INTRODUCTION TO MINIMAL SURFACES

LECTURE NOTES FOR MATH 286, STANFORD, WINTER 2025

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Contents

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1. Introduction

These are my lecture notes for Math 286 taught at Stanford, Winter 2025. They cover the basic theory of minimal surfaces. The material is taken from various sources including [Oss86, Law77, Whi16, CM11, Pér17, Whi13]. I am grateful to the attendees of the course for spotting numerous errors during the course. I am also grateful to Jianchun Chu for catching a huge number of typos and mistakes in an earlier version of the notes. Please write `ochodosh@stanford.edu` with any comments.

Part 1. Plateau's problem

We begin by discussing Plateau's problem, first studied by Lagrange in 1760. We (loosely) formulate the problem as follows:

> Given some class of "submanifolds" of a Riemannian manifold (M, g), does there exist one of least area?

The name is in honor of Joseph Plateau who studied this problem experimentally in the 1870's using soap films.

2. The direct method

Plateau's problem is a question in the calculus of variations, so we are led to the *direct method*: show that a minimizing sequence in the class converges (possibly in some weak topology) to a minimizer (still in the class). This is problematic in multiple ways. There is the issue of thin "tentacles" as illustrated in Figure 1 where a minimizing sequence that becomes dense in space is illustrated. Moreover, if we work with parametrized objects, there's also issues of diffeomorphism invariance of area (see Remark 3.2 below).

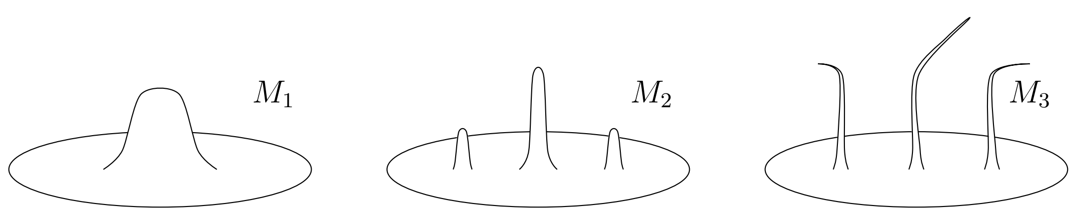

Figure 1. The least area surface bounded by a planar circle is a disk. Note that $\text{area}(M_i) = \text{area}(D) + o(1)$, but M_i is becoming dense in $\mathbb{R}^3$.

To handle these issues one must either improve the minimizing sequence somehow (one approach is discussed in Section 3 below) or else develop a sufficiently weak topology for the convergence.

3. Douglas–Radó

In this section, we discuss the classical formulation of Plateau's problem. Let D be the open unit disk in $\mathbb{R}^2$. Fix $\Gamma \subset \mathbb{R}^n$ a smooth Jordan curve (simple closed curve). Let

$$\mathcal{C}_\Gamma = \{F \in C^0(\bar{D}) \cap C^\infty(D) : F|_{\partial D} \text{ is a } \partial D \text{ weakly monotone parametrization of } \Gamma\}$$

If $F \in \mathcal{C}_\Gamma$ is an immersion, then we can define a pullback Riemannian metric on D by $g_{ij} = \partial_i F \cdot \partial_j F$. We recall that the volume form is

$$dV_g = \sqrt{\det g}\, dxdy = \sqrt{g_{xx}g_{yy} - (g_{xy})^2}\, dxdy$$

so it's reasonable to define the area of F to be

$$\mathcal{A}(F) := \int_D \sqrt{|\partial_x F|^2 |\partial_y F|^2 - (\partial_x F \cdot \partial_y F)^2} = \int_D |\partial_x F \wedge \partial_y F|. \tag{3.1}$$

Note that for arbitrary $F \in \mathcal{C}_\Gamma$, we can use (3.1) as a *definition* of the area of F.[1] By way of justification, we remark that the *area formula* in geometric measure theory says that $\mathcal{A}(F)$ agrees with the 2-dimensional Hausdorff measure of $F(D)$ (counted with multiplicity). See [Sim83, Theorem 3.3].

We set $a_\Gamma := \inf_{F \in \mathcal{C}_\Gamma} \mathcal{A}(F)$. We can now rigorously state:

Problem 3.1 (The classical Plateau problem)**.** Find $F \in \mathcal{C}_\Gamma$ attaining a_Γ.

Note that if $\varphi : \bar{D} \to \bar{D}$ is a diffeomorphism, then $\mathcal{A}(F \circ \varphi) = \mathcal{A}(F)$ (this is essentially coordinate invariance of the Riemannian volume form).

Remark 3.2. This raises following potential obstruction to the direct method. Suppose that some F attains a_Γ. For any sequence of diffeomorphisms φ_i, we have $F_i \in \mathcal{C}_\Gamma$ with $\mathcal{A}(F_i) = a_\Gamma$. However, for many choices of φ_i, F_i has no convergent subsequence.

To resolve Remark 3.2, we now introduce the energy functional. This will resolve the diffeomorphism invariance as well as the issue illustrated in Figure 1, modulo one final difficulty that we will need to address later.

We observe that

$$\sqrt{|\partial_x F|^2 |\partial_y F|^2 - (\partial_x F \cdot \partial_y F)^2} \leq |\partial_x F||\partial_y F| \leq \frac{1}{2}(|\partial_x F|^2 + |\partial_y F|^2)$$

with equality if and only if

$$|\partial_x F| = |\partial_y F| \text{ and } \partial_x F \cdot \partial_y F = 0 \tag{3.2}$$

We call F satisfying (3.2) *weakly conformal*. Note that this is equivalent to the "pullback metric" $g_{ij} = \partial_i F \cdot \partial_j F$ satisfying $g = \mu(dx^2 + dy^2)$ for $\mu \geq 0$ smooth. As such, if we define

[1] Note that this is a notion of *unsigned* area.

the energy of F by

$$\mathcal{E}(F) := \frac{1}{2}\int_D |\partial_x F|^2 + |\partial_y F|^2 = \frac{1}{2}\int_D |\nabla F|^2, \tag{3.3}$$

we've proven:

Lemma 3.3. *For $F \in \mathcal{C}_\Gamma$, we have $\mathcal{A}(F) \leq \mathcal{E}(F)$ with equality if and only if F is weakly conformal on D, i.e.* (3.2) *holds.*

We say that $F : D \to \mathbb{R}^n$ is harmonic if each component is (and write $\Delta F = 0$). We recall

Lemma 3.4. *For $G \in C^\infty(\bar{D})$, there exists $F \in C^\infty(\bar{D})$ harmonic so that $F|_{\partial D} = G|_{\partial D}$.*

Proof. Existence of $F \in C^2(D) \cap C^0(\bar{D})$ follows from the Poisson integral [GT01, Theorem 2.6]. Boundary Schauder estimates [GT01, Theorem 6.19] imply that $F \in C^\infty(\bar{D})$ □

A basic property of harmonic F is that it minimizes energy among maps with the fixed boundary data:

Lemma 3.5. *For $G, F : \bar{D} \to \mathbb{R}^n$ smooth with $G|_{\partial D} = F|_{\partial D}$ and $\Delta F = 0$ we have $\mathcal{E}(F) \leq \mathcal{E}(G)$ with equality if and only if $F = G$.*

Proof. Let $V = G - F$. Then we have

$$\mathcal{E}(G) = \mathcal{E}(F) + \mathcal{E}(V) + \int_D \nabla F \cdot \nabla V = \mathcal{E}(F) + \mathcal{E}(V) - \int_D (\Delta F) \cdot V = \mathcal{E}(F) + \mathcal{E}(V).$$

This completes the proof. □

Lemma 3.6. *For $G \in \mathcal{C}_\Gamma$, there's $G_i \in \mathcal{C}_\Gamma \cap C^\infty(\bar{D})$ with $\mathcal{A}(G_i) \leq \mathcal{A}(G) + o(1)$ as $i \to \infty$.*

This proof from the proof of the Douglas-Rado Theorem in [Whi16] (Claim 1).

Proof. Let $2\delta_0$ be sufficiently small so that the nearest point projection from the tubular neighborhood $\Pi : U_{2\delta_0}(\Gamma) \to \Gamma$ is smooth (and well-defined). For $\delta < \delta_0$ we set

$$\Phi_\delta(P) = \begin{cases} P & P \in U_{2\delta}(\Gamma)^c \\ \Pi(P) & P \in U_\delta(\Gamma) \\ \Pi(P) + (\delta^{-1} d(p, \Gamma) - 1)(P - \Pi(P)) & P \in U_{2\delta}(\Gamma) \setminus U_\delta(\Gamma) \end{cases}$$

Note that if $G \in \mathcal{C}_\Gamma$ then $\Phi_\delta \circ G \in \mathcal{C}_\Gamma$ and $\mathcal{A}(\Phi_\delta \circ G) = \mathcal{A}(G) + o(1)$ as $\delta \to 0$.

For any $\delta \in (0, \delta_0)$ we can find $r \in (0, 1)$ so that $A := \bar{D} \setminus D_r$ has $F(A) = \Gamma$ and $F|_{\partial D_r} : \partial D_r \to \Gamma$ is a smooth map homotopic to a parametrization of Γ. By replacing A by a smooth homotopy to such a parametrization we can obtain $\tilde{G}_\delta$ with $\mathcal{A}(\tilde{G}_\delta) = \mathcal{A}(\Phi_\delta \circ G)$ and $\tilde{G}_\delta \in \mathcal{C}_\Gamma \cap C^\infty(\bar{D})$. □

Proposition 3.7. *There exists $F_i \in \mathcal{C}_\Gamma \cap C^\infty(\bar{D})$ harmonic so that $\mathcal{E}(F_i) \to a_\Gamma$.*

Proof. By Lemma 3.6, we can consider a minimizing sequence $G_i \in \mathcal{C}_\Gamma \cap C^\infty(\bar{D})$ with $\mathcal{A}(G_i) \to a_\Gamma$. We now show that it's possible to find a harmonic $F_i \in \mathcal{C}_\Gamma \cap C^\infty(\bar{D})$ with $\mathcal{E}(F_i) \leq \mathcal{A}(G_i) + o(1)$ as $i \to \infty$. This will complete the proof since $\mathcal{E}(F_i) \geq \mathcal{A}(F_i) \geq a_\Gamma$.

We fix $G = G_i$. For $s \neq 0$, we set $\tilde{G}(x, y) = (G, sx, sy) \in \mathbb{R}^{n+2}$, so that $\tilde{G}$ is an embedding. Thus, the pullback metric $\tilde{g}_{ij} = g_{ij} + s^2\delta_{ij}$ defines a non-degenerate metric on $\bar{D}$. Note that $\mathcal{A}(\tilde{G}) = \mathcal{A}(G) + o(1)$ as $s \to 0$. By uniformization (cf. Theorem 3.9 below), there's a diffeomorphism $\varphi : \bar{D} \to \bar{D}$ so that $(\tilde{G} \circ \varphi)^* g_{\mathbb{R}^{n+2}} = \varphi^*\tilde{g} = \lambda(dx^2 + dy^2)$. In particular, we have that $\tilde{G} \circ \varphi$ is conformal. Let $F : \bar{D} \to \mathbb{R}^n$ denote the harmonic map agreeing with $G \circ \varphi$ on ∂D. Using Lemma 3.5 we have

$$\mathcal{E}(F) \leq \mathcal{E}(G \circ \varphi) \leq \mathcal{E}(\tilde{G} \circ \varphi) = \mathcal{A}(\tilde{G} \circ \varphi) = \mathcal{A}(\tilde{G}) \leq \mathcal{A}(G) + o(1)$$

as $s \to 0$. This completes the proof. □

Remark 3.8. Since $\mathcal{A}(F) \leq \mathcal{E}(F)$, Proposition 3.7 implies that $\inf_{F \in \mathcal{C}_\Gamma} \mathcal{E}(F) = a_\Gamma$.

We used the following uniformization result above:

Theorem 3.9 (Uniformization of disks)**.** *Suppose that (Σ, g) is a compact Riemannian surface with boundary so that Σ is homeomorphic to a disk. Then there's a smooth diffeomorphism $\varphi : \bar{D} \to \Sigma$ so that $\varphi^* g = \lambda(dx^2 + dy^2)$ for some $0 < \lambda \in C^\infty(\bar{D})$.*

See e.g. [Tay23, Proposition 6.4] for a simple proof.

We now recall the weak maximum principle for harmonic functions:

Lemma 3.10. *If $u \in C^0(\bar{D}) \cap C^\infty(D)$ then $\max_{p \in \bar{D}} u = \max_{p \in \partial D} u$.*

Applying this to the coordinates of F, we see that space-filling tentacles cannot occur for a harmonic minimizing sequence.

Choosing a harmonic minimizing sequence has also partially resolved the issue of diffeomorphism invariance of the area functional, but still some invariance remains: we recall that the Möbius transformations of the form

$$\varphi(z) = e^{i\phi_0} \frac{a + z}{1 + \bar{a}z}$$

for $a \in \mathbb{C}, |a| < 1, \phi_0 \in \mathbb{R}$ are precisely the set of (orientation preserving) conformal diffeomorphisms $\bar{D} \to \bar{D}$.

Lemma 3.11. *For $\varphi : \bar{D} \to \bar{D}$ a Möbius transformation, we have $\mathcal{E}(F \circ \varphi) = \mathcal{E}(F)$.*

Proof. Write $g = \varphi^*\delta$. Then we have

$$\int_D |\nabla_g (F \circ \varphi)|_g^2 \, dV_g = \int_D |\nabla_\delta F|_\delta^2 \, dV_\delta = \mathcal{E}(F). \tag{3.4}$$

Since φ is a conformal map we have $g = \mu\delta$ for $\mu > 0$ smooth. Then $|\nabla_g f|^2 = \mu^{-1}|\nabla_\delta f|^2$ and[2] $dV_g = \mu dV_\delta$. Thus we see that

$$\mathcal{E}(F \circ \varphi) = \int_D |\nabla_g (F \circ \varphi)|_g^2 \, dV_g,$$

completing the proof. □

Exercise 3.1. Prove (3.4) in two ways: (i) direct computation and (ii) appealing to isometry invariance of geometric quantities.

Corollary 3.12. *If $F : \bar{D} \to \mathbb{R}^n$ is harmonic and $\varphi : \bar{D} \to \bar{D}$ is a Möbius transformation then $F \circ \varphi$ is harmonic.*

Proof. Let $\hat{F}$ be the harmonic function with boundary values $F \circ \varphi|_{\partial D}$. Lemma 3.11 gives

$$\mathcal{E}(\hat{F} \circ \varphi^{-1}) = \mathcal{E}(\hat{F}) \leq \mathcal{E}(F \circ \varphi) = \mathcal{E}(F)$$

so $\hat{F} \circ \varphi^{-1} = F$. This completes the proof. □

Recalling that the set of Möbius transformations is non-compact, we still need to handle the invariance of energy. The key tool is as follows:

Lemma 3.13 (Courant–Lebesgue)**.** *For $p \in \mathbb{R}^2$ and $F : \bar{D} \to \mathbb{R}^n$ smooth, let $\ell(\rho)$ be the arc-length of $F|_{D \cap \partial B_\rho(p)}$. Then*

$$\min_{a \leq \rho \leq b} \ell(\rho)^2 \leq \frac{4\pi \mathcal{E}(F)}{\log b/a}$$

for $0 < a < b < \infty$.

Proof. We use polar coordinates centered at p. We have that $|\nabla F|^2 = |\partial_r F|^2 + r^{-2}|\partial_\theta F|^2$. Thus we have

$$\ell(r)^2 = \left(\int_{D \cap \partial B_r(p)} |\partial_\theta F| d\theta \right)^2 \leq 2\pi \int_{D \cap \partial B_r(p)} |\partial_\theta F|^2 d\theta$$

so

$$\int_a^b \frac{\ell(r)^2}{r} dr \leq 2\pi \int_a^b \int_{D \cap \partial B_r(\rho)} r^{-2} |\partial_\theta F|^2 r d\theta dr \leq 4\pi \mathcal{E}(F).$$

This proves the assertion. □

We can now solve the classical Plateau Problem 3.1.

Theorem 3.14 (Douglas–Radó)**.** *There's $F \in \mathcal{C}_\Gamma$ attaining $a_\Gamma = \mathcal{A}(F)$. The map F is harmonic and weakly conformal.*

Proof. Proposition 3.7 gives $F_i \in \mathcal{C}_\Gamma \cap C^\infty(\bar{D})$ harmonic with $\mathcal{E}(F_i) = a_\Gamma + o(1)$. Fix $a, b, c \in \partial D$ distinct and $A, B, C \in \Gamma$ with the same orientation. Recalling that Möbius

[2]Caution: the second expression would be $\mu^{\frac{k}{2}} dV_\delta$ if D was a k-dimensional domain.

transformations act transitively on ordered triples on ∂D, we can arrange that $F_i(a) = A, F_i(b) = B, F_i(c) = C$ (this will preserve F_i harmonic by Lemma 3.12 and will not change the area thanks to diffeomorphism invariance).

By the weak maximum principle (Lemma 3.10), we have $\|F_i\|_{L^\infty(D)} \leq \max_{P\in\Gamma} ||$. Thus, by higher derivative estimates for harmonic functions (Lemma 3.15) we can pass to a subsequence so that the F_i converge to F in $C^\infty_{\text{loc}}(D)$. Note that F will be harmonic.

We claim that the $F_i|_{\partial D}$ are equicontinuous. If not, there's $p_i, q_i \in \partial D$ with $\delta_i := |p_i - q_i| \to 0$ but $|F_i(p_i) - F_i(q_i)| \not\to 0$. Apply the Courant–Lebesgue lemma at p_i to find $\delta_i \leq \rho_i \leq \sqrt{\delta_i}$ so that

$$\ell_{F_i}(\rho_i)^2 \leq \frac{C(\Gamma)}{\log \delta_i} \to 0.$$

Since $\rho_i \to 0$, for i large, up to passing to a subsequence and permuting the labels, $\gamma_i := \partial D \cap B_{\rho_i}(p_i)$ is disjoint from the arc from b to c on ∂D. Thus, the arc $F_i(\gamma_i) \subset \Gamma$ is disjoint from the arc from B to C on Γ. Combined with the observation that $\ell_{F_i}(\rho_i) \to 0$, the distance between the endpoints of $F_i(\gamma_i)$ tend to zero, we find that the length of $F_i(\gamma_i)$ tends to zero as $i \to \infty$.

Putting this together we have that the length of $F_i(\partial(D \cap B_{\rho_i}(p_i))$ tends to zero as $i \to \infty$. Since $p_i, q_i \in \partial(D \cap B_{\rho_i}(p_i))$ we thus have that $|F_i(p_i) - F_i(q_i)| \to 0$, a contradiction. Thus $F_i|_{\partial D}$ are equicontinuous. Passing to a subsequence, we have that $F_i|_{\partial D}$ are Cauchy in $C^0(\partial D)$. Since $F_i - F_j$ is harmonic, the weak maximum principle (Lemma 3.10) gives

$$\max_D |F_i - F_j| = \max_{\partial D} |F_i - F_j|,$$

so F_i is Cauchy in $C^0(D)$. Thus, we have that the interior F extends to a $C^0(\bar{D})$ function and $F_i \to F$ in $C^0(\bar{D})$. Note that this preserves weak monotonicity on the boundary. Thus $F \in \mathcal{C}_\Gamma$. Fatou's lemma and Lemma 3.3 give

$$\mathcal{A}(F) \leq \mathcal{E}(F) \leq \liminf_{i\to\infty} \mathcal{E}(F_i) = a_\Gamma.$$

Thus $F \in \mathcal{C}_\Gamma$ attains a_Γ and $\mathcal{E}(F) = \mathcal{A}(F)$. Thus F is weakly conformal by Lemma 3.3. □

We used the following interior estimates for harmonic functions (cf. [GT01, Theorem 2.10]):

Lemma 3.15. *If* $\Delta u = 0$ *on* D *and* $D' \subset D$ *then* $\sup_{D'} |D^\alpha u| \leq C(\alpha, D') \sup_D |u|$.

Remark 3.16. In view the proof given above, we can view the classical Plateau problem as a geometric version generalization of the Riemann mapping theorem. Indeed, if $\Gamma \subset \mathbb{C}$ is a Jordan curve, the solution to Plateau's problem for Γ will yield a conformal diffeomorphism between D an the interior of Γ. In fact, the idea of energy minimization and the Courant–Lebesgue lemma can be used to prove the Uniformization Theorem 3.9 used above; cf. [Mor08, 366] and [Str88, p. 29].

3.1. **The holomorphic differential.** Consider $F : \bar{D} \to \mathbb{R}^n$ weakly conformal and harmonic. We recall that we can write the harmonic condition $\Delta F = 0$ in terms of complex derivatives

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

as

$$\partial_{\bar{z}}\partial_z F = 0.$$

Thus

$$(\phi_1, \ldots, \phi_n) := \partial_z F$$

is a $\mathbb{C}^n$-valued holomorphic function on D. Note that

$$4\phi^2 := 4\sum_{k=1}^n \phi_k^2 = \sum_{k=1}^n \big((\partial_x F_k)^2 - (\partial_y F_k)^2 - 2i\partial_x F_k \partial_y F_k\big) = |\partial_x F|^2 - |\partial_y F|^2 - 2i\partial_x F \cdot \partial_y F$$

and

$$4|\phi|^2 = |\partial_x F|^2 + |\partial_y F|^2$$

Thus, since ϕ is weakly conformal we find that $\phi^2 = 0$. Moreover, $|\phi|^2 = \frac{1}{2}\mu$ for μ the induced conformal factor $g = \mu(dx^2 + dy^2)$.

Corollary 3.17. *There is $\mathcal{B} \subset D$ with no limit points so that $F|_{D\setminus\mathcal{B}}$ is an immersion.*

Proof. The function ϕ is holomorphic so its zeroes are isolated. □

We call points $\mathcal{B} \subset D$ where F fails to be an immersion *branch points*. We discuss this further below.

3.2. **Branched minimal immersions.** We will call $F : \bar{D} \to \mathbb{R}^n$ weakly conformal and harmonic a *branched minimal immersion*. Take caution to note that a branched minimal immersion need not be a minimizer for Plateau's problem.

Note, however, that a branched minimal immersion $F : \bar{D} \to \mathbb{R}^n$ is automatically a critical point of the area functional in the following sense. Suppose that F_s is a 1-parameter family of maps (smooth with respect to s) with $F_0 = F$ and $F_s|_{D\setminus K} \equiv F|_{D\setminus K}$ for $K \Subset D$ compact. Then for $V = \frac{d}{ds}\big|_{s=0} F_s$, we have

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}(F_s) = \int_D \nabla F \cdot \nabla V = -\int_D \Delta F \cdot V = 0$$

using that V is compactly supported. On the other hand, we have $\mathcal{A}(F_s) \leq \mathcal{E}(F_s)$ with equality at 0. This implies that

$$\frac{d}{ds}\Big|_{s=0} \mathcal{A}(F_s) = 0,$$

as claimed.

3.3. **The reflection principle.** It's natural to ask about the regularity of the solution to the Plateau problem at the boundary. The following result shows that F is regular up to the boundary in a very special case:

Proposition 3.18. *Consider Γ a smooth Jordan curve and $F : \bar{D} \to \mathbb{R}^n$ in $\mathcal{C}_\Gamma$ a branched minimal immersion. Assume there is a line segment $L \subset \Gamma$. Then F extends by Schwarz reflection across L as a branched minimal immersion.*

Proof. Write $D^+ = \{(x,y) \in D : y > 0\}$ and $\ell = \{(x,0) : |x| < 1\}$.

We can assume that $L \subset \{x_2 = \cdots = x_n = 0\}$ and apply a conformal transformation D to obtain $F : \overline{D^+} \to \mathbb{R}^n$ with $F(\ell) = L$. Then $F_2, \ldots, F_n$ are harmonic functions with zero boundary values on ℓ. Thus, Schwarz reflection allows us to extend them a smooth harmonic function on D via

$$\tilde{F}_i(x,y) = \begin{cases} F_i(x,y) & y \geq 0 \\ -F_i(x,-y) & y < 0. \end{cases}$$

We now consider F_1. We claim that we can extend F_1 by

$$\tilde{F}_1(x,y) = \begin{cases} F_i(x,y) & y \geq 0 \\ F_i(x,-y) & y < 0. \end{cases}$$

Of course we need to check that $\tilde{F}_1$ is harmonic across ℓ. To this end, let $\tilde{\phi} = \partial_z \tilde{F}$ and note that for $y < 0$ we have

$$\tilde{\phi}_1(x,y) = \frac{1}{2}(\partial_x F_1(x,-y) + i\partial_2 F_1(x,-y)) = \overline{\tilde{\phi}_1(x,-y)}$$

Note that $F_i = 0$ along ℓ for $i \geq 2$ implies that $\tilde{\phi}_2, \ldots, \tilde{\phi}_n$ are purely imaginary along ℓ. Using that $\tilde{F}$ is conformal on $D \setminus \ell$, i.e. $\tilde{\phi}^2 = 0$, we obtain

$$\tilde{\phi}_1^2 = -(\tilde{\phi}_2^2 + \cdots + \tilde{\phi}_n^2).$$

Thus, $\operatorname{Im} \tilde{\phi}_1(z) \to 0$ as $z \to \ell$. Combining these facts, Schwarz reflection implies that $\tilde{\phi}_1$ extends holomorphically to D. Thus $\tilde{F}$ is harmonic. We have $\tilde{\phi}^2 = 0$ on D by continuity, so $\tilde{F}$ is weakly conformal. This completes the proof. □

Remark 3.19. As proved by Lewy, this can be (significantly) generalized as follows: If Γ contains a real analytic sub-arc then F can be locally extended past the boundary as a weakly conformal harmonic map. See [Nit89, p. 287].

3.4. **Boundary regularity.** A similar (but easier) argument shows that the least area map from Theorem 3.14 is strictly monotone on the boundary.

Proposition 3.20. *Consider Γ a smooth Jordan curve and $F : \bar{D} \to \mathbb{R}^n$ in $\mathcal{C}_\Gamma$ branched minimal immersion. Then $F|_{\partial D} : \partial D \to \Gamma$ is a homeomorphism.*

Proof. It suffices to prove there cannot be an arc $\gamma \subset \partial D$ so that F is constant on γ. If there was, then after a translation, we can assume that $F(x, y) = 0 \in \mathbb{R}^n$ for all $(x, y) \in \gamma$. We can then use Schwarz reflection to extend F to a harmonic and weakly conformal map $\tilde{F}$ defined on a larger domain. This is a contradiction since $\tilde{F}|_\gamma$ is constant. □

We also have the following boundary regularity result (not proven here):

Theorem 3.21 (Hildebrandt)**.** *For Γ a smooth Jordan curve and $F : \bar{D} \to \mathbb{R}^n$ in $\mathcal{C}_\Gamma$ branched minimal immersion, then $F \in C^\infty(\bar{D})$.*

See [DHKW92, §7.3], [Str88, p. 23], [Nit89, p. 274].

Note that the boundary regularity results (Remark 3.19, Theorem 3.21) assert that the *parametrization* F (not just the image $F(\bar{D})$) inherits regularity of Γ. Of course, these results can be viewed as generalizations of proofs of boundary regularity in the context of the Riemann mapping theorem.

