Lecture 6

Velocity Through Spacetime

Soon, we will want to examine momentum and energy within special relativity but first we need to discuss some properties of velocity. We want to consider now a particle moving through spacetime. The particle is moving through space, in the frame under consideration, so classically it has momentum and kinetic energy.

Consider the same particle sitting still in another inertial reference frame, Newtonian mechanics would say that this particle has no kinetic energy. Of course, in the first frame this particle is moving in space and hence has some kinetic energy. What we observe is that kinetic energy is not the same in different reference frames. Where does this kinetic energy go? Classically it just goes away. What we will find is the relativistic energy is more natural, in some sense, because we will see where this kinetic energy goes.

Of course, the particle sitting still in space is moving in spacetime. Hence, it may be worthwhile to attach some energy to this movement in spacetime. Then in different frames we can imagine this energy transforming to kinetic energy so that ‘energy’ would be conserved in this case, unlike Newtonian mechanics.

The question is, how fast is the particle moving through spacetime? Let’s look at some examples.

1) Particle at rest in frame S.
The interval for this particle between two different times is
\[ \Delta s^2 = c^2 \Delta t^2 - \Delta 0^2. \]
The speed or ‘velocity’ would be displacement in this space divided by ‘time’. The displacement in this space is the interval, \( \Delta s \). Also this particle is in a proper frame. Because the proper time is the same in every IRF we choose the proper time as the correct to use, here \( \Delta \tau = \Delta t \). Hence we have,
\[ \frac{\Delta s}{\Delta t} = \frac{c\Delta t}{\Delta t} = c \]
So the rate of motion through spacetime for this particle at rest in space is the speed of light. As you listen to this lecture, sitting in your chair, you are moving at the speed of light through spacetime.

2) Particle moving at c through space in frame S.
Now in frame S the interval between two events is,
\[ \Delta s^2 = c^2 \Delta t^2 - \Delta x^2. \]
Here it becomes clear that we need to use the proper time to determine the speed through spacetime. If we consider the frame S determination of the time between events, then this time is not invariant in other IRFs, (it is not a proper frame since the two events do not occur at the same place in this frame). If we consider the time observed by the moving clock in the moving frame, in this case, we have \( \Delta t' = \infty \) and
\( \Delta s = 0 \) so that now we would have \( \Delta s/\Delta t' = 0/c \). However if use the proper time between the two events then we have,

\[
\frac{\Delta s}{\Delta \tau} = \frac{c \Delta \tau}{\Delta \tau} = c.
\]

Again we find the particle’s speed through spacetime as equal to the speed of light (of course the speed through space in \( S \) is \( c \) as well). (This is a fine balancing act since both \( \Delta s \) and \( \Delta t \) equal zero).

For frames traveling less than \( c \) within \( S \) it should be clear that the proper time is the correct time to use, since any other would not be the same in other reference frames.

Since \( \Delta \tau = \Delta s/c \), we see that the speed through spacetime of any object is the speed of light! So even though you observe objects moving at different speeds through space, they are all moving at the speed of light through spacetime. This way of describing motion through spacetime also helps explain why you do not notice relativistic effects here on Earth. Consider a fast jet passing by you at 1000 kph = 277 m/s. You do not notice time dilation since both you and the jet have comparable speeds as compared to your movement along the \( ct \) direction, \( 3 \times 10^8 \) m/s. 277 m/s compared to \( 300,000,000 \) m/s is not much at all.

A way to visualize this is by comparing to movement in two-dimensional space. An object moving at constant speed \( v \) in the \( x \) direction has no speed in the \( y \) direction. If this object changes its direction while maintaining its speed its \( x \) component of velocity will decrease while its \( y \) component will increase, but it will still be traveling at the same speed.

Likewise, objects in spacetime all move at constant speed \( c \) in spacetime but if you change its direction, say by moving at speed \( v \) in the \( x \) direction, then spatial speed will change and so will the speed along the \( ct \) direction. Again, its total speed will still be \( c \) through spacetime.

