,

\[
\kappa_{\text{Earth}} = \left( \frac{9 \times 10^{16} \frac{m^3}{s^2}}{(0.018 \text{m})^2} \right) \frac{5 \times 10^{18} \frac{m}{x^2}}{x^2}, \quad \text{recall } R_s \text{ for Earth} = 0.009 \text{m}
\]

\[
dF_{\text{Earth}} = \frac{5.6 \times 10^{22} \text{ N}}{x^3}
\]

We can find these expressions for various mass black holes,

<table>
<thead>
<tr>
<th>( M ) (in ( M_{\odot} ))</th>
<th>( R_s ) (in m)</th>
<th>( dF ) (in Newtons)</th>
<th>( \kappa ) at ( R_s ) (m/s²)</th>
<th>( \Delta t ) for ( R_s ) to r=0 (in s)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3000</td>
<td>( (5 \times 10^{11})/x^3 )</td>
<td>1.5 \times 10^{13}</td>
<td>15 \times 10^{-6}</td>
</tr>
<tr>
<td>10</td>
<td>30,000</td>
<td>( (5 \times 10^{9})/x^3 )</td>
<td>1.5 \times 10^{12}</td>
<td>15 \times 10^{-5}</td>
</tr>
<tr>
<td>100</td>
<td>300,000</td>
<td>( (5 \times 10^7)/x^3 )</td>
<td>1.5 \times 10^{11}</td>
<td>15 \times 10^{-4}</td>
</tr>
<tr>
<td>1000</td>
<td>3 \times 10^6</td>
<td>( (5 \times 10^5)/x^3 )</td>
<td>1.5 \times 10^{10}</td>
<td>15 \times 10^{-3}</td>
</tr>
<tr>
<td>10,000</td>
<td>3 \times 10^7</td>
<td>( (5000)/x^3 )</td>
<td>1.5 \times 10^9</td>
<td>0.15</td>
</tr>
<tr>
<td>1 \times 10^8</td>
<td>3 \times 10^8</td>
<td>( (5)/x^3 )</td>
<td>1.5 \times 10^8</td>
<td>1.5</td>
</tr>
<tr>
<td>1 \times 10^9</td>
<td>3 \times 10^9</td>
<td>( (0.5)/x^3 )</td>
<td>1.5 \times 10^7</td>
<td>15</td>
</tr>
<tr>
<td>3.7 \times 10^6</td>
<td>1.1 \times 10^{10}</td>
<td>( (0.0365)/x^3 )</td>
<td>4 \times 10^8</td>
<td>~1 minute [1]</td>
</tr>
<tr>
<td>1.5 \times 10^{12}</td>
<td>3 \times 10^{17}</td>
<td>( (5 \times 10^{-17})/x^3 )</td>
<td>9.8</td>
<td>2.4 \times 10^7 s ~ 9 months. [2]</td>
</tr>
<tr>
<td>1 \times 10^{14}</td>
<td>3 \times 10^{17}</td>
<td>( (5 \times 10^{-17})/x^3 )</td>
<td>0.15</td>
<td>1.5 \times 10^9 s ~ 40 years. [3]</td>
</tr>
</tbody>
</table>

*This result was can be found in chapter 3 of “Exploring Black Holes”, this is the longest possible time to the center of a Schwarzschild black hole, performed by jumping of a shell right at the horizon.
[1] The black hole at the center of our Milky Way has been measured to have this mass.
[2] This is a black hole designed to have the same acceleration due to gravity as here on the surface of the Earth.
[3] This is a black hole discussed in “Exploring Black Holes”, designed to have a 40 year transit to the singularity.

You may object to our continual use of classical, Newtonian, laws when deriving our results. This is a valid objection, however, it can be shown that the general relativistic results for tidal acceleration and acceleration due to gravity are the same as for Newtonian physics. (For a derivation, read Project B of Exploring Black Holes).

Let’s plot the tidal forces for some of these objects.
Fig 1. Plot for 10, 100, and 1000 solar mass black holes. All decapitation occurs outside of horizon (x=1).

Fig 2. Plot for high mass black holes. For 100,000 solar mass black holes and up, you can travel a significant distance into the black hole without disruption.

Finally a log-log plot will allow us to show all at once.