Exercise 3.2. Let Γ be a C^1-regular Jordan curve in $\mathbb{R}^n$. Suppose that $F \in \mathcal{C}_\Gamma \cap C^1(\bar{D})$ is weakly conformal and harmonic. Prove the (non-sharp) isoperimetric inequality $\mathcal{E}(F) \leq \frac{1}{4}\,\mathrm{length}(\Gamma)^2$.

Exercise 3.3. Solve the Plateau problem for C^1-regular Jordan curves by approximation by smooth Jordan curves, the methods of Theorem 3.14, and Exercise 3.2 (you can assume the result from Theorem 3.21).

Remark 3.22. If Γ is an arbitrary Jordan curve (homeomorphic image of $\mathbb{S}^1$ in $\mathbb{R}^n$), it might hold that $a_\Gamma = \infty$. We note that even in this case Douglas was able to find a weakly conformal harmonic map $F : \bar{D} \to \mathbb{R}^n$ with $F|_{\partial D} : \partial D \to \Gamma$ a homeomorphism.

3.5. **Branch points.** We now return to the discussion of branch points in slightly more detail. (Recall that branch points of $F : \bar{D} \to \mathbb{R}^n$ branched minimal immersion are points in D where F fails to be an immersion.)

Definition 3.23. A *false branch point* is one where F locally factors as $F(z) = \tilde{F}(z^Q)$ for some $Q \in \mathbb{N}_{\geq 2}$ and a local immersion $\tilde{F}$. In other words, F fails to be an immersion at a false branch point due to a coordinate singularity. A *true branch point* is a branch point that is not false.

Example 3.24. Branch points can occur. Complex submanifolds in $\mathbb{C}^n$ are area-minimizing (we will discuss this later). In particular, $F(z) = (z^2, z^3) \in \mathbb{C}^2$ defines a least area map of $D \to \mathbb{R}^4$ with a true branch point at $z = 0$.

Example 3.25. Following [Law77, p. 77-78] we can construct an example of a branched minimal surface (with a true branch point) in $\mathbb{R}^3$ as follows. Let ℓ_1, ℓ_2 denote two straight line segments (starting at the origin) of length 1 and $1+\varepsilon$ respectively in $\mathbb{R}^3$ meeting at an angle $\frac{2\pi}{3}$. Choose a curve between the endpoints that does not lie in the (ℓ_1, ℓ_2)-plane to form a piecewise smooth Jordan curve Γ. Solve Plateau's problem for Γ. (One may prove that the solution is free of branch points in this case.) Reflect across the lines 5 times to close up into a branched minimal surface with a branch point at the origin.

Example 3.25 gives a branched minimal immersion, but it cannot be an area minimizer thanks to the following result.

Theorem 3.26 (Osserman [Oss70], Gulliver [Gul73])**.** *A solution to the classical Plateau Problem 3.1 in $\mathbb{R}^3$ has no branch points on D.*

Sketch of the proof. Osserman ruled out true branch points (roughly) as follows. Suppose that F has a true branch point at $z_0 \in D$. One can find distinct curves $\gamma_1, \gamma_2 : [0, \varepsilon) \to D$ with $\gamma_i(0) = z_0$ and $F(\gamma_1(t)) = F(\gamma_2(t))$ is a transversal self-intersection (as an example, consider ℓ_1, ℓ_2 in Example 3.25).[3] Then "cut" the disk D along γ_1, γ_2 to introduce $\gamma_i^{\pm}$ (cf. Figure 2). Then glue $\gamma_1^+(t)$ to $\gamma_1^-(t)$ and $\gamma_2^+(t)$ to $\gamma_2^-(t)$. This gives a new piecewise smooth map $\tilde{F} : \bar{D} \to \mathbb{R}^3$ differing only on a set of measure zero, so $\mathcal{A}(\tilde{F}) = \mathcal{A}(F)$. However, we can "round the corners" to decrease area slightly, a contradiction.

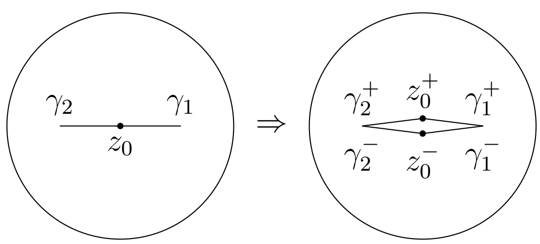

FIGURE 2. Osserman's area-decreasing modification.

Gulliver ruled out false branch points (roughly) as follows. If F locally factors through $z \mapsto z^Q$ for $Q \in \mathbb{N}_{\geq 2}$ then (away from the isolated branch point) the image of F has "multiplicity Q." Using a unique continuation argument one can extend this multiplicity all the way to ∂D, implying that F transverses the boundary Q times, a contradiction.[4] □

Exercise 3.4. Suppose that $F : \bar{D} \to \mathbb{R}^3$ is a branched minimal immersion and $z_0 \in D$.

(1) Show that up to a rotation, dilation, and translation of $\mathbb{R}^3$ we have

$$F_1 + iF_2 = (z - z_0)^Q + O(|z - z_0|^{Q+1}), \qquad F_3 = O(|z - z_0|^{Q+1})$$

where $Q \geq 2$ if and only if z_0 is a branch point.

[3]This is where $n = 3$ is used in an essential way, one should compare with $z \mapsto (z^2, z^3) \in \mathbb{C}^2$.
[4]Note that $F : z \mapsto z^2$ is a weakly conformal harmonic map $\bar{D} \subset \mathbb{C} \to \bar{D} \subset \mathbb{C}$ but $F|_{\partial D}$ is not a weakly monotone parametrization.

(2) Assuming that z_0 is a branch point, show that the unit normal $N = \frac{F_x \times F_y}{|F_x \times F_y|}$ extends continuously across z_0.
(3) Let $P = F(z_0)$. Show that F is transversal to $\partial B_\varepsilon(P)$ for all $\varepsilon > 0$ sufficiently small. Let $\gamma_\varepsilon = F^{-1}(\partial B_\varepsilon(P))$ and k_ε denote the intrinsic geodesic curvature of γ_ε. Compute $\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} k_\varepsilon$.
(4) Assuming that Γ is smooth, use (3) along with Gauss–Bonnet and Theorem 3.21 to prove that F has only finitely many branch points.

There are several open problems about branch points. The following is one of the oldest open problems in the area:

Open Question 1. For $\Gamma \subset \mathbb{R}^3$ a smooth Jordan curve and $F : \bar{D} \to \mathbb{R}^3$ a solution to the Plateau problem for Γ can Γ have branch points at the boundary?

Interestingly, it's known that boundary branch points cannot exist when $\Gamma \subset \mathbb{R}^n$ is analytic [Whi97] (even though interior branch points can exist in $\mathbb{R}^{\geq 4}$). On the other hand, the map

$$F : \{x + iy : x \geq 0\} \to \mathbb{C}^2 \qquad z \mapsto (z^3, e^{-\frac{1}{\sqrt{z}}})$$

can be seen to define a least area solution with boundary branch point along smooth $\Gamma \subset \mathbb{R}^4$. Finally, we note that a smooth Jordan curve in $\mathbb{R}^3$ bounding a branched minimal surface with a boundary branch point is constructed in [Gul91], but it's not known if F is minimizing.

Exercise 3.5. If a smooth Jordan curve Γ lies in the boundary of a convex set $K \subset \mathbb{R}^n$ prove that a branched minimal immersion F spanning Γ:

(1) has $F(D)$ contained in the interior of K and
(2) has no boundary branch points.

(Hint: for (1) use the strong maximum principle for some linear function of the coefficients of F and for (2) use the Hopf boundary point lemma.)

3.6. **Embeddedness.** We emphasize that even when it is free of branch points, the solution to Plateau's problem need not be an embedding (for example, consider $\Gamma \subset \mathbb{R}^3$ knotted). However, in certain cases one may prove that the least area disk is embedded. The following holds in greater generality than stated:

Theorem 3.27 (Meeks–Yau [MY82]). *Suppose that a smooth Jordan curve Γ lies in the boundary of a compact convex set $K \subset \mathbb{R}^3$. Any solution to the Plateau problem for Γ will be an embedding.*

See [CM11, §6] for an overview of the proof.

4. Harmonic maps

We now consider the case of the (classical) Plateau problem where $\mathbb{R}^n$ is replaced by a Riemannian manifold.

Theorem 4.1 (Morrey [Mor48])**.** *For (M,g) a closed Riemannian manifold and $\Gamma \subset (M,g)$ a homotopically trivial smooth Jordan curve, there's an element of $\mathcal{C}_\Gamma$ of least area.*

Remark 4.2. Most of the various regularity results discussed above (boundary regularity, non-existence of interior branch points in three dimensions) can be extended to the Riemannian case as well.

We won't prove Theorem 4.1. Instead we will instead discuss the work [SU81] of Sacks–Uhlenbeck concerning minimizing energy in a homotopy class of maps $\Sigma \to (M,g)$ for Σ a closed oriented surface.

For simplicity, we consider the target (M, g_M) to be isometrically embedded in some $\mathbb{R}^N$ (possible by Nash embedding). For $F : \Sigma \to M \subset \mathbb{R}^N$ we can then define area $\mathcal{A}(F)$ to be the area of $F : \Sigma \to \mathbb{R}^N$. To define the energy, fix a Riemannian metric h on Σ and set

$$\mathcal{E}(F,h) := \int_\Sigma |\nabla_h F|^2 dV_h.$$

Note that $\mathcal{E}(F,h)$ only depends on the conformal class $[h]$ (cf. Lemma 3.11). We recall that the existence of isothermal coordinates (cf. [Che55]), i.e. x, y with $h = \mu(dx^2 + dy^2)$ lets us identify a conformal class $[h]$ with a Riemann surface structure by declaring $z = x + iy$ to be a holomorphic chart.

Lemma 4.3. *$\mathcal{A}(F) \leq \mathcal{E}(F,h)$ with equality if and only if F is weakly conformal*[5]*.*

Proof. Using a partition of unity, it suffices to check the inequality in local isothermal charts where it's the same as Lemma 3.3. □

Lemma 4.4. *If $F : (\Sigma, h) \to M \subset \mathbb{R}^N$ is smooth and is a critical point of $\mathcal{E}(\cdot, h)$ among compactly supported smooth variations $F_s : \Sigma \to M \subset \mathbb{R}^N$ if and only if $(\Delta_h F)^\top = 0$.*

Definition 4.5. We call F satisfying $(\Delta_h F)^\top = 0$ a *harmonic map.*

Proof. Given a variation F_s, note that $\frac{d}{ds}\big|_{s=0} F_s(p) = \dot{F}(p) := V(p) \in T_{F(p)}M \subset \mathbb{R}^N$. Conversely, given a C^∞ map $V : \Sigma \to \mathbb{R}^N$ with

$$V(p) \in T_{F(p)}M \subset \mathbb{R}^N \text{ for all } p \in \Sigma \tag{4.1}$$

[5]with respect to $[h]$

we can let $F_s(p) = \exp_{F(p)}(sV(p)) \in M$ for s sufficiently small. Thus, F will be a critical point of energy if and only if

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}(F_s, h) = \int_\Sigma \nabla_h F \cdot \nabla_h V = -\int_\Sigma (\Delta_h F) \cdot V = 0$$

for any such V. □

Suppose that X is a vector field on $\mathbb{R}^N$ with $X \in T_P^\perp M$ for all $P \in M$. Recall that if U, V are vector fields tangent to M, then

$$D_U X \cdot V = U(X \cdot V) - X \cdot D_U V = -X \cdot \vec{A}(U, V)$$

where $\vec{A}(U,V) = (D_U V)^\perp$ is the second fundamental form of $M \subset \mathbb{R}^N$. Thus, for $\nu = X \circ F$, we compute in local isothermal coordinates:

$$\begin{aligned} \Delta F \cdot \nu &= \sum_{i=1}^2 \partial_i^2 F \cdot \nu \\ &= \sum_{i=1}^2 \partial_i (\underbrace{\partial_i F \cdot \nu}_{=0}) - \partial_i F \cdot \partial_i \nu \\ &= -\sum_{i=1}^2 \partial_i F \cdot D_{\partial_i F} X \\ &= \sum_{i=1}^2 \vec{A}(\partial_i F, \partial_i F) \cdot \nu \end{aligned}$$

where we used that $\partial_i F \in T_{F(p)} M$ in the second line. Thus, we can (somewhat imprecisely) write the harmonic map equation as

$$\Delta F = \vec{A}(dF, dF). \tag{4.2}$$

Note that (unlike in the $M = \mathbb{R}^n$ case) this is a nonlinear PDE and thus we can expect to face difficulties in establishing existence/regularity. Take note that $\vec{A}$ is actually the second fundamental form of M *evaluated* at $F(p)$ so it would be more correct to write $\vec{A} \circ F$.

4.1. **The Hopf differential.** Given a map $F : (\Sigma, h) \to M$ we can define the *Hopf differential* in isothermal coordinates by

$$\Phi = \left(|\partial_x F|^2 - |\partial_y F|^2 - 2i \partial_x F \cdot \partial_y F\right) dz^2 = 4(\partial_z F)^2 dz^2.$$

As before, we have:

Lemma 4.6. *The Hopf differential vanishes $\Phi = 0$ if and only if F is weakly conformal.*

This is a quadratic differential on Σ. We recall that dz is locally a section of the holomorphic tangent bundle and dz^2 is a section of the symmetric square of the holomorphic tangent bundle. Thus, to check that Φ is a quadratic differential we can check that under a holomorphic change of coordinates $w = w(z)$ then

$$4(\partial_z F)^2 dz^2 = 4(\partial_w F)^2 w'(z)^2 dz^2 = 4(\partial_w F)^2 dw^2.$$

We emphasize that Φ need not have holomorphic coefficients. However, we have:

Lemma 4.7. *For $F : (\Sigma, h) \to M \subset \mathbb{R}^N$ smooth harmonic map, the Hopf differential Φ is holomorphic.*

Proof. In isothermal coordinates we have

$$\partial_{\bar{z}}(\partial_z F)^2 = 2\partial_{\bar{z}}\partial_z F \cdot \partial_z F = \frac{1}{2}\Delta F \cdot \partial_z F = 0$$

since $(\Delta F)^\top = 0$ and $\partial_z F$ is the (complex linear) combination of the tangent vectors $\partial_x F$ and $\partial_y F$. □

Remark 4.8. Lemma 4.7 can be used to show that $F \in W^{1,2}(\Sigma; M)$ that satisfy $(\Delta F)^\top = 0$ in the weak sense are actually smooth. This is false for higher-dimensional domains. See [Sch84, Riv95, Hél02].

Corollary 4.9. *If $F : S^2 \to M \subset \mathbb{R}^n$ is a smooth harmonic map then it's weakly conformal.*

Proof. Combine Lemma 4.7 with the fact that a holomorphic quadratic differential on S^2 must vanish. □

Exercise 4.1. Give an alternative proof of Corollary 4.9 as follows. Write $\Phi = \phi dz^2$ in a stereographic projection chart on $S^2 \setminus \{p\}$ and show that $\phi \in L^1(\mathbb{C})$ is holomorphic. Using this prove that $\phi \equiv 0$.

Corollary 4.9 does not hold if we replaced S^2 by some other Riemann surface. For example, dz^2 is a holomorphic quadratic differential on $T^2 = \mathbb{C}/\Lambda$. However:

Exercise 4.2. By computing the first variation of $\mathcal{E}(F, \cdot)$ with respect to h, show that if $\mathcal{E}(F, h)$ is stationary for variations of h then F is weakly conformal.

5. Sacks–Uhlenbeck

We now fix (Σ, h) and try to find a harmonic map $F : (\Sigma, h) \to M$. To find a weakly conformal harmonic map (when $\Sigma \neq S^2$) we can then try to vary h.

5.1. α-harmonic maps.

One way to do this is to introduce the α-energy of Sacks–Uhlenbeck:

$$\mathcal{E}_\alpha(F) = \int_\Sigma \left((1 + |\nabla_h F|^2)^\alpha - 1\right) dV_h$$

As before, we see that F is a critical point of $\mathcal{E}_\alpha(\cdot)$ if and only if F satisfies the α-*harmonic map equation*

$$\Delta F + (\alpha - 1)\frac{d|\nabla F|^2 \cdot dF}{1 + |\nabla F|^2} = \vec{A}(dF, dF)$$

(in the weak sense).

Proposition 5.1. *Consider $G : \Sigma \to M$ smooth and the corresponding set of homotopic maps $[G] \in [\Sigma, M]$. For $\alpha \in (1, \alpha_0)$, there's a weakly α-harmonic map $F : (\Sigma, h) \to M \subset \mathbb{R}^N$ in $[G]$ that minimizes $\mathcal{E}_\alpha(F)$ among maps in $[G]$.*

Proof. We can apply the direct method. Let $F_i \in [G]$ be a minimizing sequence for $\mathcal{E}_\alpha$. A bound on $\mathcal{E}_\alpha(F)$ gives a bound on $F \in W^{1,2\alpha}$ and thus $F \in C^{\frac{\alpha-1}{\alpha}}$ by Morrey–Sobolev (C.1). Thus, a F_i converges subsequentially in C^0 and weakly in $W^{1,2\alpha}$ to F. The C^0-convergence guarantees[6] that $F \in [G]$. □

Lemma 5.2. *There's $\alpha_0 = \alpha_0(\Sigma, h) > 1$ so that if $F \in W^{1,2\alpha}(\Sigma, h)$ is a weak solution to the α-harmonic map equation for $\alpha \in (1, \alpha_0)$ then $F \in C^1(\Sigma)$. Moreover, for $\alpha \in [1, \alpha_0)$ we can estimate*

$$\|F\|_{W^{2,p}(\Sigma)} \leq C\left(1 + \|\nabla F\|_{L^\infty(\Sigma)}^{2\frac{p-1}{p}} \mathcal{E}_1(F)^{\frac{1}{p}}\right)$$

for any $p \in (1, \infty)$ and $C = C(M, \Sigma, h, p)$ independent of α.

Proof. We have[7]

$$|\Delta F| \leq 2(\alpha - 1)|D^2 F| + C|dF|^2$$

Thus $W^{2,p}$-elliptic estimates (cf. (A.1)) give

$$\|D^2 F\|_{L^p(\Sigma)} \leq C(\|F\|_{L^\infty(\Sigma)} + \|\Delta F\|_{L^p(\Sigma)}) \leq C(\|F\|_{L^\infty(\Sigma)} + (\alpha - 1)\|D^2 F\|_{L^p(\Sigma)} + \|dF\|^2_{L^{2p}(\Sigma)})$$

For $\alpha - 1$ sufficiently small, we can absorb the Hessian term to obtain

$$\|F\|_{W^{2,p}(\Sigma)} \leq C(1 + \|dF\|^2_{L^{2p}(\Sigma)}). \tag{5.1}$$

As long as $\alpha > 1$ we can take $p = \alpha$ and use $F \in W^{1,2\alpha}(\Sigma)$ yields $F \in W^{2,\alpha}(\Sigma)$. Thus, Sobolev embedding (cf. (C.2)) gives $F \in W^{1,\frac{2\alpha}{2-\alpha}}(\Sigma)$. We can take $p = \frac{\alpha}{2-\alpha} > \alpha$ in (5.1) and so on until we get $F \in W^{2,p}$ for $p > n$ in which case Morrey–Sobolev gives $F \in C^1$ as claimed. The final estimate follows by using $\int_\Sigma |\nabla F|^{2p} \leq \|\nabla F\|^{2(p-1)}_{L^\infty} \int_\Sigma |\nabla F|^2$ in (5.1). □

It's important to note that C is independent of α.

[6]If $G, \tilde{G}$ are C^0 close then $F(x)$ and $\tilde{F}(x)$ are connected by a unique minimizing geodesic in M so we can construct a homotopy by moving "linearly" along these geodesics.

[7]Strictly speaking, we do not know that $F \in W^{2,p}$ so this step is only formal. To make it rigorous, we could freeze the lower order coefficients in the α-harmonic map and mollify the Laplacian/Hessian term. The argument used here gives $W^{2,p}$ estimates for the mollified function, which then limit to corresponding estimates for F.

5.2. **Bubbling.** We now choose $\alpha_i \searrow 1$ and α_i-harmonic maps F_i (from Proposition 5.1). Note that $\mathcal{E}_1(F_i)$ is uniformly bounded since $\mathcal{E}_{\alpha_i}(F_i) \leq \mathcal{E}_{\alpha_i}(G)$ is uniformly bounded for an arbitrary fixed $G \in [G] \cap C^\infty$.

Let $\lambda_i := \max_\Sigma |\nabla_h F_i|$.

We first suppose that $\sup_i \lambda_i < \infty$. Lemma 5.2 gives that F_i is uniformly bounded in $W^{2,4}(\Sigma)$. Morrey–Sobolev (cf. (C.1)) embedding thus bounds $F_i \in C^{1,\frac{1}{2}}(\Sigma)$. Thus, passing to a subsequence, F_i converges in C^1 to $F \in [G] \cap C^{1,\frac{1}{2}}(\Sigma)$. Note that F minimizes $\mathcal{E}(\cdot, h)$ in $[G]$. Indeed, if $\tilde{F} \in [G]$ has $\mathcal{E}(\tilde{F}) \leq \mathcal{E}(F) - \delta$ then $F_i \to F$ in C^1 gives

$$\mathcal{E}(\tilde{F}) \leq \mathcal{E}(F) - \delta = \mathcal{E}_{\alpha_i}(F_i) - \delta + o(1) \leq \mathcal{E}_{\alpha_i}(\tilde{F}) - \delta + o(1) = \mathcal{E}(\tilde{F}) - \delta + o(1)$$

This is a contradiction for i sufficiently large. In particular, F is a weakly harmonic map.

On the other hand, if $\lambda_i \to \infty$ (after passing to a subsequence), we can choose $p_i \in \Sigma_i$ so that $|\nabla_h F_i|(p_i) = \lambda_i$. Let $h_i = \lambda_i^2 h$ denote the conformally changed metric. This gives $\|\nabla_{h_i} F\|_{L^\infty(\Sigma)} \leq 1$. We now note that $|D^2_{h_i} F|_{h_i} = \lambda_i^{-2} |D^2_h F|_{h_i}$ so Lemma 5.2 gives

$$\|D^2_{h_i} F_i\|_{L^p(\Sigma, h_i)} = \lambda_i^{2\frac{1-p}{p}} \|D^2_h F_i\|_{L^p(\Sigma,h)} \leq C \lambda_i^{2\frac{1-p}{p}} (1 + \lambda_i^{2\frac{p-1}{p}}) \leq C.$$

We now choose normal coordinates around p_i. As such, we can consider F_i, h_i defined on an exhaustion of $\mathbb{R}^2$ so that h_i converges to δ in C^∞_{loc}. By Morrey–Sobolev (cf. (C.1)) we have that F_i is bounded in $C^{1,\frac{1}{2}}_{\mathrm{loc}}$ and thus converges to $F : \mathbb{R}^2 \to M \subset \mathbb{R}^n$ with $|\nabla F| \leq 1, |\nabla F|(0) = 1$ and $F \in C^{1,\frac{1}{2}}_{\mathrm{loc}}$. As above, we can prove that F minimizes $\mathcal{E}(\cdot)$ among homotopic maps fixed outside of a compact set. Thus F is weakly harmonic. We also observe that Fatou's lemma gives $\mathcal{E}(F) < \infty$.

5.3. **Bootstrapping regularity.** Given $F : (\Sigma, h)$ or $\mathbb{C} \to M$ weakly harmonic with $F \in C^{1,\alpha}_{\mathrm{loc}}$, we note that $\vec{A}(dF, dF) \in C^\alpha_{\mathrm{loc}}$. Schauder estimates (cf. (A.2)) then imply that $F \in C^{2,\alpha}_{\mathrm{loc}}$. Thus $\vec{A}(dF, dF) \in C^{1,\alpha}_{\mathrm{loc}}$. Continuing this, we find that $F \in C^\infty_{\mathrm{loc}}$.

Corollary 5.3. *The harmonic map obtained in the previous section is in C^∞.*

Similarly (using Lemma 5.2 and then a similar bootstrap) we have

Lemma 5.4. *If F is a harmonic map with $\|\nabla F\|_{L^\infty(\Omega)} \leq 2$ then $\|F\|_{C^k(\Omega')} \leq C$ for $\Omega' \Subset \Omega$.*

5.4. **ε-regularity.** Given a harmonic map $F : S^2 \to M$ we can compose with a conformal diffeomorphism $S^2 \to S^2$ to obtain a new harmonic map with the same energy. Thus, controlling $\mathcal{E}(F)$ is not enough to bound F in C^k. However, if $\mathcal{E}(F)$ is sufficiently small, it does suffice:

Theorem 5.5 (ε-regularity)**.** *If a harmonic map $F : (D_2, \delta) \to M \subset \mathbb{R}^N$ has $\int_{D_2} |\nabla F|^2 \leq \varepsilon_0(M)$ then $\|\nabla F\|_{L^\infty(D_1)} \leq C(M)$.*

We give a proof based on "point-picking" as opposed to the original PDE approach.

Proof. If this fails, there's a sequence of harmonic maps $F_j : D_2 \to M$ with $\int_{D_2} |\nabla F_j|^2 \to 0$ but $\|\nabla F_j\|_{L^\infty(D_1)} \to \infty$. We can adjust the domain slightly to assume that F_j is smooth up to ∂D_2. Thus, there is some $p_j \in D_2$ so that

$$(2 - |p_j|)|\nabla F_j|(p_j) = \max_{p\in D_2} ((2 - |p|)|\nabla F_j|(p)) \to \infty$$

Let $\lambda_j = |\nabla F_j|(p_j)$ and define the dilated map

$$\tilde{F}_j(z) := F_j(p_j + \lambda_j^{-1} z).$$

If $|p_j| + \lambda_j^{-1}|z| < 2$ then z is in the domain of $\tilde{F}_j$. Rearranging this, see that the domain of $\tilde{F}_j$ includes $D_{(2-|p_j|)\lambda_j} := D_{r_j}$. Note that $r_j \to \infty$. We also observe that since

$$\nabla \tilde{F}_j(z) = \lambda_j^{-1} \nabla F_j(p_j + \lambda_j^{-1} z), \qquad \Delta \tilde{F}_j(z) = \lambda_j^{-2} \Delta F_j(p_j + \lambda_j^{-1} z),$$

we see that $\tilde{F}_j$ is still a harmonic map to M with $|\nabla \tilde{F}_j|(0) = 1$.