A fundamental often asked is: Why does times flow at the rate and direction that it does? A better question is: Why do all objects move at \( c \) through spacetime? Every second your clock ticks, you have moved \( 3 \times 10^8 \) m from where you were before.

Now recall that velocity in space is a vector. It has three components in the three directions. So our measure of the rate of movement through spacetime should be a vector as well. It will have a magnitude and it will have a direction. And, like velocity in space, our spacetime velocity should point in the direction of movement through spacetime. The velocity arrow should be tangent to the worldline.

Now we introduce some terminology and notation conventions. In three dimensions we labeled a vector either as bold face or with a little arrow over it. For vectors in our four dimensional spacetime, which we will now call \textbf{4-vectors}, the variables will be denoted differently. A three dimensional vector, \( \mathbf{v} \), has three components which we often called \( x, y, \) and \( z \). A shorthand notation is to label these components \( v_i \) where \( i = 1, 2, \) or \( 3 \). 4-vectors in spacetime will have four components, labeling these by numbers we begin with zero,
(v^0, v^1, v^2, v^3) = (v^ct, v^x, v^y, v^z), the zeroth component will correspond to the ct direction. Note, this is nothing more than a change of notation, the reason we do this is so that you may interpret equations written in more advanced texts.

As we have defined the 4-velocity as $\Delta s/\Delta \tau$, the space components are,

\[
\begin{align*}
    v^x &\rightarrow v^1 = \frac{\Delta x}{\Delta \tau} \\
    v^y &\rightarrow v^2 = \frac{\Delta y}{\Delta \tau} \\
    v^z &\rightarrow v^3 = \frac{\Delta z}{\Delta \tau}
\end{align*}
\]

And for the time component we have,

\[
v^0 = \frac{c\Delta t}{\Delta \tau}.
\]

With this notation defined, the speed squared can be expressed as,

\[
c^2 = \left(\frac{\Delta s}{\Delta \tau}\right)^2 = c^2 \left(\frac{\Delta t}{\Delta \tau}\right)^2 - \left(\frac{\Delta x}{\Delta \tau}\right)^2 - \left(\frac{\Delta y}{\Delta \tau}\right)^2 - \left(\frac{\Delta z}{\Delta \tau}\right)^2.
\]

To complete the discussion of notation we will label a 4-vector in spacetime without an arrow over it but by a greek letter representing its index, $v^\mu$. The 4-velocity is then expressed as $v^\mu$ and our result for the magnitude of the squared 4-velocity is $v^\mu v^\mu = c^2$.

**Momentum and Energy**

To develop an idea of relativistic momentum we can proceed by analogy to classical physics. Just as $p = mv$ here we define,

\[
p\mu = mv^\mu
\]

Consider just the 3-momentum for a moment, (the three spatial components of the 4-momentum). In an inertial reference frame, Newton’s Laws are still valid. As you learned in your mechanics course, this implies that (classical) momentum is conserved in interactions if no outside forces are acting upon the system. This is still valid in the relativistic regime.

To show that this is the correct form let’s examine the classical form under Galilean and Lorentz transformations, and compare to the relativistic form under Lorentz transformations. The classical definition of momentum is simply $mv$. Consider a particle which moves with $v_x,v_y,v_z$ in frame S. In frame S’ moving at speed $u$ along the x direction with respect to frame S. Then we have,

\[
\begin{align*}
    p'_x &= p_x - mu \\
    p'_y &= p_y \\
    p'_z &= p_z
\end{align*}
\]

This just shows the classical result that momentum is conserved in S and S’ since $m$ and $u$ are constants. However, consider this form under a Lorentz transformation,
\[
x' = \gamma (x - ut)
\]

\[
t' = \gamma \left( t - \frac{ux}{c^2} \right) \rightarrow v' = \frac{x' - ut}{t'} = \frac{x - ut}{t - \frac{ux}{c^2}} = \frac{x - u}{1 - \frac{ux}{c^2}} = \frac{v - u}{1 - uv/c^2}
\]

\[
y' = y \rightarrow v'_y = \frac{y'}{t'} = \frac{y}{\gamma \left( 1 - \frac{ux}{c^2} \right)} = \frac{v_y}{\gamma \left( 1 - \frac{ux}{c^2} \right)} \quad \text{(and } y \rightarrow z \text{)}
\]

Since the primed velocity in the \( y \) and \( z \) direction depend upon the speed in the (unprimed) \( x \) direction, we see that the classical definition of momentum will not be conserved under Lorentz transformations.

If we examine a collision in one frame and find the total 3-momentum of the system to be \( mv \) before and after the collision, then in another IRF the 3-momentum will be conserved as well in that frame. Why? Well the transformation from one frame to the other is simply to multiply the momentum by \( \gamma \), a constant for constant relative speeds. So in the other frame the total momentum is simply \( \gamma mv \) before and after the collision. Momentum is conserved in both frames, however the values of the total momentum will not agree in the two frames.

The spatial part of the 4-momentum is clear, but what about the temporal part? What is the meaning of \( mv^c t \)? We have, \( mv^c t = m \frac{\Delta t}{\Delta \tau} = m\gamma \). For the spatial part we have a classical analogy, the three momentum. What is the classical analog of \( m\gamma \), if there is even one? To explore this, let’s take the low velocity limit (\( v \ll c \)). To do this recall the binomial expansion, \((1 + x)^n \approx 1 + nx + n(n - 1)x^2/2! + ... \) if \( x \ll 1 \). (If this is not familiar, review it).

If \( v \ll c \) or \( v/c \ll 1 \), we can expand \( \gamma \) in this manner,

\[
\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-\frac{v^2}{c^2}) + ... \approx 1 + \frac{1}{2} \left(\frac{v^2}{c^2}\right)
\]

and,

\[
p^c \approx mc + \frac{1}{2} mc \left(\frac{v^2}{c^2}\right) \rightarrow mc + \frac{1}{2} mc \frac{v^2}{c^2} \frac{1}{c^2}
\]

Does this last term look familiar to you? It is just the classical kinetic energy divided by \( c \). Hence we see that by taking the classical limit the time component of our 4-momentum is \( mc + KE/c \). This leads us to interpret that this component is related to energy.

Passing from the classical regime to the relativistic regime, the \( ct \) component of the 4-momentum picks up the extra terms in the expansion. The classical limit gives us the one familiar term and the first term, \( mc \). What is the significance of this term? Well not much since both \( m \) and \( c \) are constants. Recall that classically we can define the zero of energy as we wish since it is only the difference in energy which matters. Hence, classically we can absorb this constant into the definition of energy by resetting the zero.
Now we define the relativistic energy, \( E \), as \( E = p^t c = mc^2 \gamma \). In the low velocity limit we regain the classical form of energy (ignoring other forms of energy for the moment).

Note that when \( v = 0 \) we are led to the most famous equation in all of physics, \( v \to 0 \), then \( \gamma \to 1 \) and \( E_0 = mc^2 \). What is the meaning of this equation? This equation can be viewed as arising from the motion of an object through spacetime, along the \( ct \) direction, sort of the kinetic energy for movement in the \( ct \) direction.

You might object that this definition is ad hoc, that the only reason we choose it was that it gave the correct nonrelativistic energy at low velocity. You could say that the energy could actually be a completely different function yielding the same low energy limit. The reply is that this is the simplest definition, it is rather beautiful and it gives the correct results experimentally. (Einstein did not follow this method in deriving this equation, he used, yet another, gedankenexperiment. This will be seen later).