Fix $R > 0$ and consider $z \in D_R$. For j large enough so that $R < r_j$, the choice of p_j gives

$$(2 - |p_j| - \lambda_j^{-1} R)|\nabla F_j|(p_j + \lambda_j^{-1} z) \le (2 - |p_j + \lambda_j^{-1} z|)|\nabla F_j|(p_j + \lambda_j^{-1} z) \le (2 - |p_j|)\lambda_j,$$

so

$$|\nabla \tilde{F}_j|(z) \le 1 + \frac{R}{(2 - |p_j|)\lambda_j - R} = 1 + o(1)$$

as $j \to \infty$. Elliptic boostrapping (Lemma 5.4) thus gives $\|\tilde{F}_j\|_{C^k(B_R)} \le C(k, R)$ for all $k \in \mathbb{Z}_{\ge 0}, R > 0$, so a subsequence converges in $C^\infty_{\text{loc}}(\mathbb{R}^2)$ to a harmonic map $\tilde{F} : \mathbb{R}^2 \to M$ with $|\nabla \tilde{F}|(0) = 1$. On the other hand, we have that (by conformal invariance of energy)

$$\int_{D_R} |\nabla \tilde{F}_j|^2 \le \int_{D_2} |\nabla F_j|^2 \to 0$$

from which we see that $\tilde{F}$ must be a constant harmonic map. This contradicts the fact that $|\nabla \tilde{F}|(0) = 1$. □

Remark 5.6. We can interpolate $L^\infty \subset C^k \cap L^2$ to improve the conclusion to $\|\nabla F\|_{L^\infty(D_1)} \le C_\delta \mathcal{E}(F)^{\frac{1}{2}-\delta}$ for all $\delta > 0$. (See [SU81, Proposition 3.1] for $\delta = 0$.)

5.5. **Removable singularity.** Recall that in the case of bubbling, we obtained a harmonic map $F : \mathbb{C} \to M \subset \mathbb{R}^N$ with $F \in C^\infty_{\text{loc}}$ and $\mathcal{E}(F) < \infty$. We claim that one can add the "point at infinity" to obtain a smooth harmonic map $F : S^2 \to M \subset \mathbb{R}^N$. This follows by inverting to $F : \mathbb{C} \setminus \{0\} \to M$ and applying Sacks–Uhlenbeck's removable singularity theorem:

Theorem 5.7 (Removable singularity)**.** *If $F : D \setminus \{0\} \to M \subset \mathbb{R}^N$ is a smooth harmonic map with $\mathcal{E}(F) < \infty$ then F extends to a smooth harmonic map on D.*

We first have the (standard) fact that we can extend weak solutions across a set of zero capacity.

Lemma 5.8. $\Delta F = \vec{A}(dF, dF)$ *in the weak sense on* D

Proof. We use a "log-cutoff." For $\varepsilon > 0$ we set

$$\varphi(r) = \begin{cases} 0 & r < \varepsilon^2 \\ 2 - \frac{\log r}{\log \varepsilon} & \varepsilon^2 \leq r \leq \varepsilon \\ 1 & r > \varepsilon. \end{cases}$$

Note that

$$\int_D |\nabla \varphi|^2 = 2\pi \int_{\varepsilon^2}^{\varepsilon} \frac{1}{r^2 (\log \varepsilon)^2} r dr = \frac{1}{|\log \varepsilon|} = o(1)$$

as $\varepsilon \to 0$. For $V \in C_c^\infty(D; \mathbb{R}^N)$ we have

$$\left| \int_D \varphi(\nabla F \cdot \nabla V - \vec{A}(dF, dF) \cdot V) \right| = \left| \int_D \nabla_V F \cdot \nabla \varphi \right| \leq C(V) \int_{\operatorname{supp} \varphi} |\nabla F|^2 + |\nabla \varphi|^2 \to 0$$

as $\varepsilon \to 0$. This completes the proof. □

We change coordinates from (r, θ) to (t, θ) defined by $r = e^{-t}$. Note that the flat metric becomes

$$dr^2 + r^2 d\theta^2 = e^{-2t}(dt^2 + d\theta^2).$$

Since energy and the harmonic map equation are both conformally invariant, we can consider a harmonic map $F : [0, \infty) \times S^1 \to M$ with finite energy where we use the metric $dt^2 + d\theta^2$ on the domain. Note that

$$\mathcal{E}(F|_{[T,\infty) \times S^1}) \to 0$$

as $T \to \infty$. In particular, ε-regularity implies that $\tilde{F}_T(t, \theta) := F(t - T, \theta)$ converges *subsequentially* in C^∞_{loc} as $T \to \infty$ to a constant map. However, this constant might *a priori* depend on the chosen subsequence. Morally, the key step in the proof of Theorem 5.7 is to prove that there is a unique limit (with a quantitative rate of convergence).

Lemma 5.9. $\int_{\{t\} \times S^1} |\partial_t F|^2 d\theta = \int_{\{t\} \times S^1} |\partial_\theta F|^2 d\theta$

Proof. We compute

$$\begin{aligned} \frac{d}{dt} \int_{\{t\} \times S^1} (|\partial_t F|^2 - |\partial_\theta F|^2) d\theta &= 2 \int_{\{t\} \times S^1} (\partial_{tt}^2 F \cdot \partial_t F - \partial_{t\theta} F \cdot \partial_\theta F) d\theta \\ &= 2 \int_{\{t\} \times S^1} ((\partial_{tt}^2 F + \partial_{\theta\theta}^2 F) \cdot \partial_t F - \partial_\theta (\partial_t F \cdot \partial_\theta F)) d\theta \\ &= 0 \end{aligned}$$

using $(\Delta F)^\top = 0$. Thus $\int_{\{t\} \times S^1} (|\partial_t F|^2 - |\partial_\theta F|^2) d\theta \equiv c$. Finiteness of energy $\mathcal{E}(F) = \int_0^\infty \int_{S^1} (|\partial_t F|^2 + |\partial_\theta F|^2) \, d\theta dt < \infty$ gives that $c = 0$. □

Lemma 5.10. $P(t) := \int_{\{t\} \times S^1} |\partial_\theta F|^2 d\theta$ *satisfies* $P(t) \to 0$ *as* $t \to \infty$ *and* $P''(t) \geq P(t)$ *for* t *sufficiently large.*

Proof. As discussed above, $|\nabla F|(t,\theta) \to 0$ as $t \to \infty$. We thus compute (writing $A * B$ to represent the product of two tensors A, B with some indices traced and then the quantity multiplied by some uniformly bounded coefficient):

$$\begin{aligned}
P''(t) &= 2\int_{\{t\}\times S^1} \left(|\partial^2_{\theta t}F|^2 + \partial_\theta F \cdot \partial^3_{\theta tt}F\right)\, d\theta \\
&= 2\int_{\{t\}\times S^1} \left(|\partial^2_{\theta t}F|^2 - \partial^2_{\theta\theta}F \cdot \partial^2_{tt}F\right)\, d\theta \\
&= 2\int_{\{t\}\times S^1} \left(|\partial^2_{\theta t}F|^2 + |\partial^2_{\theta\theta}F|^2 - A(dF,dF)\cdot \partial^2_{\theta\theta}F\right)\, d\theta \\
&= 2\int_{\{t\}\times S^1} \left(|\partial^2_{\theta t}F|^2 + |\partial^2_{\theta\theta}F|^2 + \partial_\theta(A(dF,dF))\cdot \partial_\theta F\right)\, d\theta \\
&= 2\int_{\{t\}\times S^1} \left(|\partial^2_{\theta t}F|^2 + |\partial^2_{\theta\theta}F|^2 + (dF)^2 * (\partial_\theta F)^2 + \partial F * \partial^2_{\theta *}F * \partial_\theta F\right)\, d\theta \\
&\geq \int_{\{t\}\times S^1} \left(\frac{3}{2}|\partial^2_{\theta\theta}F|^2 - \frac{1}{2}|\partial_\theta F|^2\right)\, d\theta
\end{aligned}$$

for t sufficiently large. In the second to last line we note that the third term arises from $\partial_\theta A = DA * \partial_\theta F$ since A is evaluated at F. In the final step we used $|\nabla F| \to 0$ to absorb the second term into the Hessian terms (and then discarded $\partial_{\theta t}F$). Since $\int_{\{t\}\times S^1} \partial_\theta F\, d\theta = 0$, the Poincaré inequality gives

$$\int_{\{t\}\times S^1} |\partial_\theta F|^2 \leq \int_{\{t\}\times S^1} |\partial^2_{\theta\theta}F|^2$$

the assertion follows. □

Exercise 5.1. If $P''(t) \geq P(t)$ for $t \in [T_0, \infty)$ and $P(t) \to 0$ as $t \to \infty$, show that $P(t) \leq P(T_0)e^{-t}$ for $t \in [T_0, \infty)$.

Exercise 5.2. Prove that $F(t, \theta)$ has a unique limit as $t \to \infty$ (uniformly in θ).

Proof of Removable Singularity Theorem 5.7. Combining Lemmas 5.9 and 5.10 with Exercise 5.1 we find (in cylindrical coordinates)

$$\int_T^\infty \int_{\{t\}\times S^1} |\nabla F|^2 = O(e^{-T}).$$

Returning the polar coordinates this gives

$$\int_{D_r} |\nabla F|^2 = O(r).$$

Note that $\int_{D_{|z|}(z)} |\nabla F|^2 \to 0$ as $z \to 0$. Thus, for z sufficiently small we can rescale $D_{|z|}(z)$ to D_2 (energy is unchanged) and apply ε-regularity and Remark 5.6

$$|z||\nabla F|(z) = O(|z|^{\frac{1}{2}-\delta}).$$

(The $|z|$ factor arises in the gradient when scaling back.) This implies that $F \in W^{2,p}$ for all $p \in [1,4)$. Since F is a weak solution to the harmonic map equation on all of D (Lemma 5.8), we can then repeat the argument used in the elliptic bootstrap to conclude that $F \in C^\infty(D)$ is a smooth harmonic map. □

5.6. **Existence of harmonic maps.** In sum, we've obtained:

Theorem 5.11 (Sacks–Uhlenbeck [SU81])**.** *Consider $G : \Sigma \to M$ smooth. There's either $F : (\Sigma, h) \to M$ smooth harmonic map minimizing $\mathcal{E}(\cdot)$ in $[G] \in [\Sigma, M]$ or else there's a nontrivial smooth harmonic map $F : S^2 \to M$ with $[F] \neq 0 \in \pi_2(M)$.*

Corollary 5.12. *If $\pi_2(M) = 0$ then there's a smooth energy minimizing harmonic map $F : (\Sigma, h) \to M$ in any homotopy class $[\Sigma, M]$.*

Corollary 5.13. *If $\pi_2(M) \neq 0$ there's a smooth harmonic map $F : S^2 \to M$ with $[F] \neq 0 \in \pi_2(M)$.*

Note that in the bubbling case, we only proved that F minimizes among homotopic maps that fix a neighborhood of ∞. This could be removed but it will suffice for our later applications.

5.7. **Varying the conformal structure and Douglas type conditions.** Suppose that $\Sigma \neq S^2$ and $\pi_2(M) = 0$. Fix a class in $[\Sigma, M]$. Given any Riemann surface structure (Σ, h) we can obtain a minimizing harmonic map $F_h : (\Sigma, h) \to M$. To obtain a branched minimal immersion we need to minimize $[h] \mapsto \mathcal{E}(F_h, [h])$ over all Riemann surface structures (cf. Exercise 4.2). This could pose a major problem since this set is non-compact.

In certain cases the non-compactness can be avoided:

Theorem 5.14 (Schoen–Yau [SY79])**.** *If $G : \Sigma \to M$ has $G_* : \pi_1(\Sigma) \to \pi_1(M)$ injective then there exists a branched minimal immersion $\Sigma \to M$ in $[G] \in [\Sigma, M]$.*

When $\dim M = 3$ one may obtain a least area/energy *immersion* in this manner by ruling out true[8] branch points as in Theorem 3.26. For example:

Corollary 5.15. *If (T^3, g) is any Riemannian metric on a 3-torus, then there's a least area immersion $F : T^2 \to (T^3, g)$.*

The basic idea of Theorem 5.14 is that if the conformal class degenerates then one may find a very long cylindrical isothermal chart $[-T, T] \times S^1$ in (Σ, h). Then, by an argument as in the Courant–Lebesgue lemma we can conclude that some circle $\{t\} \times S^1$ is mapped to a very short loop and is thus homotopically trivial.

[8]Since $\partial\Sigma = \emptyset$ it could happen that the minimizer F is the composition of an immersion with a branched cover $\Sigma \to \Sigma$. By discarding the branched cover, we can find a least area immersion (at the cost of changing the homotopy class).

Remark 5.16. Similar considerations hold in the classical Plateau problem with higher topology. For example, let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n$ be disjoint (smooth) Jordan curves. Let $a_{\Gamma_1 \cup \Gamma_2}$ be the minimal area among all maps of annuli. Douglas proved that if the "Douglass criterion"

$$a_{\Gamma_1 \cup \Gamma_2} < a_{\Gamma_1} + a_{\Gamma_2}$$

then there's a branched minimal annulus with boundary $\Gamma_1 \cup \Gamma_2$. (Note that $\leq$ always holds by connecting the two minimal disks by a thin tube.) The Courant–Lebesgue lemma shows that if the conformal class of the annuli are degenerating, then a minimizing sequence can be "cut" into two disks of nearly the same area.

6. The homological Plateau problem

The mapping problem does not seem to work well with higher dimensional domains [Whi83]. Instead whats works is minimizing in a homology class:

Theorem 6.1 (Federer–Flemming [FF60])**.** *For (M^n, g) a closed Riemannian manifold and $\sigma \in H_k(M; \mathbb{Z})$, there exists a singular submanifold $\Sigma \in \sigma$ of least area. When $k = n-1$ and $n \leq 7$, Σ will be completely smooth.*

One may also consider a similar problem for $\Gamma^k \subset \mathbb{R}^n$ (or a Riemannian manifold) and minimize area among all "submanifolds" Σ^{k+1} with $\partial\Sigma = \Gamma$. For $k = 1$, by [HS79], the minimizer Σ will be smooth embedded and will solve the Douglas problem for surfaces of genus g for any $g \geq \operatorname{genus}\Sigma$ (if $\operatorname{genus}\Sigma > 0$ there will also be a Douglas–Radó minimal disk with boundary Γ but it will have area $\geq \operatorname{area}(\Sigma)$).

Part 2. First variation of area

We've seen some methods for finding least area "submanifolds." We now turn to the analysis of the Euler–Lagrange equations for this problem.

7. Mean curvature and first variation

For an embedded submanifold $\Sigma \subset (M, g)$ a *vector field* along Σ is a smooth map $X : \Sigma \to TM$ with $X(p) \in T_pM$ for all $p \in \Sigma$.[9]

Definition 7.1. We define the *divergence of X along* Σ by

$$\operatorname{div}_\Sigma X = \sum_{i=1}^k g(D_{e_i}X, e_i) = \operatorname{tr}_{T\Sigma} DX$$

where $e_1, \ldots, e_k \in T_p\Sigma$ is an orthonormal basis.

[9]Alternatively: if F is the inclusion map $F : \Sigma \to M$ then vector fields along F are $X \in \Gamma(F^*TM)$.

Note that D here is the connection on M, so strictly speaking we need to extend X locally near p. However, since the e_i are tangential to Σ the value of $g(D_{e_i}X, e_i)$ is seen to be independent of this extension (exercise!).[10]

Theorem 7.2 (First variation I)**.** *Consider $F_t : \Sigma \to (M, g)$ a 1-parameter family of embeddings with $F_t = F_0$ outside of a compact set. For $\dot{F}_0 = X$ the velocity, we have*

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}_g(F_t(\Sigma)) = \int_\Sigma \operatorname{div}_\Sigma X$$

Proof for $(M, g) = \mathbb{R}^n$. In a time-independent coordinate chart $x^1, \ldots, x^k$ on Σ, the induced metric is

$$F_t^* g_{\mathbb{R}^n}(\partial_i, \partial_j) = \langle \partial_i F_t, \partial_j F_t \rangle$$

so the induced volume form becomes

$$d\mu(t) = \sqrt{\det \langle \partial_i F_t, \partial_j F_t \rangle}\, dx^1 \cdots dx^k$$

in these coordinates. We assume that the coordinates are chosen so that at $t = 0$ and at $p \in \Sigma$, $h_{ij} = \delta_{ij}$. In particular $\partial_1 F_0, \ldots, \partial_k F_0$ is an orthonormal basis for $T_p\Sigma$.

We compute[11]

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \sqrt{\det \langle \partial_i F_t, \partial_j F_t \rangle} &= \frac{1}{2} \frac{d}{dt}\Big|_{t=0} \det \langle \partial_i F_t, \partial_j F_t \rangle \\ &= \frac{1}{2} \operatorname{tr} \frac{d}{dt}\Big|_{t=0} \langle \partial_i F_t, \partial_j F_t \rangle \\ &= \sum_{i=1}^k \langle \partial_i X, \partial_i F_0 \rangle \\ &= \operatorname{div}_\Sigma X. \end{aligned}$$

Writing

$$\operatorname{area}(F_t(\Sigma)) = \int_\Sigma d\mu(t)$$

and differentiating under the integral sign completes the proof. □

Essentially same proof works for general ambient (M, g) if we choose normal coordinates near $F_t(p)$ since the first derivatives of g vanish at the center of normal coordinates.[12]

We now recall that if D is the Levi-Civita connection on (M, g) then if U, V are vector fields along $\Sigma \subset (M, g)$ tangent to Σ then

$$D_U V = \nabla_U V + \vec{A}(U, V) \tag{7.1}$$

[10] One could have defined the pullback connection F^*D on F^*TM and then define $\operatorname{div}_\Sigma X$ using this pullback connection.

[11] using $\det(I + \varepsilon A) = 1 + \varepsilon \operatorname{tr} A + O(\varepsilon^2)$

[12] Alternatively, we can use the pullback connection for the space-time map $F : \Sigma \times (-\varepsilon, \varepsilon) \to (M, g)$ (one has to check the pullback connection is symmetric, which we used above to interchange the i and t derivatives).

where ∇ is the Levi-Civita connection on Σ (with the induced metric) and $\vec{A}$ is the *second fundamental form* of Σ. In fact, this is an orthogonal decomposition of $D_U V$ into tangential $\nabla_U V$ and normal $\vec{A}(U,V)$. Recall that symmetry of the connections implies that $\vec{A}(U,V)$ is symmetric in U and V. We define the *mean curvature* vector of Σ by

$$\vec{H} = \operatorname{tr}_\Sigma \vec{A} = \sum_{i=1}^{k} \vec{A}(e_i, e_i)$$

for $e_1, \ldots, e_k$ an orthonormal basis for $T_p\Sigma$.

Theorem 7.3 (First variation II)**.** *If $\partial\Sigma$ is non-empty, let η be the outwards pointing unit co-normal. Then for any compactly supported vector field X along Σ, we have*

$$\int_\Sigma \operatorname{div}_\Sigma X = -\int_\Sigma g(\vec{H}, X) + \int_{\partial\Sigma} g(X, \eta).$$

Proof. Split X as

$$X = X^\perp + X^\top.$$

Then for $e_1, \ldots, e_k$ orthonormal basis of $T_p\Sigma$, we have

$$g(D_{e_i} X^\perp, e_i) = -g(X^\perp, D_{e_i} e_i) = -g(X, \vec{A}(e_i, e_i)).$$

We also have

$$g(D_{e_i} X^\top, e_i) = g(\nabla_{e_i} X^\top, e_i).$$

Thus,

$$\operatorname{div}_\Sigma X = \operatorname{div} X^\top - g(\vec{H}, X).$$

where the second divergence is the usual (intrinsic) divergence on Σ. We can use the divergence theorem to get

$$\int_\Sigma \operatorname{div} X^\top = \int_{\partial\Sigma} g(X, \eta).$$

This completes the proof. □

We'll define the first variation operator of Σ by

$$\delta\Sigma(X) = \int_\Sigma \operatorname{div}_\Sigma X = -\int_\Sigma g(\vec{H}, X) + \int_{\partial\Sigma} g(X, \eta). \tag{7.2}$$

It's easy to see that for any compactly supported X, there's F_t compactly supported that has velocity X at $t = 0$. The above results show that $\frac{d}{dt}|_{t=0} \operatorname{area}_g(F_t(\Sigma)) = \delta\Sigma(X)$ and then how to compute $\delta\Sigma(X)$ in terms of $\vec{H}$ and the boundary term.

Corollary 7.4. *$\Sigma \subset (M, g)$ has $\vec{H} \equiv 0$ if and only if $\frac{d}{dt}|_{t=0} \operatorname{area}_g(F_t(\Sigma)) = 0$ for any compactly supported variation F_t of the inclusion F_0 with $F_t|_{\partial\Sigma} = F_0|_{\partial\Sigma}$.*

We thus call $\Sigma \subset (M, g)$ with $\vec{H} \equiv 0$ *minimal surfaces* (more precisely, minimal submanifolds).

Example 7.5. Consider $\Sigma = \{(x,y,0) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}$. Then,

$$\delta\Sigma(X) = \int_{\partial\Sigma} g(X,\eta),$$

since the disk is totally geodesic (and thus minimal). Thus we see:

(1) If X is tangent to Σ and compactly supported then $\delta\Sigma(X) = 0$. This is the infinitesimal version of the fact that area is diffeomorphism invariant.
(2) If $X = (0,0,1)$ then $\delta\Sigma(X) = 0$, as expected since X is the velocity field of the upwards translation isometries.
(3) If $X = (x,y,z)$ is a dilation vector field, then $\delta\Sigma(X) = 2\pi = \frac{d}{dt}\big|_{t=0}\pi(1+t)^2$ as expected.

Remark 7.6. It's often useful to note that we actually proved the infinitesimal first variation formula: $\frac{d}{dt}\big|_{t=0} d\mu(t) = (\operatorname{div}_\Sigma X) d\mu(0)$. If X is a normal vector field, we find the following frequently used fact $\frac{d}{dt}\big|_{t=0} d\mu(t) = -g(\vec{H}, X) d\mu(0)$.

7.1. **Two-sided hypersurfaces.** If $\Sigma^n \subset (M^{n+1}, g)$ is a hypersurface with a unit normal N, we can write $\vec{H} = -HN$ for the *scalar mean curvature* H. The sign[13] is chosen here to simplify the first variation of area. Indeed, if we vary Σ with velocity $X = \varphi N$ then we have

$$\delta\Sigma(\varphi N) = \int_\Sigma H\varphi$$

for $u \in C_c^\infty(\Sigma)$. It's useful to also define the scalar second fundamental form by $\vec{A}(U,V) = -A(U,V)N$, so $H = \operatorname{tr} A$.

Given the unit normal, we also define the *shape operator* by $S(U) = D_U N$. Since $|N|^2 = 1$ we see that $g(S(U), N) = 0$, so $S_p : T_p\Sigma \to T_p\Sigma$.

Lemma 7.7. *The shape operator and scalar second fundamental form are related by* $g(S(U), V) = A(U,V)$. *In particular,* $S_p : T_p\Sigma \to T_p\Sigma$ *is self-adjoint.*

Proof. For U, V vector fields tangent to Σ, we can differentiate $g(V,N) = 0$ and use compatibility of the metric and connection to get

$$0 = g(D_U V, N) + g(V, D_U N) = -A(U,V) + g(V, S(U)).$$

This completes the proof. □

We call the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the shape operator the *principal curvatures*.

[13] Caution: There is reasonable agreement on the definition of vector mean curvature and convention for sign of the scalar mean curvature (anyone should agree that $S^2 \subset \mathbb{R}^3$ has vector mean curvature pointing inwards and scalar mean curvature is positive). However, there is not a uniform convention on how to relate the vector to scalar mean curvature, since one might take the *inwards* pointing unit normal, in which case the convention would need to be "$\vec{H} = H\nu$" or the *outwards* pointing unit normal in which case it's $\vec{H} = -H\nu$ (as we do in these notes).

Corollary 7.8. *The scalar mean curvature satisfies*[14] $H = \lambda_1 + \cdots + \lambda_n = \operatorname{div}_\Sigma N$.

Proof. We saw that $H = \operatorname{tr} A = \operatorname{tr} S$. The trace of a self-adjoint map is the sum of the eigenvalues, proving the first expression. For the second, choose $e_1, \ldots, e_n \in T_p\Sigma$ orthonormal and write the trace as

$$H = \sum_{i=1}^n g(S(e_i), e_i) = \sum_{i=1}^n g(D_{e_i}N, e_i) = \operatorname{div}_\Sigma N.$$

This completes the proof. □

We also note for later that $|A|^2 = \lambda_1^2 + \cdots + \lambda_n^2$.

7.2. **Gaussian curvature of minimal surface.** We recall that the Gaussian curvature[15] of $\Sigma^2 \subset \mathbb{R}^3$ can be defined extrinsically by $K = \lambda_1\lambda_2$.

Corollary 7.9. *If $\Sigma^2 \subset \mathbb{R}^{n+1}$ is minimal then $2K = -|A|^2$.*

Proof. Locally we can always choose N. Then $\lambda_1 = -\lambda_2$ using $H = 0$. □

In particular, minimal surfaces in $\mathbb{R}^3$ are negatively curved (intrinsically). Similar considerations using the traced Gauss equations give that the scalar curvature of a minimal hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ satisfies $R = -|A|^2$ is negative. Note that the sectional/Ricci curvatures of Σ need not be non-positive!

Example 7.10. For example, letting $\Sigma = \partial B_1 \subset \mathbb{R}^3$ be the unit sphere, taking $N(x) = x$ to be the outwards pointing unit normal we note that $D_U N = U$ for any $U \in T_p\mathbb{R}^3$, so we thus have $H = \operatorname{div}_\Sigma N = 2$. Alternatively, we could use the infinitesimal first variation $\frac{d}{dt}\Big|_{t=0} d\mu(t) = H\varphi d\mu(0)$ where φ is the normal speed, combined with the variation $F_t(p) = (1+t)p$ to derive the same thing.

Exercise 7.1. Consider a warped product metric

$$g := dt^2 + u(t)^2 g_N$$

for a smooth function $u(t) > 0$ on an interval I and (N, g_N) a fixed Riemannian manifold. Show that $N = \partial_t$ is a unit normal to $\Sigma_t = \{t\} \times N$. What is the (scalar) mean curvature of Σ_t?

---

[14] Note that classically H was defined to be the *average* (mean) of the principal curvature. This is less commonly used now, but "$H = \frac{\lambda_1 + \cdots + \lambda_n}{n}$" still appears in some references. We will use the "sum" not "average" convention.

[15] Recall that $2K = R$ is the scalar curvature of Σ.

8. Basic consequences of minimality

Theorem 8.1 (Coordinate functions are harmonic)**.** *A submanifold $\Sigma^k \subset \mathbb{R}^n$ is minimal if and only if the coordinate functions on $\mathbb{R}^n$ restrict to harmonic functions on Σ.*

Proof. One may compute directly (exercise!) but we will use the first variation formula. Let E_j be a coordinate vector field on $\mathbb{R}^n$ and set $X = \varphi E_j$. Since E_j is parallel, we find

$$\operatorname{div}_\Sigma X = \sum_{i=1}^k \langle D_{e_i} X, e_i\rangle = \sum_{i=1}^k D_{e_i}\varphi \langle E_j, e_i\rangle = \langle \nabla_\Sigma \varphi, E_j\rangle = \left\langle \nabla_\Sigma\varphi, \nabla_\Sigma x^j\right\rangle$$

Thus, for any $\varphi \in C^\infty_c(\Sigma\setminus\partial\Sigma)$ we find

$$\int_\Sigma \varphi \Delta_\Sigma x^j = -\int_\Sigma \left\langle \nabla_\Sigma\varphi, \nabla_\Sigma x^j\right\rangle = -\delta\Sigma(\varphi E_j) = \int_\Sigma \varphi\left\langle \vec{H}, E_j\right\rangle$$

Since φ was arbitrary, this proves that $\Delta_\Sigma \vec{x} = \vec{H}$ (meaning that the expression holds coordinate by coordinate). This proves the assertion. □

Corollary 8.2 (Convex hull property)**.** *For $\Sigma^k \subset \mathbb{R}^n$ a compact minimal surface, we let $\mathcal{C}(\partial\Sigma)$ denote the convex hull of $\partial\Sigma$. Then $\Sigma \subset \mathcal{C}(\partial\Sigma)$.*

This is clearly false for non-compact Σ as can be seen by e.g. $\{(x,y,0)\in\mathbb{R}^3 : x^2+y^2\geq 1\}$.