So we have defined 4-momentum as the following,

\[
p^\mu = \left( \frac{mc}{c^2} \frac{\Delta t}{\Delta \tau}, m\frac{\Delta x}{\Delta \tau}, m\frac{\Delta y}{\Delta \tau}, m\frac{\Delta z}{\Delta \tau} \right) = \left( \frac{E}{c}, p^x, p^y, p^z \right)
\]

and we have found the relativistic energy to be,

\[
E = p^t c = mc^2 \gamma,
\]

and we have found that the 4-momentum’s magnitude is a constant,

\[
(p^\mu)^2 = p^\mu p_\mu = \left( \frac{E}{c} \right)^2 - \left( \frac{p^x}{c} \right)^2 - \left( \frac{p^y}{c} \right)^2 - \left( \frac{p^z}{c} \right)^2
\]

We can also define another related 4-vector which we will call the energy 4-vector, or using Taylor’s terminology momenergy (we will try to stay away from this unconventional term). This is simply the 4-momentum multiplied by \( c \), \( q^\mu = cp^\mu \),

\[
q^\mu = (E, cp^x, cp^y, cp^z) \text{ with}
\]

\[
\left( q^\mu \right)^2 = E^2 - c^2 p^{x^2} - c^2 p^{y^2} - c^2 p^{z^2} = m^2 c^4 = (mc^2)^2
\]

The magnitude of this vector is \( mc^2 \) and is a constant. The momentum-energy 4 vector is conserved in collisions as well, more on this in the next lecture.

We can represent the relativistic energy in another manner now,

\[
E = \sqrt{(mc^2)^2 + (p^x c)^2 + (p^y c)^2 + (p^z c)^2}
\]

This form makes it easier to determine an object’s energy given its mass and momentum.

This energy is the total energy of the particle, \( KE \) plus all else. What this is saying is that a hot object will have more mass than a cold one, although the addition is very small, \((-1/c^2)\). From this definition of the relativistic energy, we define the kinetic energy, \( K \), as the difference between the total energy and the rest energy.

**Concepts of Mass**

**Concepts of Mass in Special Relativity**

In special relativity we have found that we can express the relativistic energy of an object in two equivalent ways:

1. \( E = \gamma mc^2 \), this is useful for cases where we know the velocity of an object in one frame.
(2) \[ E^2 = (mc^2)^2 + (cp^x)^2 + (cp^y)^2 + (cp^z)^2. \]

To explore the consequences of these equations and the interpretation of mass that now arises we examine a few systems.

**Example 1a:** An isolated box of gas consisting of identical molecules (considered as fundamental objects) of mass m. To start, let’s consider the gas to be at temperature 0 K (not possible thermodynamically but this is just a gedankenexperiment) and find the total energy of the system. (The container we take as massless).

The energy of each particle is simply their rest energy, \( \varepsilon = mc^2 \). The total energy of the system of gas molecules is the sum of all the individual energies (we will call this system \( E_0 \)),

\[ E_0 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \ldots = M_0 c^2. \]

The total mass of the system is \( E_0/c^2 \) since the system is at rest. If it were moving in this frame the energy would be \( E = \gamma E_0 = \gamma Mc^2 \). -the mass is the same.

**Example 1b:** Take the same system and add some heat energy to it (put over a flame for a period of time). The temperature of the gas will rise as the molecules acquire kinetic energy. Let’s say it reaches 1000 K. The energy of each molecule is now, \( \varepsilon_i^\ast = (mc^2)^2 + (cp^x)^2 + (cp^y)^2 + (cp^z)^2 \),

where the momenta are distributed via the Maxwell-Boltzmann equation (we’ll just consider them to be random). The energy of the system as a whole is again the sum of the individual energies,

\[ E_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \ldots = M_1 c^2. \]

As the energy of each molecule is greater than before the sum is greater and, as we expect, the energy of the system is greater (we added heat energy). Thus we have \( E_1 > E_0 \) and \( M_1 > M_0 \), The hotter system, still at rest, has a greater mass.