Proof. Suppose that $\partial\Sigma \subset \{\langle x,a\rangle \leq t\}$ for $a\in\mathbb{R}^n\setminus\{0\}$ and $t\in\mathbb{R}$. We claim that $\Sigma\subset\{\langle x,a\rangle\leq t\}$. If not, the maximum of $\langle x,a\rangle$ is attained at some point in the interior of Σ. However, we just saw that $\Delta_\Sigma\langle x,a\rangle = 0$. This contradicts the weak maximum principle[16]. □

Corollary 8.3. *If $\Sigma^k\subset\mathbb{R}^n$ is minimal and has $\partial\Sigma=\emptyset$, then Σ is non-compact.*

Corollary 8.4. *Consider $\Sigma^2\subset\mathbb{R}^n$ a compact minimal surface with boundary so that Σ is homeomorphic to the disk. For a compact convex set K with $K\cap\partial\Sigma=\emptyset$, it holds that $K\cap\Sigma$ is simply connected.*

Proof. Suppose that $\gamma\subset K\cap\Sigma$ is a simple closed curve. Then $\gamma=\partial D$ for some disk $D\subset\Sigma$. By the convex hull property and $\gamma\subset K$ we find $D\subset K$. This completes the proof. □

Exercise 8.1. Formulate and prove a generalization of Corollary 8.4 that holds for higher dimensional $\Sigma^k\subset\mathbb{R}^n$.

Proposition 8.5 (Flux)**.** *Suppose that $\Sigma^k\subset(M,g)$ is minimal and K is a Killing vector on (M,g). Then, if $\Gamma_1,\Gamma_2\subset\Sigma$ are oriented hypersurfaces with $[\Gamma_1]=[\Gamma_2]\in H_{k-1}(\Sigma)$, then*

$$\int_{\Gamma_1} g(K,\eta_{\Gamma_1}) = \int_{\Gamma_2} g(K,\eta_{\Gamma_2})$$

[16]For the Laplacian on $\mathbb{R}^n$ this is stated in Lemma 3.10, but the same result holds for a general elliptic equation of the form: $a^{ij}D^2_{ij}u + b^iD_iu\geq 0$, cf. [GT01, Theorem 3.1].

for η_{Γ_i} *the oriented co-normal to* Γ_i *in* Σ.

Thus, this defines $F_K : H_{k-1}(\Sigma) \to \mathbb{R}$, the *flux* map.

Proof. Recall that a Killing vector satisfies $g(D_X K, Y) + g(D_Y K, X) = 0$ for any vector fields X, Y, so in particular we see that $g(D_X K, X) = 0$. Tracing this over an orthonormal basis of $T_p\Sigma$ we find $\operatorname{div}_\Sigma K = 0$. Since Σ is minimal, we thus have

$$\operatorname{div}_\Sigma K^\top = 0.$$

We can then apply the divergence theorem to the vector field $K^\top$ and k-chain with Ω in Σ with $\partial\Omega = \Gamma_1 - \Gamma_2$ to prove the assertion. □

9. Monotonicity

Proposition 9.1 (Cone inequality)**.** *Suppose that* $\Sigma^k \subset B_R(0) \subset \mathbb{R}^n$ *is a minimal submanifold with* $\partial\Sigma \subset \partial B_R$. *Then* $|\Sigma| \leq \frac{R}{k}|\partial\Sigma|$.

Remark 9.2. Note that $\operatorname{cone}(\partial\Sigma) := \{tp : p \in \partial\Sigma\}$ has area $= \int_0^R \frac{r^{k-1}}{R^{k-1}}|\partial\Sigma| dr = \frac{R}{k}|\partial\Sigma|$ so if Σ had least area (among competitors fixing $\partial\Sigma$) we could prove this by observing that $|\Sigma| \leq |\operatorname{cone}(\partial\Sigma)|$ (assuming that the cone is a valid competitor). The proof below will only use stationarity (Σ is assumed to be minimal as opposed to area-minimizing).

Proof. Take $X(x) = x$. Note that $DX = \operatorname{Id}$ so $\operatorname{div}_\Sigma X = k$. Thus, the first variation formula (Theorem 7.3) yields

$$k|\Sigma| = \int_\Sigma \operatorname{div}_\Sigma X = \int_{\partial\Sigma} X \cdot \eta \leq R|\partial\Sigma|$$

This completes the proof. □

We write ω_k for the volume of the unit ball in $\mathbb{R}^k$.

Theorem 9.3 (Monotonicity)**.** *Suppose that* $\Sigma^k \subset \mathbb{R}^n$ *is a minimal submanifold with* $\partial\Sigma \cap B_R = \emptyset$. *Then for* $0 < s < r < R$ *we have*

$$r^{-k}|\Sigma \cap B_r| - s^{-k}|\Sigma \cap B_s| = \int_{\Sigma \cap (B_r \setminus B_s)} |x|^{-k-2}|x^\perp|^2,$$

so in particular

$$r \mapsto \frac{|\Sigma \cap B_r(x)|}{\omega_n r^n} := \Theta_\Sigma(x, r)$$

is monotone nondecreasing.

Proof. We can assume that Σ intersects $\partial B_r, \partial B_s$ transversally. Consider $X(x) = |x|^{-k}x$. Note that $D|x| = \frac{x}{|x|}$ so

$$DX = |x|^{-k}\operatorname{Id} - k|x|^{-2-k} x \otimes x^\flat.$$

Thus, if $e_1, \dots, e_k$ is an orthonormal basis of $T_p\Sigma$, we find that

$$\operatorname{div}_\Sigma X = k|x|^{-2-k}\left(|x|^2 - \sum_{i=1}^k (e_i \cdot x)^2\right) = k|x|^{-2-k}|x^\perp|^2$$

Below, we write η for the outwards pointing unit co-normal to $\Sigma \cap B_t$ for $t \in \{s, r\}$. Apply the first variation formula twice as follows:

$$\begin{aligned}\int_{\Sigma\cap(B_r\setminus B_s)} k|x|^{-2-k}|x^\perp|^2 &= \int_{\Sigma\cap(B_r\setminus B_s)} \operatorname{div}_\Sigma X \\ &= r^{-k}\int_{\Sigma\cap\partial B_r} \eta\cdot x - s^{-k}\int_{\Sigma\cap\partial B_s}\eta\cdot x \\ &= r^{-k}\int_{\Sigma\cap B_r}\operatorname{div}_\Sigma x - s^{-k}\int_{\Sigma\cap B_s}\operatorname{div}_\Sigma x \\ &= k\left(r^{-n}|\Sigma\cap B_r| - s^{-n}|\Sigma\cap B_s|\right).\end{aligned}$$

This completes the proof. □

Corollary 9.4. *If $\Sigma^k \subset \mathbb{R}^n$ has $\partial\Sigma \cap B_R(0) = \emptyset$ and $0 \in \Sigma$ then $|\Sigma \cap B_r(0)| \geq \omega_k r^k$ with equality if and only if $\Sigma \cap B_r(0)$ is a flat disk.*

Proof. Observe that $\lim_{s\searrow 0}\Theta_\Sigma(0, s) = 1$ since smooth embedded submanifolds are nearly flat at small scales. Thus, monotonicity gives

$$\frac{|\Sigma\cap B_r|}{\omega_n r^n} = 1 + \omega_n^{-1}\int_{\Sigma\cap B_r}|x|^{-k-2}|x^\perp|^2 \geq 1.$$

If equality held, we would get that $x^\perp \equiv 0$ along Σ, i.e. Σ is a cone centered at 0. Since Σ is smooth it must be a flat disk. □

The monotonicity formula places strong constraints on the geometry/behavior of a minimal surface. For example the following result rules out "tentacle" type behavior:

Corollary 9.5. *Suppose that Σ_i is a sequence of minimal submanifolds in $B_2 \subset \mathbb{R}^n$ with $|\Sigma_i| \leq \Lambda$. Pass to a subsequence so that the volume measures on Σ_i converge weakly, i.e. for $f \in C_0(B_2)$ it holds that*

$$\int_{\Sigma_i} f dV_{\Sigma_i} \to \int f d\mu$$

for some Radon measure on B_2. Then if $x_i \in \Sigma_i$ has $x_i \to x \in B_2$ then $x \in \operatorname{supp}\mu$.

Proof. For $t < r < d(x, \partial B_2)$, choose $f \leq \chi_{B_r(x)}$ so that for i large $\chi_{B_{r-t}(x_i)} \leq f$. Then we have

$$\limsup_{i\to\infty}|\Sigma_i \cap B_{r-t}(x_i)| \leq \limsup_{i\to\infty}\int_{\Sigma_i} f dV_{\Sigma_i} = \int f d\mu \leq \mu(B_r(x))$$

On the other hand, since $x_i \in \Sigma_i$, monotonicity (Corollary 9.4) gives

$$\omega_k(r-t)^k \leq |\Sigma_k \cap B_{r-t}(x_i)| \leq \mu(B_r(x)).$$

Letting $t \to 0$ we get $\mu(B_r(x)) \geq \omega_k r^k$. This completes the proof. □

10. The Gehring link problem

This section roughly follows [BS83].

Lemma 10.1. *If a compact minimal submanifold $\Sigma^k \subset \mathbb{R}^n$ has $\partial\Sigma \cap B_R(0) = \emptyset$ and $0 \in \Sigma$ then $k\omega_k R^{k-1} \leq |\partial\Sigma|$. Equality holds only for Σ a flat disk.*

Proof. Take $X(x) = |x|^{-k}x$ on $\Sigma \setminus B_s$. As in the proof of montonicity, we can let $s \to 0$ to find

$$k\omega_k \leq \int_\Sigma k|x|^{-2-k}|x^\perp|^2 + k\omega_k = \int_{\partial\Sigma} |x|^{-k}x \cdot \eta.$$

Using $|x| \geq R$ on $\partial\Sigma$ we find

$$k\omega_k \leq R^{1-k}|\partial\Sigma|$$

which proves the assertion. □

Corollary 10.2 (Gehring link problem)**.** *Suppose that Γ_1, Γ_2 are smooth Jordan curves in $\mathbb{R}^3$ that have non-zero linking number and* $\operatorname{dist}(\Gamma_1, \Gamma_2) \geq 1$*. Then* $\min\{|\Gamma_1|, |\Gamma_2|\} \geq 2\pi$*.*

Proof. Find a minimal surface $\Sigma_1 \subset \mathbb{R}^3$ with $\partial\Sigma_1 = \Gamma_1$. Up to a translation, $0 \in \Gamma_2 \cap \Sigma_1$, so $\Gamma_1 \cap B_1 = \emptyset$. Lemma 10.1 gives $|\Gamma_1| \geq 2\pi$. □

To find Σ_1 we could use homological area-minimization. Alternatively, we could argue that Lemma 10.1 holds for the solution to the classical Plateau problem.

Remark 10.3. Using the homology minimizers, the result extends to all dimensions, cf. [BS83]. In fact, for $\Gamma^k \subset \mathbb{R}^n$ closed submanifold, if we defined $\operatorname{FillRad}(\Gamma)$ to be the infimum of $r > 0$ so that $[\Gamma] = 0 \in H_k(U_r(\Gamma))$ (for $U_r(\Gamma)$ denote the r-tubular neighborhood) then we can prove (in the same manner) the *Euclidean filling radius inequality* $|\Gamma| \geq k\omega_k \operatorname{FillRad}(\Gamma)^k$.

See also [Mat75, ES76, Oss76, Gag80, Ere09, Gut10].

11. Fáry–Milnor

We discuss the Eckholm–White–Wienholtz proof [EWW02] of the Fáry–Milnor theorem.

Given $\Sigma^k \subset \mathbb{R}^n$ compact minimal submanifold, let

$$E(\partial\Sigma) := \{tx : x \in \partial\Sigma, t \geq 1\}$$

be the exterior cone over $\partial\Sigma$.

Proposition 11.1. *Assuming that $0 \in \Sigma \subset B_R(0)$ for a compact minimal surface, we have $k\omega_k R^{k-1} \leq |E(\partial\Sigma) \cap \partial B_R|$.*

Proof. Take $X = |x|^{-k}x$ as before. The first variation formula gives

$$\int_{\partial\Sigma} \eta_\Sigma \cdot X = k\omega_k + \int_\Sigma \operatorname{div}_\Sigma X$$

$$\int_{E(\partial\Sigma)\cap\partial B_R} |x|^{-1}x\cdot X - \int_{\partial\Sigma} |x|^{-1}x\cdot X = \int_{E(\partial\Sigma)\cap B_R} \operatorname{div}_{E(\partial\Sigma)} X + \vec{H}\cdot X.$$

Even though $\vec{H}$ may not be $= 0$ on $E(\partial\Sigma)$, we have $\vec{H}\cdot x = 0$ since x is tangent to the exterior cone. Note also that $|x|^{-1}x \cdot X = |x|^{-k+1}$ and $\eta_\Sigma \cdot X \leq |x|^{-k+1}$ along $\partial\Sigma$. Thus, since $\operatorname{div}_\Sigma X \geq 0$, we have

$$\begin{aligned} k\omega_k &\leq \int_{E(\partial\Sigma)\cap\partial B_R} |x|^{-1}x\cdot X + \int_{\partial\Sigma}(\eta_\Sigma\cdot X - |x|^{-1}x\cdot X) \\ &\leq R^{-k}\int_{E(\partial\Sigma)\cap\partial B_R} |x|^{-1}x\cdot x \\ &= R^{1-k}|E(\partial\Sigma)\cap\partial B_R| \end{aligned}$$

This completes the proof. □

Exercise 11.1. Show that the proof of Proposition 11.1 proves that $r \mapsto r^{-k}|(\Sigma\cup E(\partial\Sigma))\cap B_r|$ is non-decreasing, i.e. $\Sigma\cup E(\partial\Sigma)$ satisfies the monotonicity formula for all radii. This is due to Gromov (rediscovered in [EWW02]).

We now assume that $k = 2$.

Lemma 11.2. *Then the geodesic curvature of $E(\partial\Sigma)\cap\partial B_R \subset E(\partial\Sigma)$ satisfies $\kappa = R^{-1}$.*

Proof. Parametrize $E(\partial\Sigma)\cap\partial B_R$ by unit speed as $\gamma(t)$. Then, the (vector) curvature as a curve in $\mathbb{R}^n$ is $\vec{k} = \gamma''(t)$ and the geodesic curvature satisfies $\kappa = -\eta\cdot\vec{k}$. Note that $\eta(\gamma(t)) = |\gamma(t)|^{-1}\gamma(t)$ and since $\gamma(t)\cdot\gamma(t)$ is constant, we get $\gamma''(t)\cdot\gamma(t) = -|\gamma'(t)|^2 = -1$. Putting this together, the assertion follows. □

Thus

$$R^{-1}|E(\partial\Sigma)\cap B_R| = \int_{E(\partial\Sigma)\cap\partial B_R}\kappa = \int_{E(\partial\Sigma)\cap B_R} K - \int_{\partial(E(\partial\Sigma))}\kappa = -\int_{\partial(E(\partial\Sigma))}\kappa \leq \int_{\partial\Sigma}|\vec{k}|.$$

We used Gauss–Bonnet and that $E(\partial\Sigma)$ has $K = 0$ (since it contains a radial line, there's one zero principal curvature). As such, we obtain

$$2\pi = \omega_1 \leq R^{-1}|E(\partial\Sigma)\cap B_R| \leq \int_{\partial\Sigma}|\vec{k}|. \tag{11.1}$$

(Note that $=$ holds if and only if Σ is a flat disk.) This is not surprising, since we know that $\int_\gamma |\vec{k}| \geq 2\pi$ for any closed loop in $\mathbb{R}^n$. However, the argument we gave actually proves something stronger:

Theorem 11.3 (Fáry–Milnor [Fár49, Mil50])**.** *If $\Gamma\subset\mathbb{R}^3$ has $\int_\Gamma|\vec{k}| \leq 4\pi$ then Γ is unknotted.*

Proof. Let $F : \bar{D} \to \mathbb{R}^3$ be the Plateau solution for Γ. We will assume that F is an immersion on ∂D; this could be arranged by e.g. perturbing Γ to be real analytic. Let $\Sigma = F(\bar{D})$. The first-variation formula will hold for Σ as well (even though it's not necessarily embedded). If Γ is knotted, we can assume that 0 is an immersed point for Σ, i.e. at least two sheets of Σ cross at 0. This gives $\lim_{s\to 0} s^{-2}|\Sigma \cap \partial B_s| \geq 2\omega_1 = 4\pi$. As such, if we repeat the proof of (11.1), we will get $4\pi \leq \int_{\partial\Sigma} |\vec{k}|$. If equality held, then Γ would be a convex curve contained in a plane in which case it's unknotted. This completes the proof. □

Note that we did not actually need to assume that F is an immersion since the first-variation may be proven for branched minimal immersions (the derivation we gave works essentially verbatim). Taking more care with the above argument we have:

Theorem 11.4 (Eckholm–White–Wienholtz)**.** *If $\Gamma \subset \mathbb{R}^n$ is a smooth Jordan curve with $\int_\Gamma |\vec{k}| \leq 4\pi$ and Σ is a branched minimal immersion with $\partial\Sigma = \Gamma$ then Σ is embedded and is in particular free of interior and boundary branch points.*

See [EWW02] for further discussion and references.

Note that the example of a branched minimal surface has boundary curve approximately transversing a circle twice and thus has total curvature $= 4\pi + \varepsilon$. See also [AT77, Hub80].

It's natural to ask about minimal surfaces of other topologies. For example [EWW02, §5], there's $\Gamma \subset \mathbb{R}^3$ smooth Jordan curve with $\int_\Gamma |\vec{k}| < 4\pi$ but Γ bounds a minimal Möbius strip. To construct Γ take two copies of a convex polygon in $\mathbb{R}^2$. Joining them at a common point yields an immersed piecewise smooth of "total curvature" 4π. Rotating each slightly (in opposite directions) around a line containing the joined point yields a polygonal curve in $\mathbb{R}^3$ that has a single self-intersection. This procedure strictly decreases the total curvature since e.g. if the angle at the (unrotated) crossing is measured via $\cos\alpha = (x, y, 0) \cdot (-x, y, 0) = y^2 - x^2$ then the rotation yields $\cos\alpha_\theta = (x, \cos\theta y, \sin\theta y) \cdot (-x, \cos\theta y, -\sin\theta y) = x^2 + (\cos^2\theta - \sin^2\theta)y^2 - x^2$. See Figure 3 Thus, one may smooth the curve out to yield Γ of

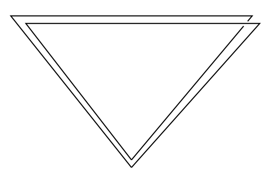

FIGURE 3. A piecewise smooth Jordan curve in $\mathbb{R}^3$ with total curvature $< 4\pi$ that bounds a least area Möbius strip.

total curvature $< 4\pi$. Any disk with boundary Γ will approximately have area at least $2 \times$ (area of the polygon). On the other hand, one may find a Möbius strip bounded by Γ with much less area. The Douglas criterion for non-orientable surfaces thus implies that Γ bounds a least area Möbius strip. In this direction, White recently proved the following result using mean curvature flow (combined with the original Milnor proof of Theorem 11.3):

Theorem 11.5 (White, [Whi22])**.** *For $\Gamma \subset \mathbb{R}^3$ smooth Jordan curve:*

(1) If $\int_\Gamma |\vec{k}| \leq 3\pi$ then any minimal surface bounded by Γ must be a disk.
(2) If $\int_\Gamma |\vec{k}| \leq (1.014) \times 3\pi$ then any orientable minimal surface bounded by Γ must be a disk.

This raises the following:

Open Question 2. What is the least K so that if $\int_\Gamma |\vec{k}| \leq K$ then any (orientable) minimal surface bounded by Γ is a disk. In the orientable case, the 4π-*conjecture* asks if $K = 4\pi$ (this is the largest possible in light of examples [AT77, Hub80]).

12. The isoperimetric inequality

Recall the classical isoperimetric inequality says that if $\Omega \subset \mathbb{R}^n$ is a compact region with smooth boundary then $|\partial\Omega|^{\frac{n}{n-1}} \geq |\partial B|^{\frac{n}{n-1}}|B|^{-1}|\Omega|$ (i.e. the ball has least surface area for fixed volume). An important property of minimal submanifold is that they continue to satisfy the isoperimetric inequality (in many cases with the optimal constant). This section draws from the exposition in [Bre23].

Theorem 12.1 (Carleman [Car21], Reid [Rei59], Hsiung [Hsi61])**.** *Consider $\Sigma^2 \subset \mathbb{R}^n$ compact minimal submanifold with one boundary component $\partial\Sigma$. Then $|\partial\Sigma|^2 \geq 4\pi|\Sigma|$*

Proof. By scaling, we can assume that $|\partial\Sigma| = 2\pi$. Parametrize $\partial\Sigma$ by unit speed $\alpha : S^1 \to \partial\Sigma \subset \mathbb{R}^n$. Up to a translation, we can assume that $\int_0^{2\pi} \alpha(s)ds = 0$. Thus, Wirtinger's inequality (Poincaré inequality on $\mathbb{S}^1$) gives

$$\int_0^{2\pi} \alpha_i(s)^2 \leq \int_0^{2\pi} \alpha_i'(s)^2 \Rightarrow \int_{\partial\Sigma} |x|^2 \leq 2\pi.$$

Thus, letting $X = x$, we have

$$2|\Sigma| = \int_\Sigma \operatorname{div}_\Sigma X = \int_{\partial\Sigma} X \cdot \eta \leq |\partial\Sigma|^{\frac{1}{2}} \left(\int_{\partial\Sigma} |x|^2\right)^{\frac{1}{2}} \leq 2\pi,$$

so $|\Sigma| \leq \pi$. This completes the proof. □

The generalization to higher dimensions (or all topological types) has been a long-standing open problem with many partial results (cf. [Bre23] for citations). A recent breakthrough of Brendle resolved this for minimal surfaces of co-dimension ≤ 2. For hypersurfaces:

Theorem 12.2 (Brendle [Bre21])**.** *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a compact minimal hypersurface then $|\partial\Sigma|^{\frac{n}{n-1}} \geq |\partial B^n|^{\frac{n}{n-1}}|B^n|^{-1}|\Sigma|$.*

In co-dimension ≥ 3 we have:

(1) Almgren [Alm86]: if $\Sigma^k \subset \mathbb{R}^n$ is[17] area minimizing then Σ satisfies the sharp Euclidean isoperimetric inequality $|\partial\Sigma|^{\frac{k}{k-1}} \geq |\partial B^k|^{\frac{k}{k-1}}|B^k|^{-1}|\Sigma|$.
(2) Michael–Simon [MS73]: If $\Sigma^k \subset \mathbb{R}^n$ is[18] a minimal submanifold then Σ satisfies a Euclidean-type isoperimetric inequality with a non-sharp constant $|\partial\Sigma|^{\frac{k}{k-1}} \geq c(k)|\Sigma|$.

Part 3. Examples of minimal surfaces

We now discuss examples of minimal submanifolds, particularly those without boundary.

13. Examples via isometries

Recalling that isometries preserve the Levi-Civita connection, we have:

Lemma 13.1. *Suppose that $F : (M_1, g_1) \to (M_2, g_2)$ is an isometry. If $\Sigma \subset (M_1, g_1)$ is a submanifold then $dF(\vec{A}_\Sigma(U,V)) = \vec{A}_{F(\Sigma)}(dF(U), dF(V))$ for $U, V \in T_p\Sigma$ and $dF(\vec{H}_\Sigma) = \vec{H}_{F(\Sigma)}$.*

For $(M_1, g_1) = (M_2, g_2)$ and $F(\Sigma) = \Sigma$ this can be used to give a computation free proof that certain submanifolds are minimal.

(1) For $F(x, y, z) = (x, y, -z)$, dF_p preserves $T_p\mathbb{R}^2$ but is a reflection in the normal bundle, so we see that $\mathbb{R}^2 \subset \mathbb{R}^3$ is totally geodesic ($\vec{A} = 0$) and thus minimal. The same proof works for any $\Pi \subset \mathbb{R}^n$ affine subspace.
(2) The helicoid $\Sigma \subset \mathbb{R}^3$ is defined by rotating a line while moving upwards. We can define a global chart $X(t,\theta) := (t\cos\theta, t\sin\theta, \theta)$ (note that then $\Sigma = \{y = x \tan z\}$). See Figure 4. Let F denote the 180°-rotation around the line $\ell := \{(t, 0, 0) : t \in \mathbb{R}\}$, i.e. $F(x, y, z) = (x, -y, -z)$. We observe that $F(\Sigma) = \Sigma$ and for $p \in \ell$, $F(p) = p$ but $dF(N(p)) = -N(p)$. Thus, the above lemma gives that $-\vec{H}_\Sigma(p) = dF(\vec{H}_\Sigma(p)) = \vec{H}_{F(\Sigma)}(p) = \vec{H}_\Sigma(p)$, so $\vec{H}_\Sigma(p) = 0$. We could have done this for any other of the lines in Σ so we see the helicoid is minimal. Note that $dF_p|_{T_p\Sigma}$ is not the identity (it has one $+1$ eigenvalue and one -1) so we do not conclude that Σ is totally geodesic.
(3) Write $\mathbb{R}^{2n+2} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and define the Simons cone $C_{n,n} = \{(x, y) \in \mathbb{R}^{2n+2} : |x| = |y|\}$. We claim that $C_{n,n}$ is minimal. Note that $O(n+1) \times O(n+1)$ acts isometrically on $C_{n,n}$ so it suffices to show that the mean curvature of $C_{n,n}$ at $p = ((r, 0, \dots, 0), (r, 0, \dots, 0))$ vanishes for all $r > 0$. Note that a normal vector to $C_{n,n}$ at p is given by $N(p) = ((\frac{1}{\sqrt{2}}, 0, \dots, 0), (-\frac{1}{\sqrt{2}}, 0, \dots, 0))$. The isometry $F(x, y) = (y, x)$ preserves $C_{n,n}$ as a set and has $F(p) = p$ but $dF(N(p)) = -N(p)$, so this proves that $C_{n,n}$ is minimal.

Exercise 13.1. Find $J \in \mathfrak{so}(2n+2)$ so that $\cup_{\theta\in\mathbb{R}}(e^{\theta J}C_{n,n}) \times \{\theta\}$ is minimal.

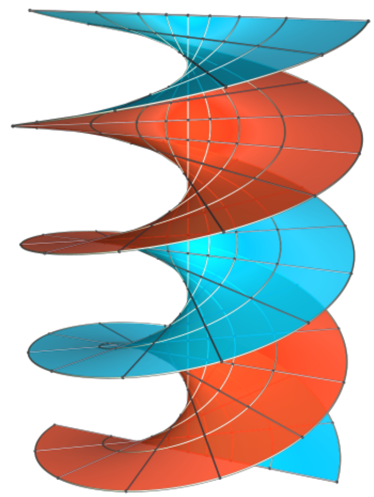

FIGURE 4. The helicoid. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`

A related method can be described as follows (cf. [Law77, pp. 20–25]). Let G be a compact connected group of isometries of a Riemannian manifold. Recall that an orbit of $p \in M$ is $G(p) = \{g(p) \in M : g \in G\}$ is a smooth embedded submanifold diffeomorphic to G/G_p where $G_p = \{g \in G : g(p) = p\}$ is the isotropy subgroup (cf. [Lee13, Proposition 21.7]); in the examples below it will be easy to verify this by hand. We say that two orbits $G(p)$ and $G(q)$ are of the *same type* if the isotropy subgroups G_p, G_q are conjugate. i.e. there's $g \in G$ with $G_q = gG_pg^{-1}$.

Theorem 13.2 (Hsiang [Hsi66])**.** *An orbit $G(p)$ is a minimal submanifold if and only if it's a critical point of volume among all nearby orbits (of the same type).*

Proof. Suppose that $G(p)$ is critical among orbits of the same type. Let $\vec{H}$ denote the mean curvature vector along $G(p)$. Note that $g_*\vec{H}_p = \vec{H}_{g(p)}$ for $g \in G$ so $g(\exp_p(t\vec{H}_p)) = \exp_{g(p)}(t\vec{H}_{g(p)})$. As such,

$$\Sigma_t := \{\exp_q(t\vec{H}_q) : q \in G(p)\}$$

is a G orbit. Moreover for $p_t = \exp_p(t\vec{H}_p)$, we have

$$g \in G_{p_t} \quad \Leftrightarrow \quad p_t = g(p_t) \quad \Leftrightarrow \quad \exp_p(t\vec{H}_p) = g(\exp_p(t\vec{H}_p)) = \exp_{g(p)}(t\vec{H}_{gp}).$$

Since $\vec{H}$ is a normal vector (and $g_*\vec{H}_p = \vec{H}_{g(p)}$), for t small this is equivalent to $g(p) = p$ i.e. $g \in G_p$. Thus, $G_{p_t} = G_p$, so Σ_t are orbits of the same type.

[17]This also holds true for "minimizing currents," i.e. singular minimizers.

[18]This also holds true for "stationary varifolds," i.e. singular minimal surfaces.

The velocity vector of Σ_t at $t = 0$ is precisely $\vec{H}$, and thus the first variation gives

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(\Sigma_t) = -\int_{G(p)} |\vec{H}|^2.$$

This completes the proof. □

Examples:

(1) $G = SO(p+1) \times SO(q+1)$ acts on $\mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q+2}$. The orbit of $(x, y) \in \mathbb{R}^{p+q+2}$ is $\mathbb{S}^p(|x|) \times \mathbb{S}^q(|y|)$ with volume proportional to $|x|^p|y|^q$. The critical points of this function on the sphere $|x|^2+|y|^2 = 1$ are easily computed to be when $|x|^2 = \frac{p}{p+q}$, $|y|^2 = \frac{q}{p+q}$, so $\mathbb{S}^p(\sqrt{\frac{p}{p+q}}) \times \mathbb{S}^q(\sqrt{\frac{q}{p+q}}) \subset \mathbb{S}^{p+q-1}$ is minimal for all $p, q \in \mathbb{Z}_{\geq 1}$.

(2) Since $\Sigma \subset \mathbb{S}^n$ is minimal if and only if the cone over Σ is minimal (Exercise!) we can use (1) to generalize the Simons cone $C_{n,n}$ to the set of "quadratic cones" $C_{p,q} = \{(x, y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 = p|y|^2\}$.

See [Law77, p. 24] for an example that constructs a non-totally geodesic $S^3 \to S^4$.

14. The catenoid

In this section, we look for axially symmetric minimal hypersurfaces Σ in $\mathbb{R}^{n+1}$ given by a parametrization $F : I \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ (for $I \subset \mathbb{R}$ an open interval) of the form

$$F(s, \omega) := (s, r(s)\omega) \in \mathbb{R} \times \mathbb{R}^n.$$

Fixing some local coordinates on $\mathbb{S}^{n-1}$ we find that

$$\partial_s F(s, \omega) = (1, r'(s)\omega), \qquad \partial_{\omega^i} F(s, \omega) = (0, r(s)\partial_{\omega^i})$$

so the induced metric satisfies

$$g = (1 + r'(s)^2)ds^2 + r(s)^2 g_{\mathbb{S}^{n-1}}.$$

The induced volume form satisfies

$$d\mu_\Sigma = \sqrt{1 + r'(s)^2}\, r(s)^{n-1} ds\, d\mu_{\mathbb{S}^{n-1}}.$$

If we vary $r(s)$ to $r(s) + t\rho(s)$ with $\rho(s)$ compactly supported, then if $\Sigma = \Sigma_{r(s)}$ is minimal, then the first variation formula gives[19]

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \operatorname{area}(\Sigma_{r(s)+t\rho(s)}) \\ &= |\mathbb{S}^{n-1}| \int_I \left(\frac{r'(s)r(s)^{n-1}}{(1 + r'(s)^2)^{\frac{1}{2}}} \rho'(s) + (n-1)(1 + r'(s)^2)^{\frac{1}{2}} r(s)^{n-2} \rho(s) \right) ds \end{aligned}$$

[19]note that $\left(\frac{x}{(1+x^2)^{\frac{1}{2}}}\right)' = \frac{1}{(1+x^2)^{\frac{3}{2}}}$

$$= |\mathbb{S}^{n-1}| \int_I \left(-\frac{r''(s)r(s)^{n-1}}{(1+r'(s)^2)^{\frac{3}{2}}} - \frac{(n-1)r'(s)^2 r(s)^{n-2}}{(1+r'(s)^2)^{\frac{1}{2}}} + (n-1)(1+r'(s)^2)^{\frac{1}{2}} r(s)^{n-2} \right) \rho(s)ds$$
$$= |\mathbb{S}^{n-1}| \int_I \left(-\frac{r''(s)r(s)^{n-1}}{(1+r'(s)^2)^{\frac{3}{2}}} + \frac{(n-1)r(s)^{n-2}}{(1+r'(s)^2)^{\frac{1}{2}}} \right) \rho(s)ds.$$

Thus, since $\rho(s)$ is arbitrary, we can see that if Σ is minimal then

$$r''(s)r(s) = (n-1)(1+r'(s)^2).$$

We want to find a first integral for this equation. One may do this by brute force or appealing to Noether's theorem, but a geometric way to do so is to use conservation of flux. Let $\Gamma_s = \{s\} \times \mathbb{S}^{n-1}$. Then the (upwards) conormal satisfies

$$\eta_s = \frac{(1, r'(s)\omega)}{\sqrt{1+r'(s)^2}}$$

Let $K = (1,0)$ denote the upwards pointing parallel vector field. Then, we have that the flux

$$\int_{\Gamma_s} \langle K, \eta_s \rangle = \frac{r(s)^{n-1}}{\sqrt{1+r'(s)^2}}$$

is independent of s. Call this constant F_0. Solving for $r'(s)$ we find

$$r'(s)^2 + 1 = F_0^2 r(s)^{2(n-1)}. \tag{14.1}$$

Exercise 14.1. Show that a solution to (14.1) yields an axially symmetric minimal hypersurface in $\mathbb{R}^{n+1}$.

14.1. **Catenoid in $\mathbb{R}^3$.** When $n = 2$, one may check that the general solution is $r(s) = F_0^{-1}\cosh(F_0(s-s_0))$ for $s_0 \in \mathbb{R}$ arbitrary. This yields the *catenoid* in $\mathbb{R}^3$ (changing F_0 and s_0 represents a scaling and vertical translation). See Figures 5 and 6. Note that for $s_0 = 0$, we can write

$$F_0 r = \cosh(F_0 s) \approx \frac{e^{F_0 s}}{2}$$

so if we delete the circle of smallest radius $r = F_0^{-1}$, the catenoid can be written as the union of two graphs on $\mathbb{R}^2$ of the form $z \approx \pm \log(2F_0 r)$. In particular, the catenoid in $\mathbb{R}^3$ is not contained in a slab of bounded height.

14.2. **Catenoid in $\mathbb{R}^{\geq 4}$.** When $n \geq 3$ there is no closed form solution, but we can solve (14.1) to find the profile function implicitly. Instead of doing this, we'll consider the qualitative behavior (which is very different in higher dimensions as compared to in $\mathbb{R}^3$). When $r(s) \gg 1$ (assuming $r'(s) > 0$) we can estimate

$$r'(s)^2 \approx F_0^2 r(s)^{2(n-1)} \Rightarrow \left(r(s)^{2-n}\right)' \approx -F_0(n-2) \Rightarrow r(s) \approx (C - F_0(n-2)s)^{-\frac{1}{n-2}}.$$

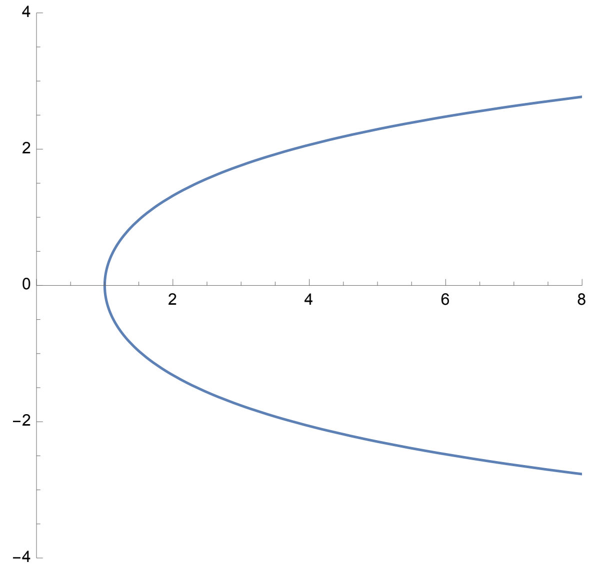

FIGURE 5. The profile curve of the $\mathbb{R}^3$ catenoid (with $F_0 = 1$).

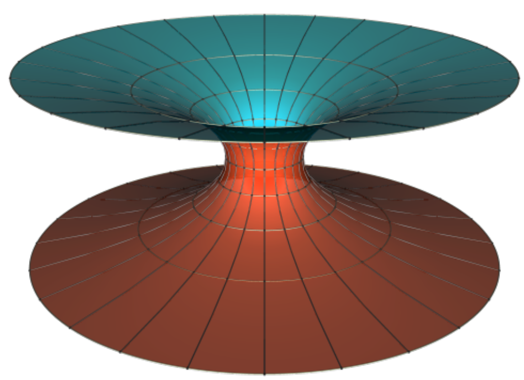

FIGURE 6. The catenoid. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`

In particular, separation of variables suggests that $r(s) \to \infty$ at some finite s. It's not hard to prove this rigorously:

Exercise 14.2. Show that for $n \geq 3$ and fixed flux F_0, there's a solution to (14.1), unique up to translation in s, implicitly given by

$$s = \int_{F_0^{\frac{1}{n-1}}}^{r(s)} \frac{d\rho}{\sqrt{F_0^2 \rho^{2(n-1)} - 1}}$$

for $s \geq 0$. Conclude that the catenoid in $\mathbb{R}^{n+1}$, for $n \geq 3$ is contained in a slab $\{|x^{n+1}| \leq S\}$ and find an integral relating S and F_0.

Exercise 14.3. For any $n \geq 2$, show that a catenoid in $\mathbb{R}^{n+1}$ that's symmetric with respect to the x^{n+1}-plane can be written as the union of two graphs (deleting the central $\mathbb{S}^{n-1}$) over $\mathbb{R}^n \times \{0\}$. Show that the graphical function is approximately equal to the Green's function on $\mathbb{R}^n$.

See Figure 7 for a comparison between the $\mathbb{R}^3$ and $\mathbb{R}^4$ profile curves. In particular, we emphasize that the $\mathbb{R}^3$ catenoid is not contained in any half-space, while an $\mathbb{R}^{\geq 4}$ catenoid is contained in a slab.

15. The Weierstrass–Enneper representation

We now describe a powerful method (based on Riemann surface theory) for finding two-dimensional minimal surfaces in Euclidean space. The basic idea is to combine the induced Riemann surface structure with the fact that the coordinate functions are harmonic (Theorem 8.1) and the fact that the Gauss map is conformal (described below).

We first recall that we called weakly conformal harmonic maps $F : \bar{D} \to \mathbb{R}^n$ "branched minimal immersions" in Section 3.2. The following lemma (to be used later) shows that

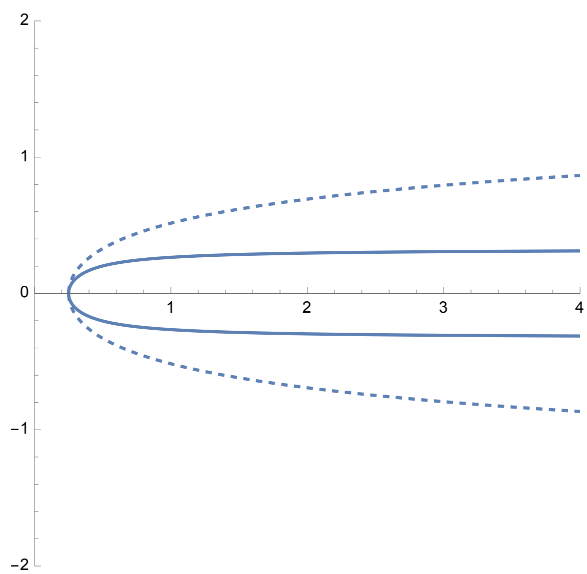

Figure 7. The profile curve of the $\mathbb{R}^4$ catenoid (for comparison, the dashed plot is the $\mathbb{R}^3$ profile curve with the same smallest radius).

this terminology is consistent with the terminology used in later sections, i.e. a minimal immersion is one with vanishing mean curvature $\vec{H} = 0$.

Lemma 15.1. *Given a Riemann surface Σ, consider a conformal immersion $F : \Sigma \to \mathbb{R}^n$. Then F has $\vec{H} = 0$ if and only if F is harmonic in the sense that each coordinate function is a harmonic function on Σ.*

Proof. Let $g = F^* g_{\mathbb{R}^n}$ denote the induced metric. Theorem 8.1 gives $\Delta_g F = \vec{H}$. Since F is conformal, this completes the proof. □

Exercise 15.1. Prove this remains true (with "harmonic map" in place of "harmonic") for conformal maps $F : \Sigma \to (M, g)$.

15.1. **The Gauss map.** For $\Sigma^2 \subset \mathbb{R}^3$ minimal, there are just two principal curvatures, so Corollary 7.8 gives $0 = H = \lambda_1 + \lambda_2$. Thus, in some (oriented) orthonormal basis of $T_p\Sigma$, the shape operator $S = DN : T_p\Sigma \to T_p\Sigma = T_{N(p)}\mathbb{S}^2$ becomes $\mathrm{diag}(\lambda, -\lambda)$.

Corollary 15.2. *If $\Sigma^2 \subset \mathbb{R}^3$ is minimal, the unit normal defines a weakly conformal orientation reversing map $N : \Sigma \to \mathbb{S}^2$.*

15.2. **Holomorphic differential.** We briefly review and generalize the discussion from Section 3.1 on the holomorphic differential. Fix a Riemann surface Σ and smooth map $F : \Sigma \to \mathbb{R}^n$. Let x, y be local oriented coordinates and let $\zeta = x + iy$ (note that ζ may not be compatible with the Riemann surface structure).

Let $g_{ij} = \partial_i F \cdot \partial_j F$. We define a $\mathbb{C}^n$-valued 1-form by $\phi = (\phi_1, \ldots, \phi_n) := \partial_\zeta F d\zeta = \frac{1}{2}(\partial_x F - i\partial_y F)d\zeta$. As in Section 3.1, we have

$$4(\partial_\zeta F)^2 = g_{xx} - g_{yy} - 2ig_{xy}, \qquad 4|\partial_\zeta F|^2 = g_{xx} + g_{yy}.$$

Thus, we find:

(1) ϕ_k is holomorphic if and only if F_k is harmonic
(2) x, y are isothermal if and only if $\phi^2 = 0$.
(3) if x, y are isothermal, then F is an immersion if and only if $|\phi| \neq 0$.

As such, using Lemma 15.1, we see that finding (2-dimensional) minimal surfaces in $\mathbb{R}^n$ can be viewed as a problem in Riemann surface theory:

Lemma 15.3. *If $F : \Sigma \to \mathbb{R}^n$ is a minimal immersion and x, y are local oriented isothermal coordinates then ϕ is a $\mathbb{C}^n$-valued holomorphic 1-form with*

$$\phi^2 = 0 \qquad \textit{and} \qquad |\phi|^2 \neq 0. \tag{15.1}$$

Conversely, if Σ *is simply connected and* $\phi = (\phi_1, \ldots, \phi_n)$ *satisfies*[20] (15.1) *then*

$$F = \operatorname{Re} \int \phi \tag{15.2}$$

defines a conformal minimal immersion $F : \Sigma \to \mathbb{R}^n$.

To be precise, the integral we defined above is a path integral from some fixed basepoint $z_0 \in \Sigma$ (often this is ignored, since a change of basepoint is equivalent to applying a translation to the image of X).

Note that the integral (15.2) is path-independent since the ϕ_k are holomorphic and Σ was assumed to be simply connected. If Σ is not simply connected then (15.2) will be well-defined if and only if $\int_\gamma \phi_k \in i\mathbb{R}$ is purely imaginary for all closed loops γ in Σ. This is called the *period problem*. Of course, since the ϕ_k are holomorphic, it suffices to check this on a basis of $H_1(\Sigma)$.

Example 15.4. Consider the cylindrical coordinate parametrization of the catenoid

$$F(u,v) = \begin{pmatrix} \cosh v \cos u \\ \cosh v \sin u \\ v \end{pmatrix}$$

for $(u,v) \in \mathbb{R} \times \mathbb{S}^1$. Note that

$$\partial_u F = \begin{pmatrix} -\cosh v \sin u \\ \cosh v \cos u \\ 0 \end{pmatrix}, \qquad \partial_v F = \begin{pmatrix} \sinh v \cos u \\ \sinh v \sin u \\ 1 \end{pmatrix}$$

so using $\cosh^2 = 1 + \sinh^2$ we find that $|\partial_u F|^2 = |\partial_v F|^2 = \cosh^2 v$ and $\partial_u F \cdot \partial_v F = 0$, i.e. F is a conformal immersion. Setting $\zeta = u + iv \in \mathbb{C}/\mathbb{Z}$ we thus set

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = (\partial_u F - i\partial_v F) d\zeta = \begin{pmatrix} -\cosh v \sin u - i \sinh v \cos u \\ \cosh v \cos u - i \sinh v \sin u \\ -i \end{pmatrix} d\zeta = \begin{pmatrix} -\sin \zeta \\ \cos \zeta \\ -i \end{pmatrix} d\zeta$$

Note that

$$\int_{\{*\} \times \mathbb{S}^1} \phi = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \\ -i \end{pmatrix} dt = \begin{pmatrix} 0 \\ 0 \\ -2\pi i \end{pmatrix}$$

is purely imaginary, so the periods are all zero (as expected).

15.3. **Conjugate minimal surfaces.** We now assume that ϕ is a $\mathbb{C}^n$ valued holomorphic 1-form on Σ so that $\int_\gamma \phi = 0$ for all closed curves $\gamma \subset \Sigma$ (compare with the period problem

[20]As with the Douglas–Radó maps, one may drop the assumption that $|\phi|^2 \neq 0$ and thus consider *branched minimal surfaces*.

where we just require that each period is purely imaginary). Given any (branched) minimal immersion $F : \Sigma \to \mathbb{R}^n$, this can always be achieved by considering a small patch $D \subset \Sigma$ or else by passing to an appropriate cover.[21]

Then

$$\Phi := \int \phi$$

(choosing a basepoint) defines a holomorphic map $\Phi : X \to \mathbb{C}^n$ with $F = \operatorname{Re}\Phi$. As such, we can define $F_\theta := \operatorname{Re}(e^{i\theta}\Phi)$. It's easy to see that F_θ is a (branched) minimal immersion.

Lemma 15.5. *The metric induced on Σ by F_θ is independent of θ.*

Proof. We have $|e^{i\theta}\phi|^2 = |\phi|^2$. □

Example 15.6. We found that

$$\phi = \begin{pmatrix} -\sin\zeta \\ \cos\zeta \\ -i \end{pmatrix} d\zeta$$

for the catenoid $\Sigma = \mathbb{C}/\mathbb{Z}$. Passing to the universal cover $\tilde{\Sigma} = \mathbb{C}$, we can integrate to find

$$\Phi = \begin{pmatrix} \cos\zeta \\ \sin\zeta \\ -i\zeta \end{pmatrix} = \begin{pmatrix} \cosh v \cos u \\ \cosh v \sin u \\ v \end{pmatrix} + i \begin{pmatrix} -\sinh v \cos u \\ \sinh v \sin u \\ -u \end{pmatrix}$$

(up to a fixed translation which we ignore) where we recall $\zeta = u + iv \in \mathbb{C}$. Observe that (up to an ambient isometry of $\mathbb{R}^3$), $F_{\frac{\pi}{2}}$ is a parametrization of the helicoid! In particular, we conclude that the helicoid and catenoid are locally isometric. See `https://en.wikipedia.org/wiki/Catenoid#/media/File:Helicatenoid.gif`.

15.4. **Weierstrass representation.** We now assume that $n = 3$, i.e. we have $F : \Sigma \to \mathbb{R}^3$. We relate the integrand $\phi = (\phi_1, \phi_2, \phi_3)$ to the Gauss map. In the sequel, it's useful to recall that the ratio of two holomorphic 1-forms is meromorphic (assuming the denominator does not vanish identically) and that conformality gives $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$. We may assume that ϕ_3 is not identically zero. Let

$$g = \frac{\phi_3}{\phi_1 - i\phi_2} = -\frac{\phi_1 + i\phi_2}{\phi_3}$$

(the second equality follows from conformality). We note that

$$\phi_1 = \frac{1}{2}(\phi_1 - i\phi_2 + \phi_1 + i\phi_2) = \frac{1}{2}(g^{-1} - g)\phi_3.$$

Similarly,

$$\phi_2 = \frac{i}{2}(g^{-1} + g)\phi_3,$$

[21]Note that the catenoid does not satisfy this condition without either restricting to a simply connected coordinate patch or else passing to the universal cover.

Following [HK97] we write $\phi_3 = dh$ (note the mild abuse of notation: $\operatorname{Re}\phi_3 = dx^3$ is exact but $\phi_3 = dx^3 + i(dx^3)^*$ need not be exact unless we pass to an appropriate cover). Thus we have

$$\phi = \left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1\right) dh. \tag{15.3}$$

We may easily reverse this calculation:

Lemma 15.7. *Given a meromorphic function g and holomorphic 1-form dh on Σ, the $\mathbb{C}^3$-valued 1-form $\phi = (\phi_1, \phi_2, \phi_3)$ defined by (15.3) satisfies $\phi^2 = 0$ and thus $F = \operatorname{Re}\int\phi$ defines a branched minimal immersion, possibly after passing to a cover of Σ to resolve the period problem.*

Note that $|\phi|^2 = \frac{1}{2}(|g| + |g|^{-1})^2|dh|^2$ so we find that F is an immersion if and only if at any zero of dh there's a zero/pole of g of the same order.

Recall that the Gauss map $N : \Sigma \to \mathbb{S}^2$ is orientation reversing conformal and inverse stereographic projection $\sigma : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is conformal and orientation reversing. We now show that g is the Gauss map:

Lemma 15.8. *$g = \sigma \circ N$.*

Proof. Note that

$$dF = \operatorname{Re}\left[\left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1\right) dh\right]$$

and thus we find that

$$\operatorname{Re}\left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1\right), \qquad \operatorname{Im}\left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1\right)$$

are (linearly independent) tangent vectors to Σ. On the other hand, we have

$$\sigma^{-1} \circ g = \frac{(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2 - 1)}{|g|^2 + 1}.$$

so it suffices to observe that

$$\begin{aligned}
&\left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1\right) \cdot (2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2 - 1)\\
&= \frac{\bar{g}\operatorname{Re} g - |g|^2 g\operatorname{Re} g + i\bar{g}\operatorname{Im} g + i|g|^2 g\operatorname{Im} g}{|g|^2} + |g|^2 - 1\\
&= 0.
\end{aligned}$$

This completes the proof. ∎

We'll call g and dh the *Weierstrass data* and/or the Gauss map and height differential.

Exercise 15.2. Compute the Gaussian curvature and second fundamental form in terms of the Weierstrass data.

15.5. **Examples.** We can easily find the Weierstrass data for the minimal surfaces in $\mathbb{R}^3$ already discussed:

Exercise 15.3. Show that:

(1) the catenoid has Weierstrass data $g = z, dh = \frac{dz}{z}$ on $\mathbb{C} \setminus \{0\}$
(2) the helicoid has Weierstrass data $g = e^{iz}, dh = dz$.

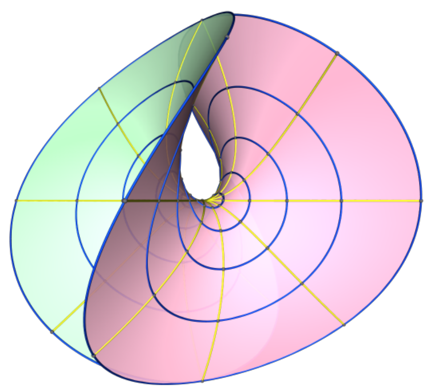

FIGURE 8. Enneper's surface. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`

The next simplest example is *Enneper's surface* given by $g = z, dh = zdz$. See Figure 8 (note that Enneper's surface is immersed, not embedded). An interesting feature is that the induced metric

$$2|\phi|^2 = (|z| + |z|^{-1})^2|z|^2|dz|^2 = (1 + |z|^2)^2|dz|^2$$

is thus rotationally symmetric (even though the embedding is not rotationally symmetric).

Exercise 15.4. Determine $\theta \in \mathbb{S}^1$ so that $t \mapsto e^{i\theta}t \in \mathbb{C}$ is mapped to a straight line in $\mathbb{R}^3$ contained in Enneper's surface. Conclude that "half of Enneper's surface" is an embedded surface with straight line boundary.

The Enneper–Weierstrass representation can be used to construct many examples of embedded/immersed minimal surfaces in $\mathbb{R}^3$ (and $\mathbb{R}^n$ when appropriately generalized). See Figure 9 (and Weber's minimal surface archive `https://minimal.sitehost.iu.edu/archive/`).