If you introduce a notion of relativistic mass for the molecules, you run into trouble. With this concept the mass of each molecule is now different and the kinetic energy of each molecule is absorbed into the mass. In order to include this, and have our original definitions, (1) and (2), to still hold, there are only two possibilities:

a) The total energy of the system (in example 1b) is the same and each molecule’s mass increases.

b) We increase the mass of each molecule and still ascribe the same momentum to it.

If first scenario is taken to be what occurs then there is no energy associated with the molecules movement (it is now in the mass of the molecules). If there is no energy associated with movement the momenta are zero and the molecules are not moving. The gas remains at 0 K! This violates at least two of the laws of thermodynamics. This can not be what occurs.

If the second scenario is considered now each molecule increases in mass while they are still moving. If they are moving at the same speed (as when there is no increase of the molecules’ masses) then the energy is not that given by formula (2) which disagrees with formula (1). If you adjust the speed of the molecules so that energies given by formulas (1) and
agree the actual speeds of the molecules is much less than in the situation where there is no relativistic mass.

Example 2: Consider two particles in a box. In any random frame the center of mass of the system, which consists of the two particles only, is most likely moving in some direction. The mass of the system is then given by

\[ E = \gamma M c^2 = \gamma \left( (\varepsilon_1 + \varepsilon_2)/c^2 \right)c^2 = \gamma (\varepsilon_1 + \varepsilon_2), \]

where the energies of the particles are those in the center of mass frame. In the CM frame the energy of each particle is,

\[ \varepsilon_i = (mc^2)^2 + (cp_i x)^2 + (cp_i y)^2 + (cp_i z)^2. \]

Example 3: Consider a solitary particle of mass m. In any frame where the particle is not at rest the system itself is not at rest (since the CM follows the particle). It is not conceivable to think of this as a box of gas consisting of one particle since the system will not be at rest in any frame.

What we see is that the famous equation \( E = mc^2 \) must be considered with some care. It is indeed true that it suggests that mass and energy are intimately related however energy associated with movement must be considered separately. The main reason is that the relation (1), \( E = \gamma mc^2 \), specifies that the additional energy beyond the rest energy is due to movement (since \( \gamma \) is a function of the velocity). If any other form of energy is added to a system it cannot be attributed to \( \gamma \), it must go in the mass (for the system considered in its rest frame, i.e. \( \gamma = 1 \)).

The conclusion is that energy associated with movement must be treated specially. Much of the distinction arises for systems considered at rest, an object or system at rest is described by only one relation, \( E_0 = mc^2 \). Any change in the energy of this system at rest is reflected by a change in mass only. If the system begins to move, its mass does not change and its energy is governed by \( \gamma mc^2 \).

There is no such thing as relativistic mass!!

One of the more significant conclusions is that mass does not change with its velocity! The concept of relativistic mass (RM) does not correspond to reality. The problem is easy to see from our earlier examples. Again, if the molecules of the gas gain mass via this concept then either the energy increases beyond what is observed or energy is conserved and the momenta vanish, yielding a system with zero temperature in disagreement with observations.

We can see this explicitly from the formula, \( E = \sqrt{(mc^2)^2 + (p^x c)^2 + (p^y c)^2 + (p^z c)^2} \), that if RM is taken as a primitive (objects really do get more massive when they move) then each occurrence of m must be replaced with \( \gamma m_0 \). Doing so in the above equation leads to a difference, i.e. this expression must reduce to \( E = \gamma m_0 c^2 \).

\[
E = \sqrt{(\gamma m_0 c^2)^2 + (\gamma^2 c^2 m_0 \frac{\Delta x}{\Delta t})^2} \\
= \gamma \sqrt{(m_0 c^2)^2 + (\gamma c^2 m_0 \frac{\Delta x}{\Delta t})^2} = \gamma \sqrt{(m_0 c^2)^2 + (p^x)^2} \neq \gamma m_0 c^2
\]

Thus these two relations are incompatible. The problem arises when one considers the \( \gamma m_0 c^2 \) form to be more fundamental and simply associates \( \gamma \) with m. However the other form
stems explicitly from the energy-momentum 4-vector and in this form one can not make such an identification. The magnitude of $q^\mu$ is a Lorentz invariant (same in all frames) however RM forces it to change (to $\gamma mc^2$). For those who follow the geometric interpretation for relativity, this notion of RM is abhorrent!

There are other problems with this concept. One of the main reasons for its introduction is to supposedly explain why objects can not move at or beyond the speed of light. The problem with this view is that if this is the sole reason for the restriction unnatural action at a distance must occur. Consider the Stanford Linear Accelerator (SLAC) where electrons are accelerated to within a few meters per second of the speed of light and you in a car in Singapore at a few meters per second. If RM is the cause of the prevention of exceeding $c$, then somehow there must be a causal connection between the electron in Stanford and you in Singapore. What is that causal connection? The statement that the mass becomes infinite must somehow be related to your movement, restricting your measuring the velocity of the electron. Even if you are several galaxies away, then somehow the mass of the electron is affecting your relative velocity. Thus RM can not be the only cause restricting velocities. It is the geometry of spacetime that is the cause of the prohibition of traveling at $c$.

Now, RM could be considered a result of the prohibition of traveling at $c$ (due to geometry) but a careful analysis shows that you can not have both effects. That is, if the effects of relativity are due to geometry you can not have relativistic mass and the converse (RM but no geometry) is insufficient to explain all of the effects.

Many influential people continue to use the concept of RM when describing relativity to general audiences but do not in their own professional research. This behavior borders on an ethical violation in my opinion. If the purpose of such expositions is to bring more people (especially the young) into the formal study of physics, a disservice is being performed since one will need to unlearn the idea to progress. It is unfortunate that this one idea is deeply engrained in the educated public.

**Conservation of Momentum-Energy**

We previously defined the relativistic energy and expressed it as,

$$E = \sqrt{(mc^2)^2 + (p_x c)^2 + (p_y c)^2 + (p_z c)^2}$$

This form was found from the energy 4-vector $q^\mu = cp^\mu$, (Taylor’s momenergy),

$$q^\mu : (E, cp^x, cp^y, cp^z) \quad \text{with}$$

$$(q^\mu)^2 = E^2 - c^2 p^2 = m c^4 = (mc^2)^2$$

The definition of the energy-momentum 4-vector is very close to our definition of the interval. The energy 4-vector squared is similar in that there is a ‘time’ part, $E^2$, with the ‘space’ parts subtracted off, $p^2$, and this squared 4-vector is constant in all frames, just as the interval is constant. Hence it displays the same metric properties if we relate energy to time and momentum to space. (Natural since momentum is associated with movement through space, and as we said previously, energy is associated with movement through time). In fact, we can make a similar spacetime diagram relating energy and momentum (again, with the conversion constant $c$ to ensure that all dimensions have the same units).

The vertical axis will be the energy axis and the horizontal will be momentum times the speed of light, $pc$, direction. Now all of the analytical techniques we developed for spacetime
applies here. Example, In the proper frame the energy 4-vector is equal to the rest energy and has the same magnitude in all IRFs.

We see that energy is always greater than the rest energy in other frames, as we should expect. And most importantly, notice that in order for a particle of mass \( m \) to travel at \( c \) would require an infinite amount of energy. (See this from the diagram, in the rest frame the particle has energy \( mc^2 \). As the velocity is increased, (shifted up one side of the hyperbola), to get the line to have a slope of 45 degrees would have \( E = \infty \ ).