16. CALIBRATIONS

Definition 16.1. Suppose that $\Sigma^k \subset (M, g)$ is an oriented submanifold. A k-form $\alpha \in \Omega^k(M)$ is a *calibration* for M if:

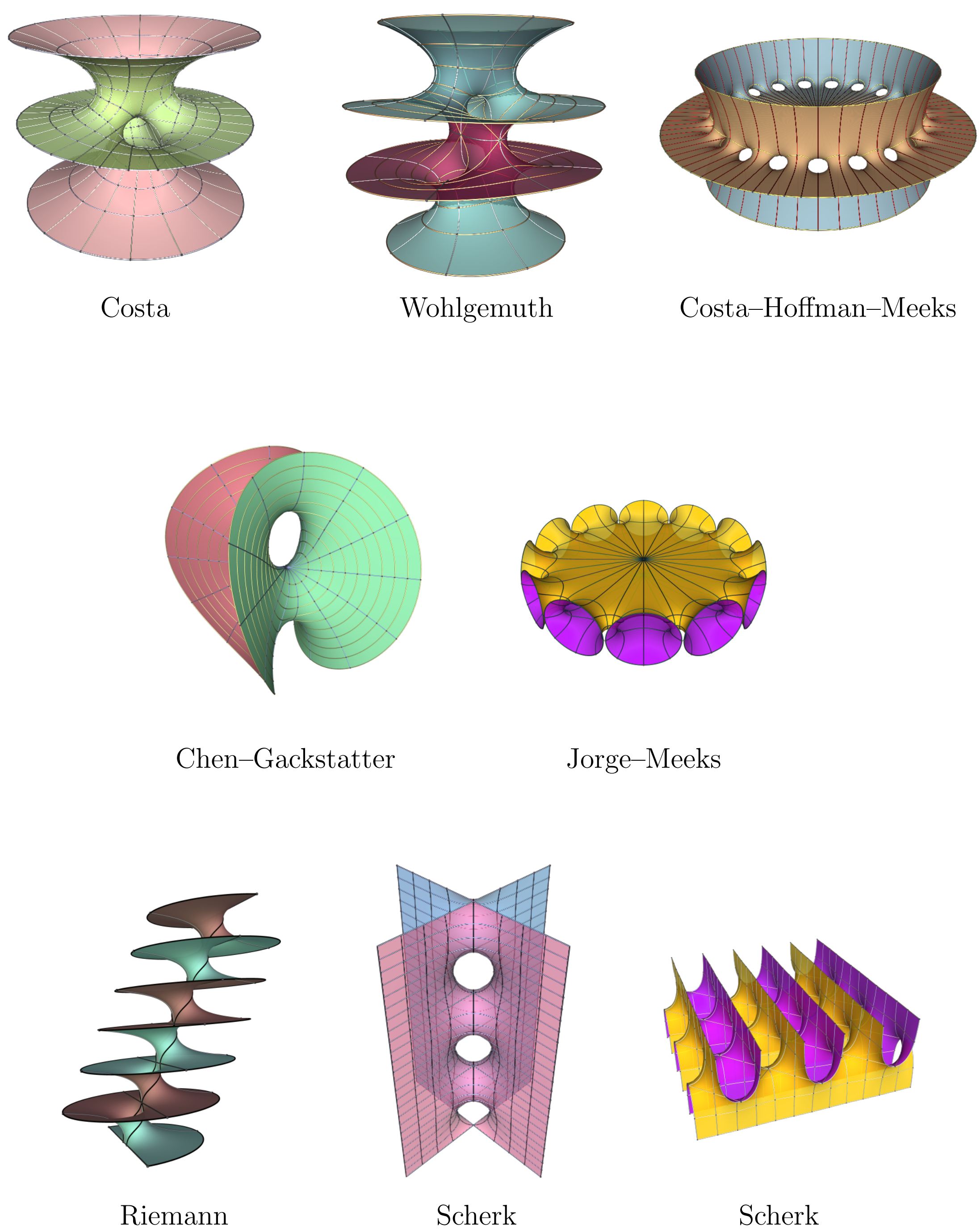

FIGURE 9. More examples of (complete) minimal surfaces in $\mathbb{R}^3$. The top and bottom row are embedded while the middle row are immersed. The bottom row has infinite total curvature, while the top two have finite total curvature. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`

(1) $d\alpha = 0$
(2) If $e_1, \dots, e_k \in T_pM$ are orthonormal then $\alpha(e_1, \dots, e_k) \leq 1$.
(3) If $e_1, \dots, e_k$ is an oriented basis for $T_p\Sigma$ then $\alpha(e_1, \dots, e_k) = 1$.

Theorem 16.2. *If α calibrates Σ then Σ is homologically area-minimizing on compact sets.*

Proof. It suffices to prove that if Σ is compact and calibrated by α then $|\Sigma| \leq |\tilde{\Sigma}|$ for all $\tilde{\Sigma}$ with $\partial\tilde{\Sigma} = \partial\Sigma$ and $[\Sigma] - [\tilde{\Sigma}] = 0 \in H_k(M)$. Find a $(k+1)$-chain Ω with $\partial\Omega = \Sigma - \tilde{\Sigma}$. Then, Stokes theorem gives

$$0 = \int_\Omega d\alpha = \int_\Sigma \alpha - \int_{\tilde{\Sigma}} \alpha.$$

We have that $\alpha|_\Sigma = d\,\mathrm{Vol}_\Sigma$ and $\alpha|_{\tilde{\Sigma}} \leq d\,\mathrm{Vol}_{\tilde{\Sigma}}$. This completes the proof. □

Example 16.3. Suppose that (M, g) is foliated by oriented minimal surfaces. Let ν denote the unit normal to the leaves of the foliation and ω denote the volume form of g. Let $\alpha = \iota_\nu\omega$. Since $d\omega = 0$ and $\mathcal{L}_X\omega = (\operatorname{div} X)\,\omega$, Cartan's magic formula[22] gives

$$d\alpha = \operatorname{div} \nu \ \omega$$

On the other hand, we have $g(D_\nu\nu, \nu) = 0$ (differentiate $|\nu|^2 = 1$) and thus if Σ is a leaf of the foliation then $\operatorname{div} \nu = \operatorname{div}_\Sigma \nu = 0$.

Remark 16.4. Implicit in the previous example is the observation that codimension one calibrations are the same as vector fields X so that

(1) $\operatorname{div} X = 0$
(2) $|X| \leq 1$
(3) X is a unit normal along Σ.

Indeed we can set $\alpha = \iota_X\omega$ for ω the volume form.

Example 16.5. Let $\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j = \sum_{k=1}^n dx_j \wedge dy_j$ on $\mathbb{C}^n$. One may check that $\frac{\omega^k}{k!}$ calibrates dimensional complex submanifolds (this follows from the so-called *Wirtigner's inequality* [Wir36]) so they are all area-minimizing on compact sets (Federer [Fed65]). We just check for $k = 1$ which follows from $\omega(v_1, v_2) = g(Jv_1, v_2)$ so $|\omega(v_1, v_2)| \leq 1$ with equality if and only if v_1, v_2 span a complex plane in $\mathbb{C}$. More generally, the same thing holds for a Kähler manifold (complex submanifolds are calibrated by powers of the Kähler form).

It's interesting to ask if all area-minimizing surfaces in $\mathbb{C}^n$ are holomorphic (up to a rotation/reflection). For example, we have the following question of White [Whi16]:

Open Question 3. If $\Sigma^2 \subset \mathbb{C}^n$ is an area-minimizing surface with a true branch point then is Σ holomorphic?

[22] $\mathcal{L}_X = d\iota_X + \iota_X d$

See also [Mor82, Mic84, MW95, MW06].

Remark 16.6. There are many other important classes of calibrated submanifolds. For example in a Calabi–Yau manifold, the so-called *special Lagrangian* submanifolds are calibrated. See [HL82].

17. Minimality of the Simons cone

We recall that the Simons cone

$$C_{n,n} := \{(x,y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : |x| = |y|\}$$

was seen to be a minimal hypersurface in Section 13.

Theorem 17.1. *For $n \geq 3$ the Simons cone $C_{n,n} \subset \mathbb{R}^{2n+2}$ minimizes area on compact sets.*

This was first proven by Bombieri–De Giorgi–Giusti [BDGG69]. We note that in $\mathbb{R}^7$ and below no non-flat minimizing cones exist (proven by Almgren, Simons [Alm66, Sim68]).

Corollary 17.2. *There exists $\Gamma^6 \subset \mathbb{R}^8$ closed oriented submanifold so that the least area "submanifold" Σ^7 with $\partial\Sigma = \Gamma$ is not smooth.*

Proof. Let $\Gamma = \partial C_{n,n}$ and note that $C_{n,n}$ is not smooth at 0. □

Remark 17.3. One might find this result unsatisfying since we did not prove that no smooth area-minimizer exists. This stronger statement is true. In fact one can prove that Γ does not bound any smooth minimal surfaces.

Remark 17.4. In fact, one may show that (for $p, q \in \mathbb{Z}_{\geq 1}$) the quadratic cones $C_{p,q} = \{(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 = p|y|^2\}$ are minimizing if and only if $p + q + 2 \geq 9$ or else $p + q + 2 = 8$ and $(p,q) \notin \{(1,5),(5,1)\}$.

Since $C_{n,n}$ is not smooth, we should state Theorem 17.1 more precisely. We let

$$\Omega := \{|x| < |y|\} \subset \mathbb{R}^{2n+2}$$

so that $\partial\Omega = C_{n,n}$. Consider $\Omega' \subset \mathbb{R}^{2n}$ open with $\partial\Omega'$ smooth and

$$\Omega\Delta\Omega' := (\Omega \setminus \Omega') \cup (\Omega' \setminus \Omega) \Subset B_R$$

for some $R > 0$. We'll prove that

$$|\partial\Omega \cap B_R| < |\partial\Omega' \cap B_R| \tag{17.1}$$

following the "sub-calibration" method from [DPP09].

Let $f(x,y) = \frac{1}{4}(|x|^4 - |y|^4)$ and set $X = \frac{\nabla f}{|\nabla f|}$. Note that X is smooth away from $(0,0)$ and obviously satisfies $|X| \leq 1$ and is normal along $C_{n,n}$. We compute

$$f_{x_i} = |x|^2 x_i$$

$$
\begin{aligned}
|\nabla f|^2 &= |x|^6 + |y|^6 \\
f_{x_i x_j} &= 2x_i x_j + \delta_{ij}|x|^2 \\
(|\nabla f|^2)_{x_i} &= 6|x|^4 x_i
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\sum_{i=1}^{n+1}\left(\frac{f_{x_i}}{|\nabla f|}\right)_{x_i} &= \sum_{i=1}^{n+1}\frac{f_{x_i x_i}|\nabla f|^2 - \frac{1}{2}f_{x_i}(|\nabla f|^2)_{x_i}}{|\nabla f|^3} \\
&= \sum_{i=1}^{n+1}\frac{(|x|^6+|y|^6)(2x_i^2+|x|^2) - 3|x|^6 x_i^2}{|\nabla f|^3} \\
&= \frac{(|x|^6+|y|^6)(n+3)|x|^2 - 3|x|^8}{|\nabla f|^3} \\
&= \frac{n|x|^8 + (n+3)|x|^2|y|^6}{|\nabla f|^3}.
\end{aligned}
$$

The derivatives with respect to y are the same but the sign flips. Thus we find

$$
\begin{aligned}
|\nabla f|^3 \operatorname{div} X &= n(|x|^8 - |y|^8) + (n+3)|x|^2|y|^2(|y|^4 - |x|^4) \\
&= (|x|^4 - |y|^4)(n(|x|^4 + |y|^4) - (n+3)|x|^2|y|^2).
\end{aligned}
$$

Note that

$$
(n+3)|x|^2|y|^2 \leq \frac{n+3}{2}|x|^4 + \frac{n+3}{2}|y|^4
$$

so as long as $\frac{n+3}{2} \leq n$ (i.e. $n \geq 3$), $\operatorname{div} X$ has the same sign as f. We now have

Proof of Theorem 17.1. We prove Theorem 17.1 in the special case that $\Omega \subset \Omega'$, i.e. the competitor "lies to one side." (See Figure 10.) The general proof is similar but requires us to keep track of more signs. By assumption we have that $f > 0$ on $\Omega' \setminus \Omega$ and thus $\operatorname{div} X \geq 0$ on $\Omega' \setminus \Omega$ (in fact > 0 a.e. if we keep careful track of the equality in AM-GM above). We also observe that X is *inwards pointing* with respect to $\Omega' \setminus \Omega$ along $C_{n,n}$. Thus, the divergence theorem gives

$$
0 < \int_{\Omega' \setminus \Omega} \operatorname{div} X = \int_{C_{n,n} \cap B_R} N \cdot X + \int_{\partial\Omega' \cap B_R} N \cdot X.
$$

Since $N \cdot X = -1$ along $C_{n,n}$ and ≤ 1 along $\partial\Omega'$ we thus conclude

$$
|C_{n,n} \cap B_R| < |\partial\Omega' \cap B_R|
$$

completing the proof. □

Exercise 17.1. Prove Theorem 17.1 without assuming that $\Omega \subset \Omega'$. Also, use a cutoff function to justify the choice of X in the divergence formula (since X is not smooth across $(0,0)$).

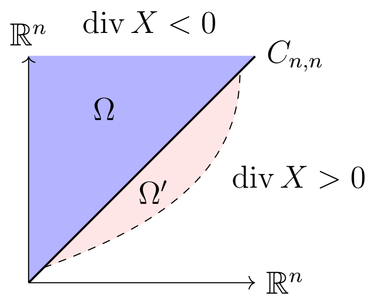

FIGURE 10. The vector field X is a "sub-calibration" proving that the Simons cone $C_{n,n}$ minimizes area.

18. Minimal graphs

Recall that for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ the area of the graph of u (denoted Γ_u) is

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}.$$

Suppose we vary u to $u + t\varphi$ for $\varphi \in C_c^\infty(\Omega)$. Then

$$\frac{d}{dt}\Big|_{t=0} \mathcal{A}(u + t\varphi) = \int_\Omega \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1 + |\nabla u|^2}} = -\int_\Omega \varphi \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

Comparing with the first variation, we thus find that Γ_u is minimal if and only if it satisfies the *minimal surface equation*

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0. \tag{18.1}$$

We note that (18.1) is a *second order quasilinear elliptic PDE* since one can equivalently write

$$\Delta u - \frac{D^2 u(\nabla u, \nabla u)}{1 + |\nabla u|^2} = \sum_{i,j=1}^n \left(\delta_{ij} - \frac{D_i u D_j u}{1 + |\nabla u|^2} \right) D_{ij}^2 u = 0$$

and observe that the equation is linear with respect to the second derivatives of u (quasilinear) and that the matrix with coefficients

$$a_{ij} := \left(\delta_{ij} - \frac{D_i u D_j u}{1 + |\nabla u|^2} \right) \tag{18.2}$$

is positive definite (elliptic).

Exercise 18.1. Show that the eigenvalues of $(a_{ij})_{i,j=1}^n$ defined in (18.2) are 1 with multiplicity $n - 1$ and $\frac{1}{1+|\nabla u|^2}$ with multiplicity 1. Conclude that if $|\nabla u| \to \infty$, the minimal surface equation is not uniformly elliptic.

Theorem 18.1. *If $u : \Omega \to \mathbb{R}$ satisfies the minimal surface equation then its graph Γ_u minimizes area in $\Omega \times \mathbb{R}$ with $\partial \Gamma_u$ fixed.*

Proof. The vertical translation of Γ_u foliates $\Omega \times \mathbb{R}$ and thus forms a calibration (Example 16.3). ☐

Corollary 18.2. *If $u : \mathbb{R}^n \to \mathbb{R}$ is an entire solution to the minimal surface equation then $|\Gamma_u \cap B_R(x)| \leq CR^n$.*

Proof. Assume that $\partial B_R(x)$ intersects Γ_u transversely. Then Γ_u cuts $\partial B_R(x)$ into two regions, both with area $O(R^n)$. Since Γ_u minimizes area in $\mathbb{R}^{n+1}$, this completes the proof. ☐

Exercise 18.2. Solve the minimal surface equation on a square in $\mathbb{R}^2$ by making the ansatz $u(x,y) = X(x) + Y(y)$. Using a reflection, construct a (complete) doubly periodic minimal surface in $\mathbb{R}^3$ (it will not be a graph).

Part 4. The maximum principle for minimal surfaces

19. The maximum principle for the minimal surface equation

Suppose that $\Sigma^n \subset \mathbb{R}^{n+1}$ is an embedded minimal hypersurface. For any $p \in \Sigma$, the implicit function theorem gives $r > 0$ so that

$$\Sigma \cap B_r(p) = \operatorname{graph} u$$

for u defined on a subset of $T_p\Sigma$ (taking values in $(T_p\Sigma)^\perp$). Since Σ is minimal, we thus have that u satisfies the minimal surface equation.

19.1. **Regularity of minimal surfaces (warmup).** This observation gives that local properties of Σ can be reduced to the study of the minimal surface equation, for which one has many tools such as the maximum principle. As a basic example, we have:

Lemma 19.1. *Suppose that $\Sigma^n \subset \mathbb{R}^{n+1}$ is a hypersurface that's C^2-smooth in the sense that it can be locally written as the graph of a C^2-function over its tangent plane at any point. Assume that Σ is minimal. Then Σ is C^∞-smooth.*

Proof. It suffices to prove that if $u \in C^2_{\mathrm{loc}}(B)$ solves the minimal surface equation, then $u \in C^\infty_{\mathrm{loc}}(B)$. Write the minimal surface equation as

$$0 = \mathcal{M}(u) = \sum_{i,j=1}^n a_{ij}(Du) D^2_{ij} u = 0$$

for a_{ij} as in (18.2). Since $u \in C^2_{\mathrm{loc}}(B)$ we see that $a_{ij} \in C^1_{\mathrm{loc}}(B)$. Thus, Schauder theory (A.2) implies that $u \in C^{2,\alpha}_{\mathrm{loc}}(B)$, so in particular $a_{ij} \in C^{1,\alpha}_{\mathrm{loc}}(B)$. Differentiating the minimal surface equation and writing $u_k = D_k u$ we get

$$\sum_{i,j=1}^n a_{ij}(Du) D^2_{ij} u_k + \sum_{\ell=1}^n \left(\sum_{i,j=1}^n D_\ell a_{ij}(Du) D^2_{ij} u \right) D_\ell u_k = 0$$

The lower order term (in parenthesis) will be in $C^{\alpha}_{\mathrm{loc}}(B)$ since $u \in C^{2,\alpha}_{\mathrm{loc}}(B)$. Thus, we can apply Schauder estimates to this equation satisfied by u_k to get $u \in C^{3,\alpha}_{\mathrm{loc}}(B)$. This (inductively) gives $u \in C^{\infty}_{\mathrm{loc}}(B)$. □

Remark 19.2. In fact, it's possible to conclude that Σ is real analytic. See e.g. [Mor08, §5.8]. Note that if Σ is C^1 then we can already ask if it's minimal in the sense that $\int_\Sigma \operatorname{div}_\Sigma X = 0$ for any compactly supported vector field along Σ since all we need is the volume form of Σ and the unit normal (to define $\operatorname{div}_\Sigma X$). Both of these only depend on first derivatives of u. Lemma 19.1 is still true under this assumption. See e.g. `https://cmouhot.wordpress.com/wp-content/uploads/1900/10/mse.pdf`.

19.2. **Maximum principle.**

Proposition 19.3 (Maximum principle for minimal surface equation)**.** *Suppose that $u_1, u_2 \in C^{\infty}_{\mathrm{loc}}(B)$, $B \subset \mathbb{R}^n$ solve the minimal surface equation. Assume that $u_1 \leq u_2$ in B and $u_1(0) = u_2(0)$. Then $u_1 = u_2$ in B.*

Proof. We write the minimal surface equation as $\mathcal{M}(u) = \sum_{i,j=1}^n a_{ij}(Du)D^2_{ij}u$. Let $v = u_2 - u_1$ so that $v \geq 0$ in B and $v(0) = 0$. We want to show that v satisfies a linear elliptic PDE (so we can apply the maximum principle). We have

$$
\begin{aligned}
0 &= \mathcal{M}(u_2) - \mathcal{M}(u_1) \\
&= \sum_{i,j=1}^n a_{ij}(Du_2)D^2_{ij}v + \sum_{i,j=1}^n (a_{ij}(Du_2) - a_{ij}(Du_1))D^2_{ij}u_1.
\end{aligned}
$$

The second term looks troublesome, but we can resolve it by setting $u_t = u_1 + (t-1)(u_2 - u_1)$ and observing that

$$
\begin{aligned}
a_{ij}(Du_2) - a_{ij}(Du_1) &= \int_1^2 \frac{d}{dt}(a_{ij}(Du_t))dt \\
&= \int_1^2 D_k a_{ij}(Du_t) D_k(\tfrac{d}{dt}u_t)dt \\
&= \left(\int_1^2 D_k a_{ij}(Du_t)dt\right) D_k v.
\end{aligned}
$$

Thus, if we set $\tilde{a}_{ij} = a_{ij}(Du_2)$ and

$$
\tilde{b}_k := \sum_{i,j=1}^n \left(\int_1^2 D_k a_{ij}(Du_t)dt\right) D^2_{ij}u_1
$$

we get

$$
0 = \sum_{i,j=1}^n \tilde{a}_{ij}D^2_{ij}v + \sum_{k=1}^n \tilde{b}_k D_k v.
$$

Note that $\tilde{a}_{ij}$ is uniformly elliptic on compact subsets of B (using Exercise 18.1 and $u \in C^\infty_{\text{loc}}(B)$). The assertion thus follows from the strong maximum principle. $\square$

Corollary 19.4 (Uniqueness of solutions to the minimal surface equation)**.** *If $u_1, u_2 \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ solve the minimal surface equation on a bounded domain $\Omega \subset \mathbb{R}^n$ and $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$ then $u_1 = u_2$ in Ω.*

Proof. Since $\bar{\Omega}$ is compact, if $t \gg 0$, then $u_1 < u_2 + t$ on $\bar{\Omega}$. Let

$$t^* = \inf\{t : u_1 \leq u_2 + t\}.$$

Note that since the boundary values agree, we must have $t^* \geq 0$. Suppose that $t^* > 0$. We have that $u_1 \leq u_2 + t^*$ and there's $x^* \in \bar{\Omega}$ so that $u_1(x^*) = u_2(x^*) + t^*$ (otherwise we could take t^* smaller). Since the boundary values agree x^* is in the interior. We can thus apply the maximum principle (Proposition 19.3) to u_1 and $u_2 + t^*$ in a small ball $B \ni x^*$. This is a contradiction. Thus, $t^* = 0$ so $u_1 \leq u_2$. Repeating this argument with u_1 and u_2 swapped, this proves the assertion. $\square$

We can now prove that not all domains and boundary values admit solutions to the minimal surface equation (compare with Theorem 21.1).

Corollary 19.5. *There exists $\varphi \in C^\infty(\partial\Omega)$, with $\Omega = B_2 \setminus B_1 \subset \mathbb{R}^n$ so that there's no solution u to the minimal surface equation on Ω with $u|_{\partial\Omega} = \varphi$*

Proof. Let $\varphi = 0$ on ∂B_1 and λ on ∂B_2. Since the boundary condition is rotationally symmetric, uniqueness of solutions (Corollary 19.4) implies that a solution u with this boundary condition must be rotationally symmetric. By the analysis in Section 14, the graph of u must be a portion of the catenoid. However, among all catenoids that are graphical over Ω, the maximal heigh λ at $r = 2$ occurs precisely for the catenoid with neck at $s_0 = 0$ of radius $F_0 = 1$. See Exercise 19.1. $\square$

Exercise 19.1. Prove the final statement in Corollary 19.5.

Recalling that any submanifold can be locally written as a graph over it's tangent plane, this also proves that two minimal hypersurfaces cannot make "one-sided" (interior) contact (unless they agree). More precisely:

Corollary 19.6 (Geometric maximum principle)**.** *Suppose that $\Sigma_1^n, \Sigma_2^n \subset \mathbb{R}^{n+1}$ are properly embedded minimal hypersurfaces and $U \subset \mathbb{R}^{n+1}$ is an open set so that $\partial\Sigma_i \cap U = \emptyset$ for $i = 1, 2$. Assume that*

- *$\Sigma_1 \cap U = \partial\Omega$ for $\Omega \subset U$ open,*
- *$\Sigma_1 \cap U, \Sigma_2 \cap U$ are connected, and*
- *$\Sigma_2 \cap U \subset \overline{\Omega}$.*

Then, either $\Sigma_1 \cap U = \Sigma_2 \cap U$ or else $\Sigma_2 \cap U \subset \Omega$ is disjoint from Σ_1.

Proof. If $\Sigma_1 \cap \Sigma_2 \cap U \neq \emptyset$ then we find that $\Sigma_1 \cap \Sigma_1 \cap U \subset \Sigma_1 \cap U$ is a non-empty subset. It's the intersection of two (relatively) closed sets and thus (relatively) closed. It's (relatively) open by the maximum principle. Thus the assertion follows □

Note that the same thing holds in a Riemannian manifold. To generalize this one would need to obtain a form of the minimal surface equation that holds with a non-flat ambient metric and check that the above proof applies.

Exercise 19.2. Suppose that Σ_1^n, Σ_2^n are two compact minimal hypersurfaces with $\Sigma_1 \cap \Sigma_2 = \emptyset$. Prove that $d(\Sigma_1, \Sigma_2) = \min\{d(\Sigma_1, \partial\Sigma_2), d(\Sigma_2, \partial\Sigma_1)\}$ where $d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$. Is this still true if $\Sigma_1 \cap \Sigma_2 \neq \emptyset$?

20. Hoffman–Meeks halfspace theorem

Theorem 20.1 (Hoffman–Meeks [HM90a])**.** *Suppose that $\Sigma^2 \subset \mathbb{R}^3$ is a complete properly embedded minimal surface contained in a half-space. Then Σ is a flat plane.*

Note that this fails in a dramatic way in $\mathbb{R}^4$, since the catenoid is contained in a slab.

Proof. Assume that $\Sigma \subset \{z \geq 0\}$ but $\Sigma \not\subset \{z \geq t\}$ for any $t > 0$. The maximum principle implies that $\Sigma \cap \{z = 0\} = \emptyset$. For $\varepsilon > 0$, let $\Gamma_{\varepsilon,r}$ denote the bottom half of the catenoid with neck $\{(x, y, \varepsilon) : x^2 + y^2 = r\}$. Note that $\Gamma_{\varepsilon,r} \cap \{z \geq 0\}$ is compact. As such, since Σ is properly embedded, we can find $\varepsilon > 0$ so that Σ is disjoint from $\Gamma_{\varepsilon,1} \cup (\cup_{r\in(0,1]} \partial\Gamma_{\varepsilon,r})$. Let

$$\mathcal{R} := \{r \in (0, 1] : \Sigma \cap \Gamma_{\varepsilon,s} = \emptyset \text{ for all } s \in (r, 1]\}.$$

Let $r = \inf \mathcal{R}$. If $r > 0$, the maximum principle gives a contradiction. On the other hand, as $r \to 0$, the catenoid $\Gamma_{\varepsilon,r}$ "limits" to $\{z = \varepsilon\}$ from which get that $\Sigma \subset \{z \geq \varepsilon\}$. This is a contradiction. □

Exercise 20.1. If $\Sigma \subset \mathbb{R}^3$ is a properly embedded minimal surface with compact boundary so that $\Sigma \subset H := \{z \geq 0\}$ show that $d(\Sigma, \partial H) = d(\partial\Sigma, \partial H)$.

For $\mathbb{R}^3$, the proof would work essentially the same for a proper immersion. On the other hand, the half-space theorem is false for non-proper immersions. For example:

Theorem 20.2 (Jorge–Xaiver [JX80])**.** *There exists a complete minimal immersion $\Sigma^2 \to \mathbb{R}^3$ so that the image is contained between two planes.*

Sketch of the proof. Consider Weierstrass data with Gauss map $g = e^f$ (where f is a holomorphic function to be chosen) and height differential $dh = dz$ on $D \subset \mathbb{C}$. For any choice of holomorphic function f, the Weierstrass data gives an (unbranched) immersion with x^3 is bounded. It thus remains to choose f so that the immersion is complete, i.e. ∂D has infinite distance from $0 \in D$ with respect to the induced metric

$$\frac{1}{2}(|g| + |g|^{-1})^2 |dz|^2.$$

We consider a sequence of regions $K_1, K_2, \ldots$ in the disk D as in Figure 11. The key observation is that any path to ∂D with finite Euclidean length must eventually start crossing all even K_n or all odd K_n. Using Runge's approximation theorem (cf. [Rud87, Theorem 13.9] and [Hof88, p. 96]) we can find a holomorphic function f on D with $f \approx c_n$ (freely chosen) on K_n, which allows us to force such a curve to have infinite length. □

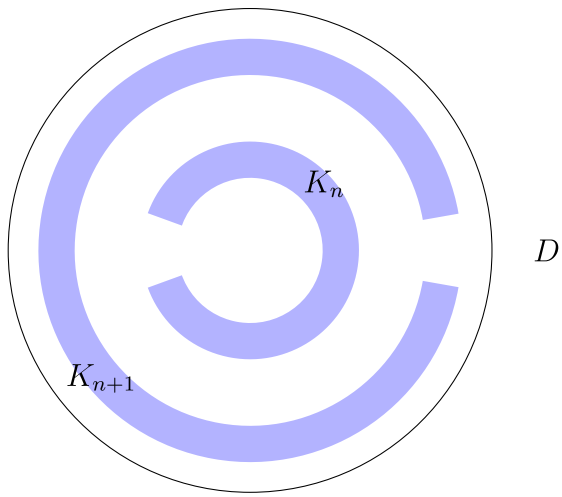

FIGURE 11. The regions used by Jorge–Xavier to construct the complete minimal immersion in a slab in $\mathbb{R}^3$.

Nadirashvili has generalized this to a complete minimal immersion $\Sigma \to \mathbb{R}^3$ with image contained in a ball [Nad96]. On the other hand, we remark that Colding–Minicozzi have proven [CM08] that the half-space theorem holds under the assumption that Σ is embedded (not necessarily properly) and a topological disk.

Exercise 20.2. Prove that if Σ_1, Σ_2 are two minimal hypersurfaces in $\mathbb{S}^n$ then $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ (this is known as Frankel's theorem). Find disjoint minimal surfaces in $\mathbb{H}^3$.

21. Radó's theorem

Consider $\Omega \subset \mathbb{R}^2$ convex. We would like to solve the minimal surface equation on Ω. This can be done in all dimensions using PDE methods (cf. [Sim97]):

Theorem 21.1. *For $\Omega \subset \mathbb{R}^n$ bounded domain with smooth strictly mean-convex*[23] *boundary, if $\varphi \in C^0(\partial\Omega)$ then there's $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ so that u solves the minimal surface equation* (18.1) *on Ω with $u|_{\partial\Omega} = \varphi$.*

Instead here we describe Radó's solution using the classical Plateau problem.

Proof of Theorem 21.1 for $n = 2$ and $\varphi \in C^\infty$. Let $F : \bar{D} \to \mathbb{R}^3$ be the Douglas–Radó solution to the Plateau problem for $\Gamma = \operatorname{graph}_{\partial\Omega} \varphi$. Using the Gulliver–Osserman Theorem 3.26, we have that F has no interior branch points. Since Γ lies on the boundary of a convex set we have that F also has no boundary branch points by Exercise 3.5. For simplicity we assume that F is an embedding so $\Sigma = F(\bar{D})$ is a smooth embedded minimal surface with $\partial\Sigma = \Gamma$. (It's easy to modify the proof below to cover the case where F is a branched immersion.)

The convex hull property implies that $\Sigma \subset \Omega \times \mathbb{R}$. The maximum principle (let ℓ be a supporting line for Ω and consider the plane $\ell \times \mathbb{R}$ which cannot make interior contact with Σ since it would necessarily be one-sided) implies that $\Sigma \setminus \partial\Sigma \subset \Omega \times \mathbb{R}$.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection. Assume there's two points $p \neq q \in \Sigma$ with $\pi(p) = \pi(q)$. Then, we can consider $\Sigma_t = \Sigma + te_3$. For $t \gg 0$, $\Sigma_t \cap \Sigma = \emptyset$. Thus, we can decrease t (slide Σ_t down) until the first time there's $x \in \Sigma_t \cap \Sigma$. By assumption, $t > 0$. Thus, the contact is one-sided and in the interior, a contradiction to the maximum principle.

To prove that $\Sigma = \operatorname{graph}_\Omega u$ it remains to show that there's no point $p \in \Sigma$ with horizontal unit normal. Let P be the horizontal plane at such a point. Since P is transverse to Σ at p, we can choose a small neighborhood $p \in U \subset \Sigma$ so that P divides U into two smooth connected components U_1, U_2 with common boundary curve γ. Let U_1^* be the reflection of U_1 across P. Then U_1^* and U_2 are disjoint (by the previous paragraph) minimal surfaces with a common boundary curve so that they are tangent at p. This contradicts the boundary version of the maximum principle. ☐

Exercise 21.1. State and prove a boundary version of the maximum principle used in the previous proof.

More generally, we have that if $\Omega \subset \mathbb{R}^n$ is mean convex then the solution to the minimal surface equation for given boundary data is the unique compact minimal hypersurface with the same boundary (essentially the same proof works).

22. Shiffman's theorem and the convex curve conjecture

Theorem 22.1 (Shiffman [Shi56]). *Suppose that $\Sigma^2 \subset \mathbb{R}^2 \times [0,1]$ is a minimal embedded annulus in a slab with $\partial\Sigma = \Gamma_0 \cup \Gamma_1$ convex curves in $\mathbb{R}^2 \times \{0,1\}$. Then Σ is transversal to $\mathbb{R}^2 \times \{t\})$ and the intersection $\Gamma_t := \Sigma \cap (\mathbb{R}^2 \times \{t\}$ is strictly convex for $0 < t < 1$.*

[23] Mean-convex means that $\vec{H}_{\partial\Omega}$ points into Ω at every point. If Ω is strictly convex, then it's strictly mean-convex. In $\mathbb{R}^2$, the notions are the same.

Proof. We can assume that Σ is conformally equivalent to $A = \{z \in \mathbb{C} : 1 \leq |z| \leq r\}$ and consider Σ as the image of $F : A \to \mathbb{R}^3$. Up to a homothety we can assume that $F(\{|z| = 1\})$ is a convex curve in $\mathbb{R}^2 \times \{0\}$ and $F(\{|z| = r\})$ is a convex curve in $\mathbb{R}^2 \times \{\log r\}$. In particular F_3 is a harmonic function with the same boundary values as $\log |z|$ so $F_3(z) = \log |z|$. This implies that Σ is transversal to each parallel plane. In particular, $\Gamma_{\log c}$ is parametrized by $\theta \mapsto F(ce^{i\theta})$ and the Gauss map g is never 0 or ∞. Thus, the angle $\phi = \arg g \in S^1$ is also the angle of $\Gamma_{\log c}$ in $\mathbb{R}^2 \times \{\log c\}$. Thus, convexity is equivalent to $\frac{d}{d\theta}\phi(ce^{i\theta})$ non-vanishing. Since the argument of a holomorphic function is harmonic, this follows from the fact that it does not vanish on ∂A and the maximum principle. □

This is known as Shiffman's first theorem. His second theorem implies that if Γ_0, Γ_1 are round circles then so are Γ_t. This, in turn, allows one to appeal to a classification by Riemann of such Σ: it's either a part of a catenoid (if the circles are co-axial) or else a part of Riemann's minimal surface (see Figure 12). See [HM90b, §3.3].

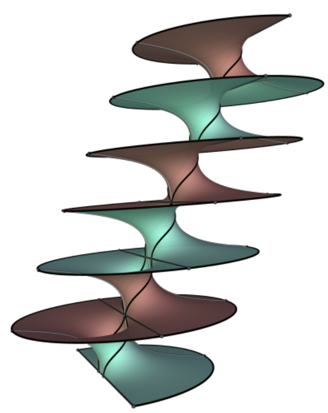

FIGURE 12. Riemann's minimal surface intersects each parallel plane in a round circle or straight line. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`.

Sketch of the proof of Shiffman's second theorem. We sketch a proof due to Meeks–White [MW91]. Choose a deformation $\Gamma_i(t)$ so that $\Gamma_i(0) = \Gamma_i$, $\Gamma_i(1)$ are co-axial circles, and $\Gamma_i(t)$ are moving "to the outside." By analyzing the moduli space of minimal annuli with boundary in parallel planes, Meeks–White prove that[24] there's a smooth family $\Sigma(t)$ with $\partial\Sigma(t) = \Gamma_0(t) \cup \Gamma_1(t)$ and $\Sigma(0) = \Sigma$. Theorem 22.3 (proven below) implies that any minimal surface bounded by $\Gamma_0(1) \cup \Gamma_1(1)$ is axially symmetric and thus part of a catenoid. Since we know the catenoid (and Riemann example) explicitly, we can see that any nearby minimal

[24]We have skipped over the most difficult part of the argument. Note that we must use the precise nature of the deformation, since, for example, if we shrink co-axial circles in parallel planes then eventually there's no catenoid with that boundary.

annulus (with round boudary circles) is again of the same type. Thus, a continuity argument from $t = 0$ to 1 gives that $\Sigma(0)$ is of this type. □

Remark 22.2. For $n \geq 3$ if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a minimal hypersurface that's foliated by round spheres then it's a part of the catenoid, as proven by Jagy [Jag91]. A generalization of the Riemann example to higher dimensions (just not foliated by round spheres) was obtained by Kaabachi–Pacard [KP07].

Exercise 22.1. Prove Jagy's theorem: a minimal hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ (with $n \geq 3$) that intersects each parallel plane in a round sphere is a part of the catenoid.

Closely related to the 4π-conjecture (Open Question 2) is:

Open Question 4 (Convex curve conjecture, Meeks)**.** If Γ_0, Γ_1 are convex curves in parallel planes then they cannot bound a compact minimal surface of positive genus.

Since the total curvature of $\Gamma_0 \cup \Gamma_1$ is 4π, the method of Eckholm–White–Wienholtz (Theorem 11.4) implies that any (branched, immersed) compact minimal surface bounded by Γ_0, Γ_1 is embedded.

We have the following partial result (see also [MW91] for other partial results):

Theorem 22.3 (Schoen [Sch83])**.** *For $\Gamma_t \subset \{z = t\} \subset \mathbb{R}^3$, $t = 0, 1$ convex curves, assume that Γ_0, Γ_1 are invariant under reflection in the xz- and yz-planes. If Σ is a compact minimal surface with $\partial\Sigma = \Gamma_0 \cup \Gamma_1$ then Σ is either the disks in parallel planes bounded by the Γ_i or else an annulus that's invariant under the same reflections.*

Proof. The convex hull property implies that $\Sigma \subset \{z \in [0, 1]\}$ and the maximum principle gives $\Sigma \setminus \partial\Sigma \subset \{z \in (0, 1)\}$ (unless Σ are disks in parallel planes). We now use the method of moving planes. Let $\Pi_s = \{y = s\}$ denote a translation of the xz-plane. Let Σ_s^* denote the reflection of $\Sigma_s := \Sigma \cap \{y \leq s\}$ over Π_s. Starting from $s \gg 0$ we decrease s until the first time that Σ_s^* is not "strictly inside" Σ_s. If $s > 0$ then since $\Sigma^* s$ is "weakly inside," there cannot be interior contact. By assumption on Γ_0, Γ_1 there cannot be contact on the $\{z = 0\}$ or the $\{z = 1\}$ planes.

Thus, we consider Σ_s, Σ_s^* along their common boundary $\gamma_s = \partial\Sigma_s \setminus \{z \in \{0, 1\}\}$. On one hand, the boundary version of the maximum principle (Exercise 21.1) says that Σ_s, Σ_s^* cannot be tangent along γ_s. On the other hand, the assumptions on Γ_0, Γ_1 imply that they cannot be tangent at $\partial\gamma_s$. As such, continuity gives that Σ_s, Σ_s^* meet with a definite angle on $\overline{\gamma_s}$. Since there is no contact elsewhere we could thus decrease s slightly keeping Σ_s^* "strictly inside" of Σ_s, a contradiction. Thus $s = 0$. We can repeat the same argument from the other side to get $\Sigma_0 = \Sigma_0^*$.

This gives that $\Gamma_t \cap \Pi$ consists of graphs of bounded slope over $\Pi \cap \{z = t\}$. By considering the orthogonal plane, we see that $(\Sigma \cap \{z = t\}) \setminus \Pi$ has exactly component on both sides of $\Pi \cap \{z = t\}$, so Σ is an annulus. This completes the proof. $\square$

Part 5. **Second variation of area**

23. Computing the second variation

Theorem 23.1 (Second variation I)**.** *Consider $F_t : \Sigma^k \to (M, g)$ a 1-parameter family of embeddings with $F_t = F_0$ outside of a compact set. For $D_t F_t|_{t=0} = X, D_t^2 F_t|_{t=0} = Y$ the velocity and acceleration, we have*

$$\begin{aligned}
&\frac{d^2}{dt^2}\Big|_{t=0} \operatorname{area}_g(F_t(\Sigma)) \\
&= \int_\Sigma |(DX)^\perp|^2 + (\operatorname{div}_\Sigma X)^2 - \sum_{i,j=1}^k g(D_{e_i} X, e_j) g(D_{e_j} X, e_i) - \sum_{i=1}^k R_g(X, e_i, e_i, X) \\
&\qquad + \int_\Sigma \operatorname{div}_\Sigma Y
\end{aligned}$$

where e_i is an orthonormal frame and our curvature convention is that $R_g(e_i, e_j, e_j, e_i)$ is a sectional curvature.

Exercise 23.1. If M_t is a 1-parameter family of $n \times n$ matrices with $M_0 = \mathrm{Id}$ show that $(\det M_t)''(0) = \operatorname{tr} \ddot{M} + (\operatorname{tr} \dot{M})^2 - \operatorname{tr} \dot{M}^2$.

Proof for $(M, g) = \mathbb{R}^n$. This is similar to the first variation formula (Theorem 7.2). In the Euclidean case we can set $Y = \ddot{F}_0$.

Letting $M_{ij} = \langle \partial_i F_t, \partial_j F_t \rangle$ we have

$$\begin{aligned}
\dot{M}_{ij} &= \langle \partial_i X, \partial_j F_0 \rangle + \langle \partial_i F_0, \partial_j X \rangle \\
\ddot{M}_{ij} &= 2 \langle \partial_i X, \partial_j X \rangle + \langle \partial_i Y, \partial_j F_0 \rangle + \langle \partial_i F_0, \partial_j Y \rangle
\end{aligned}$$

Note that

$$\begin{aligned}
\operatorname{tr} \dot{M} &= 2 \operatorname{div}_\Sigma X \\
\operatorname{tr} \dot{M}^2 &= \sum_{i=1}^k (\dot{M})^2_{ii} = \sum_{i,j=1}^k \dot{M}^2_{ij} \\
&= 2 \sum_{i,j=1}^k \langle \partial_i X, \partial_j F_0 \rangle^2 + 2 \sum_{i,j=1}^k \langle \partial_i X, \partial_j F_0 \rangle \langle \partial_j X, \partial_i F_0 \rangle \\
&= 2 |(DX)^\top|^2 + 2 \sum_{i,j=1}^k \langle \partial_i X, e_i \rangle \langle \partial_j X, e_j \rangle
\end{aligned}$$

Using $(f(t)^{\frac{1}{2}})''(0) = -\frac{1}{4}f'(0)^2 + \frac{1}{2}f''(0)$ for $f(0) = 1$ we have

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\sqrt{\det M_t} &= -\frac{1}{4}\left(\frac{d}{dt}\Big|_{t=0}\det M_t\right)^2 + \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\det M_t \\ &= \frac{1}{2}\operatorname{tr}\ddot{M} + \frac{1}{4}(\operatorname{tr}\dot{M})^2 - \frac{1}{2}\operatorname{tr}\dot{M}^2 \\ &= \operatorname{div}_\Sigma Y + |DX|^2 + (\operatorname{div}_\Sigma X)^2 - |(DX)^\top|^2 - \sum_{i,j=1}^k \langle\partial_i X, e_i\rangle\langle\partial_j X, e_j\rangle \\ &= \operatorname{div}_\Sigma Y + |(DX)^\perp|^2 + (\operatorname{div}_\Sigma X)^2 - \sum_{i,j=1}^k \langle D_{e_i} X, e_i\rangle\left\langle D_{e_j} X, e_i\right\rangle.\end{aligned}$$

This completes the proof. □

Exercise 23.2. Generalize the proof to a non-flat Riemannian manifold.

If Σ is minimal, then the first variation formula gives that the Y term does not matter. (This is a geometric version of the usual fact that for a smooth manifold with no fixed Riemannian metric, the Hessian of a function is well-defined at a critical point but not elsewhere.) We thus define

$$\delta^2\Sigma(X) := \int_\Sigma |(DX)^\perp|^2 + (\operatorname{div}_\Sigma X)^2 - \sum_{i,j=1}^k g(D_{e_i}X, e_i)g(D_{e_j}X, e_i) - \sum_{i=1}^k R_g(X, e_i, e_i, X).$$

Theorem 23.2 (Second variation II)**.** *Assume that $\Sigma^k \subset (M, g)$ is a minimal submanifold and X is a normal vector field along Σ. Then*

$$\delta^2\Sigma(X) = \int_\Sigma |(DX)^\perp|^2 - |X \cdot \vec{A}|^2 - \operatorname{tr}_{T\Sigma} R_g(X, \cdot, \cdot, X).$$

Proof. Since X is normal and $\vec{H} = 0$ we have $\operatorname{div}_\Sigma X = 0$. Moreover,

$$g(D_{e_i}X, e_j) = -g(X, D_{e_i}e_j) = -X \cdot \vec{A}(e_i, e_j).$$

This completes the proof. □

Remark 23.3. Note the proof also gives $|X \cdot \vec{A}|^2 = |(DX)^\top|^2$.

Theorem 23.4 (Second variation III)**.** *Assume that $\Sigma^n \subset (M^{n+1}, g)$ is a minimal hypersurface with unit normal ν*

$$\delta^2\Sigma(\varphi) := \delta^2\Sigma(\varphi\nu) = \int_\Sigma |\nabla\varphi|^2 - |A|^2\varphi^2 - \operatorname{Ric}_g(\nu, \nu)\varphi^2$$

for any $\varphi \in C_c^\infty(\Sigma)$.

Proof. Since ν is a unit vector field, we see that $g(D\nu, \nu) = 0$ so $(D\nu)^\perp = 0$. □

24. Second variation and curvature

We call $\Sigma^n \subset (M^{n+1}, g)$ *two-sided* if it admits a smooth unit normal.

Corollary 24.1. *If (M^{n+1}, g) has $\operatorname{Ric}_g > 0$ then there are no closed, two-sided stable minimal hypersurfaces. In particular if M is oriented then $H_n(M;\mathbb{Z}) = 0$.*

Proof. Take $\varphi = 1$ to get
$$\int_\Sigma |A|^2 + \operatorname{Ric}_g(\nu,\nu) \leq 0.$$
This is a contradiction. If $H_n(M;\mathbb{Z}) \neq 0$ then we could minimize area in a homology class to find a two-sided stable minimal hypersurface. □

Lemma 24.2 (Doubly traced Gauss equations)**.** *If $\Sigma \subset (M, g)$ is a two-sided hypersurface then*

$$R_g = R_\Sigma + 2\operatorname{Ric}_g(\nu,\nu) + |A|^2 - H^2 \tag{24.1}$$

along Σ. Here R_g, R_Σ is the ambient, intrinsic scalar curvatures, Ric_g is the ambient Ricci curvature, $|A|^2$ is the norm of the scalar second fundamental form of Σ and H the scalar mean curvature.

Proof. Recall that
$$D_X Y = \nabla_X Y - A(X,Y)\nu$$
for X, Y tangent to Σ. We assume that X, Y, Z, W are tangent vector fields that are ∇-parallel at the point under consideration (so in particular $[X, Y] = 0$ and so on) and compute the ambient curvature as

$$\begin{aligned}
\operatorname{Rm}_g(X,Y,Z,W) &= g(D_X D_Y Z - D_Y D_X Z, W)\\
&= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, W)\\
&\qquad - g(D_X(A(Y,Z)\nu) - D_Y(A(X,Z)\nu), W)\\
&= \operatorname{Rm}_\Sigma(X,Y,Z,W) + A(X,Z)A(Y,W) - A(Y,Z)A(X,W).
\end{aligned}$$

(We used $g(\nu, W) = 0$ twice and $\nabla_X \nu = A(X,\cdot)$ as proven in Lemma 7.7.) For $e_1, \ldots, e_n$ an orthonormal basis of $T_p\Sigma$ we can trace $Y = Z = e_i$ to get

$$\operatorname{Ric}_g(X,W) - \operatorname{Rm}_g(X,\nu,\nu,W) = \operatorname{Ric}_\Sigma(X,W) + \sum_{i=1}^n A(X,e_i)A(e_i,W) - A(X,W)H.$$

(Note that this shows that for $\Sigma \subset \mathbb{R}^{n+1}$ minimal we have $\operatorname{Ric}_\Sigma \leq 0$.) Tracing again for $X, W = e_j$ we get
$$R_g - 2\operatorname{Ric}_g(\nu,\nu) = R_\Sigma + |A|^2 - H^2$$
This proves the assertion. □

Theorem 24.3 (Schoen–Yau). *If (M^3, g) has positive scalar curvature $R > 0$ then any connected two-sided stable minimal surface $\Sigma^2 \subset (M, g)$ is*[25] *topologically S^2.*

Proof. As before, we take $\varphi = 1$ to get

$$\int_\Sigma |A|^2 + \mathrm{Ric}_g(\nu, \nu) \leq 0.$$

Since we have not assumed that $\mathrm{Ric} > 0$, this is not a contradiction. Instead we recall that the intrinsic Gaussian curvature is related to the intrinsic scalar curvature as $K_\Sigma = 2R_\Sigma$. Thus the (doubly traced) Gauss equations (24.1) (and $H = 0$) give

$$2(\mathrm{Ric}_g(\nu, \nu) + |A|^2) = R_g + |A|^2 - 2K$$

Thus

$$\int_\Sigma R_g + |A|^2 \leq 2 \int_\Sigma K.$$

Since we assumed that $R_g > 0$, Gauss–Bonnet gives $\chi(\Sigma) > 0$. □

Corollary 24.4 (Schoen–Yau, Gromov–Lawson). *There's no metric g on T^3 with positive scalar curvature.*

Proof. We saw (Corollary 5.15) that there is a least area immersion $F : T^2 \to (T^3, g)$ (among maps homotopic to $T^2 \to T^2 \times \{*\}$). This minimal surface will be two-sided stable (immersion) and the previous analysis can apply to give that it must be a sphere (contradiction).

Alternatively, one may minimize in $H_2(T^3) = \mathbb{Z}^3$ to find $\Sigma^2 \subset (T^3, g)$ of least area. Each component of Σ would need to be a sphere. However, a sphere in T^3 bounds a ball (lift to the universal cover) and is thus homologically trivial. □

25. The Bernstein problem

Theorem 25.1 (Bernstein). *Suppose that u is an entire solution to the minimal surface equation on $\mathbb{R}^2$ then u is affine, i.e. the graph of u is a flat plane.*

The proof given below is not the original proof (which used complex analysis and PDE methods).

Proof. Let Σ be the graph of u. We have that Σ is area-minimizing (Theorem 18.1) and thus stable. It's two-sided so Theorem 23.4 gives

$$\int_\Sigma |A|^2 \varphi^2 \leq \int_\Sigma |\nabla \varphi|^2$$

[25] $\mathbb{R}P^2$ is possible if M is nonorientable

for all $\varphi \in C_c^\infty(\Sigma)$. We also recall that Corollary 18.2 gives $|\Sigma \cap B_R| \leq CR^2$. We use a log-cutoff function as in Lemma 5.8:

$$\varphi(x) = \begin{cases} 1 & |x| \leq R \\ 2 - \frac{\log |x|}{\log R} & |x| \in [R, R^2] \\ 0 & |x| \geq R^2. \end{cases}$$

It's easy to see that φ can be used in the stability inequality (even though it's only piecewise smooth). Note that

$$|\nabla_\Sigma \varphi|^2 \leq |\nabla_{\mathbb{R}^3} \varphi|^2 = \begin{cases} \frac{1}{|x| \log R} & |x| \in [R, R^2] \\ 0 & \text{otherwise.} \end{cases}$$

We thus need to estimate $\int_{\Sigma \cap (B_{R^2} \setminus B_R)} \frac{1}{|x|^2}$ using the quadratic area growth. We do this dyadically for $R = 2^k$

$$\begin{aligned} \int_{\Sigma \cap (B_{2^{2k}} \setminus B_{2^k})} \frac{1}{|x|^2} &\leq \sum_{j=k}^{2k-1} \int_{\Sigma \cap (B_{2^{j+1}} \setminus B_{2^j})} \frac{1}{|x|^2} \\ &\leq \sum_{j=k}^{2k-1} 2^{-2j} |\Sigma \cap (B_{2^{j+1}} \setminus B_{2^j})| \\ &\leq \sum_{j=k}^{2k-1} 2^{-2j} |\Sigma \cap B_{2^{j+1}}| \\ &\leq C \sum_{j=k}^{2k-1} 2^{-2j} 2^{2j} \\ &\leq Ck = C \log 2^k. \end{aligned}$$

Thus we get (for $R = 2^k$) that

$$\int_\Sigma |\nabla \varphi|^2 \leq \frac{1}{(\log R)^2} \log R = o(1).$$

Since $\varphi \to 1$ pointwise as $R \to \infty$ Fatou's lemma thus gives

$$\int_\Sigma |A|^2 \leq 0.$$

Thus $|A| = 0$ so Σ is flat. This completes the proof. □

There are several natural generalizations:

(1) Is an entire minimal graph over $\mathbb{R}^n$ a flat hyperplane? (This is "Bernstein's problem"). This is true for $n \leq 7$ [Fle62, DG65, Alm66, Sim68] but non-flat examples exist for $n \geq 8$ [BDGG69].

(2) Is a complete stable minimal hypersurface a hyperplane? (This is the "stable Bernstein problem"). Since minimal graphs are stable this generalizes the Bernstein problem. This is true in $\mathbb{R}^{n+1}$ for $n+1 \leq 6$ [Pog81, FCS80, dCP79, CL24, CL23, CMR24, CLMS24, Maz24] and false for $n+1 \geq 8$ ([BDGG69]. The problem is open in $\mathbb{R}^7$ (cf. [SSY75, CSZ97, SS81, Bel25]).

A related problem is:

Open Question 5. If $\Sigma^n \to \mathbb{R}^{n+1}$ is a complete, two-sided stable minimal immersion must Σ have intrinsic/extrinsic volume $O(R^n)$ volume growth estimates?