**Thus massive objects can never travel at \( c \).**

Another interesting point is that for massless objects (photons, gravitons, gluons, maybe neutrinos but probably not) we have \( 0 = E^2 - p^2c^2 \rightarrow E = pc \). As was discussed in our discussion on regions of spacetime, events which are timelike, spacelike, or lightlike separated remain that way for all reference frames. This implies that massless objects remain on the light cone for all frames. This gives us another fact; all massless particles must travel at the speed of light. This should be clear, for if there were a massless particle which traveled less than the speed of light, then we can transform to the frame where it is at rest. In this frame \( m = E = 0 \). The object no longer exists.

**Collisions**

Now we want to consider small systems of particles which can collide. Two examples will suffice to demonstrate the point.
In the first example we examine the elastic collision of two 8 kg masses, each traveling at $\beta=3/5$ ($v/c = 3/5$) heading at each other along a line. What is the energy and momentum of each after the collision?

First let’s find the rest energy of each object, $E = mc^2 = (8 \text{ kg})(3 \times 10^8 \text{ m/s}) = 7.2 \times 10^{17} \text{ J}$.

Let’s examine the energy and momentum of these two objects prior to the collision.

Just as for classical momentum problems we must treat momentum as a vector and examine the vector sum of these two quantities. However we need to use the Lorentzian geometry when examining the components.

Let’s examine the x component first,
\[ p^2 c = mc \frac{\Delta x}{\Delta t} = m c \gamma \frac{\Delta x}{\Delta t} = m c \gamma v = mc^2 \gamma \beta. \]

\[ \beta = v / c = 3 / 5 = 0.6 \]

\[ \gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\sqrt{1 - (\frac{3}{5})^2}} = 1.25 \]

\[ p^1 c = mc^2 (1.25)(0.6) = 0.75mc^2 = (6 kg)c^2. \]

\[ p^2 c = -(6 kg)c^2, \text{ since it is traveling in the opposite direction}. \]

Now we can find the total energy of each of these particles prior to the collision.

\[ E_1 = \sqrt{(mc^2)^2 + (p^1 c)^2} = c^2\sqrt{(8 kg)^2 + (6 kg)^2} = (10 kg)c^2 = E_2 \text{ from symmetry}. \]

So each particle has an energy of \(10c^2 = 9 \times 10^{17} \text{ J}\) and a magnitude of momentum of \(pc = 6c^2 = 5.4 \times 10^{17} \text{ J}\).

Now, conservation of momentum demands that they will rebound at the same speeds. Conservation of energy in this one frame demands that the total energy of each after the collision is \(10c^2\) as well, (this is because they will scatter at the same speeds).

So we have found the energies and momenta after the collision.

Now consider an inelastic collision.

In the center of mass frame that we are examining these collisions in, the two particles come together at equal speeds, collide, and clump together to form one object. Conservation of momentum will tell us that the final speed of this clump will be zero in this frame. (This is the same as the classical case). The total energy before the collision will be the same as the previous case, \(E = 10c^2 + 10c^2 = 20c^2\). This will equal the energy after the collision as well. The total momentum before the collision is 0 and it is 0 after the collision.

**Momentum:** This is trivial. After the collision, the object is at rest, hence \(p = 0\) the same before the collision.

**Energy:** Now from the relativistic energy formula we find,

\[ E_{\text{final}} = 20c^2 \text{ J} = m_f c^2 \rightarrow m_f = 20 \text{ kg} \]

The total mass prior to the collision was 8 kg + 8 kg = 16 kg and after the collision the total mass is 20 kg.

What we observe is that the kinetic energy prior to the collision went into the mass of the final object. Classically we said that this energy went into heat and sound energy but now these forms of energy have mass equivalents and we absorb those energies into \(m\). You should convince yourself that this result holds in other IRFs. In the elastic case they maintain their masses (8 kg) since the kinetic energy before is equal to the kinetic energy after the collision.

**Example e⁻ + e⁺ collision, annihilation.**
Consider an electron and a positron next to each other. Since these are antiparticles of each other they can be completely converted in energy (photons). Let’s examine the kinematics.

**Definition of electron volt (eV) as a measure of energy.**

The energy of each of these is $m_e c^2$. Convenient units to use for energy of such light particles is the electron-volt (eV). (Recall that this is the amount of energy which an electron gains after being accelerated through 1 Volt difference).

\[
m_e = 9.11 \times 10^{-31} \text{ kg}
\]
\[
m_e c^2 = (9.11 \times 10^{-31} \text{ kg})(3 \times 10^8 \text{ m/s}) = 8.199 \times 10^{-14} \text{ J}
\]
\[
= \frac{8.199 \times 10^{-14} \text{ J}}{1.6 \times 10^{-19} \frac{1}{\text{eV}}} = 5.11 \times 10^5 \text{ eV} = 0.511 \text{ MeV}
\]

This is an often quoted result for the electron’s rest energy (0.511 MeV or 511 keV), its mass is then 0.511 MeV/c^2.

We will carry out this example in units of eV.

Initial energy: 2(0.511 MeV) = 1.022 MeV

Final energy = 1.022 MeV from energy conservation.

This energy is split among the two photons which travel of at c.

\[
E = 0.511 \text{ MeV} \quad m_{\text{after}} = 0
\]
\[
p_c = 0.511 \text{ MeV} \quad \text{since} \quad 0 = E^2 - (pc)^2
\]

So that $E = pc$ (or more precisely

\[
E = p_x c \quad \text{or} \quad \sqrt{(p_x c)^2 + (p_y c)^2 + (p_z c)^2}
\]

**Question:** Why is mass not conserved?

**Answer:** This is because of our new relation $E = mc^2$. As you know, energy can change from one form to another (say thermal energy to kinetic energy). Here we can have rest energy (the energy associated with the mass, or to the ‘kinetic energy’ of your movement along the c-time direction) be converted into other forms, here kinetic.

**Example: Stanford Linear Accelerator Center (SLAC)**
At SLAC positrons and electrons are accelerated such that they can have energies up to 50 GeV each (50,000 MeV) in the center of mass frame. This energy can go into the creation of new massive particles. The $Z^0$ particles is an example, but now they have fined tuned the accelerator to make large amounts of these particles.

We can see how much energy the electrons and positrons must have to create a $Z^0$ particle and how fast the $e^-$ and $e^+$ are traveling. The rest energy of a $Z^0$ particle is 91 GeV (91,000 MeV). The rest energy of the $e^-$ and $e^+$ is 0.511 MeV each. To find the energy required, we will work backwards and assume the $e^-$ and $e^+$ have the same energy.

Remember we must use Lorentzian geometry to find the magnitudes. From conservation of energy we have,

$$91 \text{GeV} = E_{e^-} + E_{e^+} = 2E \rightarrow E = 45.5 \text{GeV}$$

for each. We then have,

$$E = 45.5 \text{GeV} = \sqrt{(0.511 \text{MeV})^2 + (pc)^2}$$

which is too small to tell a difference. However, there is another way to determine the speed. Recall that $E = \gamma mc^2$. Hence, we have,

$$45,000 \text{MeV} = \gamma (0.511 \text{MeV}) \rightarrow \gamma = 89,041$$
From this we can find the required speed, $\beta = \frac{v}{c}$,

$$\gamma^2 = \frac{1}{1 - \beta^2} = 7.928 \times 10^9$$

$$\gamma^2 = 1 - \frac{1}{89041^2}$$

$$\rightarrow \beta = \frac{v}{c} = \sqrt{1 - \frac{1}{89041^2}} \approx 1 - 6.3 \times 10^{-11}$$

These particles are traveling at $6 \times 10^{-9}$% less than $c$, extremely close.
(Notice that in getting the particles up to this speed we have accelerated them up to this value.
They initially started at a much slower speed).

Discussion questions:

Suggested Reading:

Read sections 8-5 and 8-8. Read Problem 8-5, worked example. Read sections 9-1 through 9-5.