26. Stable minimal cones

Recall that for $\Sigma \subset \mathbb{S}^n$ the cone $C(\Sigma) \subset \mathbb{R}^{n+1}$ is minimal if and only if Σ is. For $\Sigma^{n-1} \subset \mathbb{S}^n$ minimal we define

$$\lambda_0(-\Delta_\Sigma - |A|^2) := \inf_{0 \neq \psi \in C^\infty(\Sigma)} \frac{\int_\Sigma |\nabla \psi|^2 - |A|^2 \psi^2}{\int_\Sigma \psi^2}$$

It's standard to see that λ_0 is the lowest eigenvalue of the operator $-\Delta_\Sigma - |A|^2$. Note that this is not exactly the second variation of area since we have dropped the $\operatorname{Ric}_{\mathbb{S}^n} = n-1$ term.

Proposition 26.1 (Simons [Sim68])**.** *The cone $C(\Sigma) \subset \mathbb{R}^{n+1}$ is stable if and only if $\lambda_0 \geq -\frac{(n-2)^2}{4}$.*

We consider only test functions $\varphi \in C^\infty_c(C(\Sigma) \setminus \{0\})$, i.e. those who fix the tip of the cone. It's easy to consider a larger class of φ via a cutoff argument.

Proof. It's easy to check that $|A_C|^2 = r^{-2}|A_\Sigma|^2$. Thus, $C(\Sigma)$ is stable if and only if

$$\int_0^\infty \int_\Sigma \varphi^2 r^{-2} |A_\Sigma|^2 r^{n-1} d\mu_\Sigma dr \leq \int_0^\infty \int_\Sigma ((\partial_r \varphi)^2 + r^{-2}|\nabla_\Sigma \varphi|^2) r^{n-1} d\mu_\Sigma dr$$

for all $\varphi \in C^\infty_c(C(\Sigma))$. We change variables $t = \log r$. (One could view the calculation below as re-writing stability in the conformal metric $\tilde{g} = r^{-2} g = dt^2 + g_\Sigma$ which is the product metric $\mathbb{R} \times \Sigma$.) Note that $\partial_r \varphi = r^{-1} \partial_t \varphi$ and $dr = r dt$ so we get

$$\int_{-\infty}^\infty \int_\Sigma \varphi^2 |A_\Sigma|^2 e^{(n-2)t} d\mu_\Sigma dt \leq \int_{-\infty}^\infty \int_\Sigma ((\partial_t \varphi)^2 + |\nabla_\Sigma \varphi|^2) e^{(n-2)t} d\mu_\Sigma dt$$

We can replace φ by $e^{-\frac{n-2}{2}t}\varphi$ (since $\varphi \in C^\infty_c(\mathbb{R} \times \Sigma)$ this does not change stability). Note that

$$(\partial_t e^{-\frac{n-2}{2}t}\varphi)^2 e^{(n-2)t} = (\partial_t \varphi)^2 - (n-2)\varphi \partial_t \varphi + \frac{(n-2)^2}{4}\varphi^2$$

and

$$\int_{-\infty}^\infty \varphi \partial_t \varphi = \frac{1}{2} \int_{-\infty}^\infty \partial_t \varphi^2 = 0.$$

Thus we find that stability is equivalent to

$$-\frac{(n-2)^2}{4}\int_{-\infty}^{\infty}\int_{\Sigma}\varphi^2 d\mu_\Sigma dt \le \int_{-\infty}^{\infty}\int_{\Sigma}\left(|\nabla_\Sigma\varphi|^2 - |A_\Sigma|^2\varphi^2\right)d\mu_\Sigma dt + \int_{-\infty}^{\infty}\int_{\Sigma}(\partial_t\varphi)^2 d\mu_\Sigma dt$$

We're now ready to prove the assertion.

Suppose that $C(\Sigma)$ is stable. For any $\psi \in C^\infty(\Sigma)$ we can take a cutoff function $\eta \in C_c^\infty(\mathbb{R})$ with $\eta = 1$ on $[-R,R]$, $\eta = 0$ on $[-2R,2R]^c$, and $|\eta'| \le CR^{-1}$. We get

$$-\frac{(n-2)^2}{4}\int_{\Sigma}\psi^2 d\mu_\Sigma dt \le \int_{\Sigma}\left(|\nabla_\Sigma\psi|^2 - |A_\Sigma|^2\psi^2\right)d\mu_\Sigma + \frac{\int_{-\infty}^{\infty}(\eta')^2 dt}{\int_{-\infty}^{\infty}\eta^2 dt}\int_{\Sigma}\psi^2 d\mu_\Sigma$$

Letting $R \to \infty$ we find $\lambda_0 \ge -\frac{(n-2)^2}{4}$.

Conversely, suppose that $\lambda_0 \ge -\frac{(n-2)^2}{4}$. For $\varphi \in C_c^\infty(\mathbb{R}\times\mathbb{C})$ we can apply this to $\varphi_t = \varphi(t,\cdot)$ for t fixed to get

$$-\frac{(n-2)^2}{4}\int_{\Sigma}\varphi_t^2 d\mu_\Sigma \le \int_{\Sigma}\left(|\nabla_\Sigma\varphi_t|^2 - |A_\Sigma|^2\varphi_t^2\right)d\mu_\Sigma$$

Integrating this over $t \in \mathbb{R}$ we get

$$-\frac{(n-2)^2}{4}\int_{-\infty}^{\infty}\int_{\Sigma}\varphi^2 d\mu_\Sigma dt \le \int_{-\infty}^{\infty}\int_{\Sigma}\left(|\nabla_\Sigma\varphi|^2 - |A_\Sigma|^2\varphi^2\right)d\mu_\Sigma dt$$

which implies stability. □

Minimal cones in $\mathbb{R}^3$ are flat (since the link is a geodesic in $\mathbb{S}^2$) so we next conisder:

Theorem 26.2 (Almgren [Alm66])**.** *If $C(\Sigma) \subset \mathbb{R}^4$ is a stable minimal cone then it's flat $\mathbb{R}^3$.*

Proof. Note that $\Sigma \subset \mathbb{S}^3$ is a minimal surface. By Exercise 20.2, Σ is connected. Taking $\psi = 1$ in $\lambda_0 \ge -\frac{(n-2)^2}{4} = -\frac{1}{4}$ gives

$$\int_{\Sigma}|A_\Sigma|^2 d\mu_\Sigma \le \frac{1}{4}|\Sigma|$$

On the other hand, the (doubly traced) Gauss equations (24.1) give

$$6 = 2K + 4 + |A_\Sigma|^2 \quad \Rightarrow \quad |A_\Sigma|^2 = 2 - 2K$$

since $\mathrm{Ric}_{\mathbb{S}^3} = 2$ and $R_{\mathbb{S}^3} = 6$. Thus

$$\int_{\Sigma}|A_\Sigma|^2 d\mu_\Sigma = 2|\Sigma| - 4\pi\chi(\Sigma).$$

Putting these facts together we get

$$|\Sigma| \le \frac{16}{3}\pi\chi(\Sigma)$$

so Σ is a topological sphere. Thus, the assertion follows from Theorem 26.3 below. □

Theorem 26.3 (Almgren [Alm66])**.** *If $\Sigma \subset \mathbb{S}^3$ is a minimal sphere then it's totally geodesic.*

Proof. We need some preliminary results. We consider isothermal coordinates $z = x + iy$ on Σ. Recall the complex tangent vectors $\partial_z, \partial_{\bar{z}}$. We need to understand the action of the connection on these vector fields (i.e. we need to compute the Christoffel symbols). Recall that (extending the metric complex bilinearly) $g(\partial_z, \partial_z) = g_{xx} - g_{yy} - ig_{xy} = 0$ and $g(\partial_z, \partial_{\bar{z}}) = g_{xx} + g_{yy}$ so

$$0 = g(\nabla_{\partial_{\bar{z}}}\partial_z, \partial_z)$$

Thus $\nabla_{\partial_{\bar{z}}}\partial_z = A\partial_z$. Symmetry of the connection gives $\nabla_{\partial_z}\partial_{\bar{z}} = A\partial_z$. On the other hand, differentiating $g(\partial_{\bar{z}}, \partial_{\bar{z}}) = 0$ in the ∂_z direction gives $\nabla_{\partial_z}\partial_{\bar{z}} = B\partial_{\bar{z}}$. Thus $A = B = 0$ so $\nabla_{\partial_{\bar{z}}}\partial_z = \nabla_{\partial_z}\partial_{\bar{z}} = 0$. Finally, we have

$$0 = g(\nabla_{\partial_z}\partial_z, \partial_z)$$

so we get $\nabla_{\partial_z}\partial_z = \Gamma\partial_z$ (and $\nabla_{\partial_{\bar{z}}}\partial_{\bar{z}} = \bar{\Gamma}\partial_{\bar{z}}$).[26]

We also need the *Codazzi equations* We recall that for Σ a two-sided hypersurface, the ambient connection decomposes as

$$D_X Y = \nabla_X Y - A(X,Y)\nu.$$

We consider vector fields X, Y tangent to Σ that are ∇-parallel at the point under consideration:

$$\begin{aligned}
\mathrm{Rm}_g(X,Y,Z,\nu) &= g(D_X D_Y Z - D_Y D_X Z, \nu)\\
&= g(D_Y(A(X,Z)\nu) - D_X(A(Y,Z)\nu), \nu)\\
&= Y(A(X,Z)) - X(A(Y,Z))\\
&= (\nabla_Y A)(X,Z) - (\nabla_X A)(Y,Z).
\end{aligned}$$

We used $[X,Y] = \nabla_X Y - \nabla_Y X$ above. Since the ambient space is $\mathbb{S}^3$, the left-hand side is $= g(X,\nu)g(Y,Z) - g(X,Z)g(Y,\nu) = 0$. Thus ∇A is totally symmetric in all three indices.

We now compute

$$\begin{aligned}
\partial_{\bar{z}}(A(\partial_z,\partial_z)) &= (\nabla_{\partial_{\bar{z}}}A)(\partial_z,\partial_z) && \text{(mixed Christoffel symbols vanish)}\\
&= (\nabla_{\partial z}A)(\partial_z,\partial_{\bar{z}}) && \text{(Codazzi equations)}\\
&= \partial_z(A(\partial_z,\partial_{\bar{z}})) - A(\nabla_{\partial_z}\partial_z,\partial_{\bar{z}}) && \text{(mixed Christoffel symbols vanish)}\\
&= \partial_z(A(\partial_z,\partial_{\bar{z}})) - \Gamma A(\partial_z,\partial_{\bar{z}}) && \text{(mixed Christoffel symbols vanish).}
\end{aligned}$$

We have

$$4A(\partial_z,\partial_{\bar{z}}) = A(\partial_x,\partial_x) + A(\partial_y,\partial_y) = 0$$

[26]This is a general fact that holds in a Kähler manifold: Christoffel symbols that mix holomorphic and anti-holomorphic coordinates vanish.

since $\operatorname{tr} A = 0$ and x, y are isothermal (so $|\partial_x| = |\partial_y| = 0$). Thus we find that

$$4A(\partial_z, \partial_z) = A(\partial_x, \partial_x) - A(\partial_y, \partial_y) - 2iA(\partial_x, \partial_y)$$

is holomorphic. As such

$$\Phi = A(\partial_z, \partial_z)dz^2$$

defines a holomorphic quadratic differential on $\mathbb{S}^2$, so $\Phi = 0$. Thus, Σ is totally geodesic. □

Exercise 26.1. Show that $\Sigma^2 \subset \mathbb{R}^3$ a topological sphere with constant mean curvature is a round sphere (Hopf).

Remark 26.4. Brendle proved [Bre13a] the Lawson conjecture which says that an *embedded* minimal torus in $\mathbb{S}^3$ is a rotation of $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) = C_{2,2} \cap \mathbb{S}^3$ (the Clifford torus). On the other hand, there's infinitely many *immersed* minimal tori in $\mathbb{S}^3$ so this proof combines a Hopf differential argument as in Theorem 26.3 with a geometric argument that brings in embeddedness. Note that Lawson has constructed [Law70] an embedded minimal surface in $\mathbb{S}^3$ of every genus. See [Bre13b] for an overview.

To generalize Theorem 26.2 to the full range we need (not proven here) the following inequality

Lemma 26.5 (Simons [Sim68])**.** *For $\Sigma^{n-1} \subset (\mathbb{S}^n, g)$ a minimal hypersurface, the second fundamental form satisfies*

$$|A|\Delta_\Sigma|A| + |A|^4 \geq (n-1)|A|^2.$$

This easily gives the sharp dimensional restriction on stable minimal cones.

Corollary 26.6 (Simons [Sim68])**.** *If $C(\Sigma) \subset \mathbb{R}^{n+1}$ is a stable non-flat minimal hypercone then $n \geq 7$.*

Proof. Take $\psi = |A|$ in $\lambda_0(-\Delta - |A|^2) \geq -\frac{(n-2)^2}{4}$. We have

$$\begin{aligned} -\frac{(n-2)^2}{4}\int_\Sigma |A|^2 &\leq \int_\Sigma |\nabla|A||^2 - |A|^4 \\ &= \int_\Sigma -|A|\Delta_\Sigma|A| - |A|^4 \\ &\leq -(n-1)\int_\Sigma |A|^2. \end{aligned}$$

As such, if Σ is non-flat then

$$4(n-1) \leq (n-2)^2 \Rightarrow n \geq 7.$$

This completes the proof. □

Part 6. Limits of minimal surfaces

27. Examples

We first describe several possible examples of limiting behaviors of minimal surfaces.

Let Σ be a catenoid in $\mathbb{R}^3$ (see Figure 6). We have the following examples:

(1) Let $x_i \in \Sigma$ diverge and consider the shifted catenoid $\Sigma_i := \Sigma - x_i$. Then Σ_i becomes flatter and flatter so we would like to say that Σ_i converges to a flat plane "smoothly on compact sets." Note that we need to say "on compact sets" since Σ_i always has a region of high curvature, it's just further and further away from the origin.

(2) Let $\lambda_i \to 0$ and consider the scaled catenoid $\Sigma_i = \lambda_i \Sigma$. Since the catenoid grows sublinearly at infinity, away from the origin Σ_i will consist of two sheets that are close to planar. However, the curvature of Σ_i satisfies $|A_{\Sigma_i}|(x) = \lambda_i^{-1} |A_\Sigma|(\lambda_i x)$ and thus Σ_i has curvature blowing up at the origin. We can thus say that Σ_i "smoothly" converges to $\mathbb{R}^2 \setminus \{0\}$ on compact subsets of $\mathbb{R}^3 \setminus \{0\}$ with "multiplicity two."

More general phenomenon are possible.

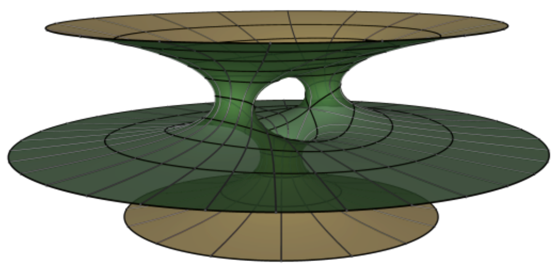

Figure 13. The Hoffman–Meeks deformation family of the Costa surface. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`

(3) Let Σ_i be (diverging) members of the "Hoffman–Meeks deformation family" of the Costa surface (See Figure 13). Then for appropriate λ_i, x_i, $\Sigma_i := \lambda_i \Sigma - x_i$ can "converge" to one of:
   (a) a flat plane with multiplicity one,
   (b) a flat plane with multiplicity three,
   (c) a flat plane with multiplicity three punctured at one or three points (where the convergence is on compact sets away from these points), or
   (d) a catenoid with multiplicity one.

(4) Let Σ_i be a sequence of "genus $g_i \to \infty$" Costa–Hoffman–Meeks surfaces (Figure 14). Then Σ_i "converges" to a catenoid $\cup$ plane smoothly away from the intersection circle. Appropriate rescalings/translations $\tilde{\Sigma}_i := \lambda_i \Sigma_i - x_i$ would converge to:
   (a) a flat plane with multiplicity one,

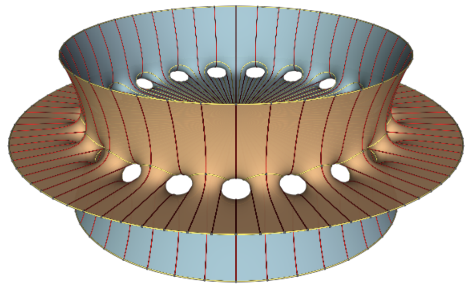

FIGURE 14. The high genus Costa–Hoffman–Meeks surface. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`

(b) a flat plane with multiplicity three,
(c) a flat plane with multiplicity three punctured at one point,
(d) two planes meeting orthogonally, each with multiplicity one, or
(e) Sherk's singly periodic surface (Figure 15).

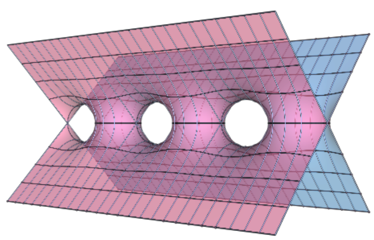

FIGURE 15. Scherk's singly periodic surface. Credit: Matthias Weber, `https://minimal.sitehost.iu.edu/archive/`

(5) Let Σ be the helicoid (Figure 4). Then appropriate scalings and translations Σ_i converge to
  (a) the helicoid,
  (b) a flat plane, or
  (c) the foliation of $\mathbb{R}^3$ by parallel planes (where the convergence may be non-smooth on a line).

28. Minimal surfaces with bounded curvature and area

We first show that the second fundamental form of a graph is related to the Hessian of the graphing function. (Loosely speaking $|A| \sim |D^2 w|$.)

Lemma 28.1. *The second fundamental form of the graph of w satisfies*

$$\frac{|D^2 w|}{(1+|\nabla w|^2)^{\frac{3}{2}}} \leq |A| \leq \frac{|D^2 w|}{(1+|\nabla w|^2)^{\frac{1}{2}}}$$

at $(x, w(x))$.

Proof. The vector fields $E_i = e_i + \partial_i w e_{n+1}$ form a basis of $T_p\Sigma$ at each point. Note that $g_{ij} = E_i \cdot E_j = \delta_{ij} + \partial_i w \partial_j w$ has eigenvalues 1 (with multiplicity $n-1$) and $1 + |\nabla w|^2$ (with multiplicity n). As such $\mathrm{Id} \leq g \leq (1+|\nabla w|^2)\,\mathrm{Id}$. We now compute

$$D_{E_i} E_j = \partial^2_{ij} w \, e_{n+1}.$$

so using $\nu = \frac{-\nabla w + e_{n+1}}{\sqrt{1+|\nabla w|^2}}$ we get

$$A(E_i, E_j) = -\frac{\partial^2_{ij} w}{\sqrt{1+|\nabla w|^2}}.$$

This proves the assertion. □

We now consider $\Sigma^n \subset \Omega \subset \mathbb{R}^{n+1}$ a properly embedded hypersurface in an open subset of $\mathbb{R}^{n+1}$ with $\partial\Sigma \cap \Omega = \emptyset$. (One could also consider a Riemannian manifold in place of Ω but some computations will become more involved.) For $p \in \Sigma$ suppose there's $w : B_r \subset T_p\Sigma \to (T_p\Sigma)^\perp$ so that $w(0) = 0$,

$$r^{-1}|w| + |\nabla w| + r|\nabla^2 w| \leq 1, \tag{28.1}$$

and graph $w \subset \Sigma$. In this case we say that $\Sigma(p; r) :=$ graph w exists. Since every hypersurface can locally be written as a graph of w with $w(0) = \nabla w(0) = 0$, we can always find $r > 0$ depending on p so that such a graphical region $\Sigma(p; r)$ exists.

Lemma 28.2. *Suppose that $r < \frac{1}{\sqrt{2}} d(p, \partial\Omega)$. Then if $\Sigma(p, r)$ exists we have*

$$\Sigma'(p; r) \subset \Sigma(p; r) \subset \Sigma \cap B_{\sqrt{2}r}(p)$$

where the left-hand-side is the connected component of $\Sigma \cap B_r(p)$ containing p.

Lemma 28.3. *Suppose that a hypersurface $\Sigma \subset \Omega$ with $\partial\Sigma \cap \Omega = \emptyset$ satisfies $\sup_{\Sigma\cap K} |A_\Sigma| \leq \beta(K)$ for all $K \Subset \Omega$. Then for any $K \Subset \Omega$ there's $r = r(K) < \frac{1}{\sqrt{2}} d(K, \partial\Omega)$ so that $\Sigma(p; r)$ exists for all $p \in \Sigma \cap K$.*

Proof. Choose $K \Subset K' \Subset \Omega$. Below we'll always work in balls contained in K'. We'll choose $r \leq r_0(K, K', \beta)$ below.

For $p \in \Sigma \cap K$, consider the intrinsic ball $B^\Sigma_{2r}(p) \subset \Sigma \cap K'$. For $q \in B^\Sigma_{2r}(p)$ let $\gamma : [0, 2r] \to B^\Sigma_{2r}(p)$ be a geodesic from p to q. Integrating $|\partial_t \nu(\gamma(t))| \leq |A|(\gamma(t)) \leq \beta$ we get

$$|\nu(q) - \nu(p)| \leq 2r\beta,$$

so taking r small, we find that $\pi : \Sigma \to T_p\Sigma$ is a local diffeomorphism at any point in $B^{\Sigma}_{2r}(p)$. Let $\gamma(t)$ be a geodesic from p to any $q \in \partial B^{\Sigma}_{2r}(p)$. We have

$$|\partial_t \langle \gamma'(t), \gamma'(0)\rangle| \leq |A|(\gamma(t)) \leq \beta$$

which yields

$$|q - p| \geq \langle q - p, \gamma'(0)\rangle \geq 2r - 2r^2\beta$$

after integrating twice. Taking r sufficiently small, this is $> \frac{3r}{2}$. Finally, a similar argument gives

$$|\langle q - p, \nu(p)\rangle| \leq 2r^2\beta < \frac{r}{2}$$

for any $q \in B^{\Sigma}_{2r}(p)$ (after taking r smaller if necessary). Let π be the projection to $T_p\Sigma$. Note that for $q \in \partial B^{\Sigma}_{2r}(p)$ we have

$$\frac{9r^2}{4} \leq |p - q|^2 = |\pi(q) - p|^2 + \langle p - q, \nu(p)\rangle^2 \leq |\pi(q) - p|^2 + \frac{r^2}{4}$$

so this gives $d(\pi(\partial B^{\Sigma}_{2r}(p)), p) > r$. Thus, putting this together we find that the connected component of $\Sigma \cap (B_r \subset T_p\Sigma) \times (T_p\Sigma)^{\perp}$ containing p is the graph of a function w defined on $B_r \subset T_p\Sigma$. We also have

$$\frac{1}{\sqrt{1 + |\nabla w|^2}} \leq 1 - 2r\beta$$

so taking r sufficiently small, we have $|\nabla w| \leq C$. This lets us use Lemma 28.1 to write $|D^2 w| \leq (1 + |\nabla w|^2)^{\frac{3}{2}} \sup_{\Sigma \cap K'} |A| \leq (1 + C^2)^{\frac{3}{2}} \beta(K') := B$ on B_r. Taylors theorem gives $|\nabla w| \leq rB$ and $r^{-1}|w| \leq rB$ so

$$r^{-1}|w| + |\nabla w| + r|D^2 w| \leq 3Br$$

Taking r even smaller, this is < 1. This completes the proof. □

Corollary 28.4. *Suppose that Σ_j is a sequence of minimal hypersurfaces $\Sigma_j \subset \Omega$ with $\partial\Sigma \cap \Omega = \emptyset$ and $\sup_{\Sigma_j \cap K} |A_\Sigma| \leq \beta(K)$ for all $K \Subset \Omega$. If $p_j \in \Sigma_j \cap K$ then up to passing to a subsequence, $p_j \to p_\infty$, $T_{p_j}\Sigma_j$ converges to Π and the graphical functions $w_j : B_r \to \mathbb{R}$ converge to w_∞ in C^∞ on $B_{r/2}$. The graph of w_∞ over Π at p_∞ is a minimal hypersurface Σ_∞ with $\partial\Sigma_\infty \cap B_{r/2}(p_\infty) = \emptyset$.*

Proof. The functions w_j satisfy the minimal surface equation and have bounded C^2-norm, so the assertion follows from Schauder theory (cf. Lemma 19.1). □

Theorem 28.5. *Consider $\Sigma^n_j \subset \Omega \subset \mathbb{R}^{n+1}$ minimal hypersurfaces with $\partial\Sigma_j \cap \Omega = \emptyset$. Assume that for all $K \Subset \Omega$ we have:*

(1) $\text{area}(\Sigma_j \cap K) \leq \alpha(K)$ *and*
(2) $\sup_{\Sigma_j \cap K} |A_{\Sigma_j}| \leq \beta(K)$.

Then up to passing to a subsequence, there's $\Sigma_\infty \subset \Omega$ smooth minimal hypersurface so that Σ_j limits to Σ "with finite multiplicity" in the following sense:

1. *there's a locally constant function $M : \Sigma_\infty \to \mathbb{Z}_{\geq 1}$ measuring the "multiplicity,"*
2. *there's an exhaustion $W_1 \Subset W_2 \Subset \dots \Sigma_\infty$ of pre-compact open sets,*
3. *there's a collection of ordered functions $w_{1,j} < \cdots < w_{M,j}$ defined on W_j so that $w_{\ell,j} \to 0$ in C^∞_{loc}, and*
4. *$\Sigma_j \cap K_j \subset \cup_\ell \operatorname{graph}_{W_j} w_{\ell,j}$ for $K_1 \Subset K_2 \Subset \dots$ an exhaustion of Ω by compact sets.*

(Note the mild abuse of notation since M may change between components of Σ_∞.) Note that the notation for graph means the graph defined using a fixed choice of unit normal:

$$\operatorname{graph}_W w := \{p + w(p)\nu(p) : p \in W\}.$$

Proof. Let μ_j denote the area measure of Σ_j. By (1) we can pass to a weakly convergent limit $\mu_j \rightharpoonup \mu_\infty$. Let $\Sigma_\infty = \operatorname{supp} \mu_\infty$. Pick $p_\infty \in \Sigma_\infty$ and choose r (depending on p_∞) a uniform graphical radius as in Lemma 28.3 (choosing r sufficiently small so that the subsequent arguments remain a bounded distance from $\partial\Omega$).

Considering Σ_j, pick a maximal collection of points $p_{1,j}, \dots, p_{L,j} \in \Sigma_j \cap B_{r/4}(p_\infty)$ so that $\Sigma_j(p_{\ell,j}; r/10)$ are pairwise disjoint.

Claim 28.6. $\Sigma_j \cap B_{r/4}(p_\infty) \subset \cup_{\ell=1}^L \Sigma_j(p_{\ell,j}; r/2)$

Proof. For $z \in \Sigma_j \cap B_{r/4}(p_\infty)$ we have $\Sigma_j(z; r/10) \cap \Sigma_j(p; r/10)$ for some $p = p_{\ell,j}$. Using Lemma 28.2 we get that $B_{\sqrt{2}r/10}(z) \cap B_{\sqrt{2}r/10}(p) \neq \emptyset$ so $\Sigma_j(z; r/10) \subset B_{\sqrt{2}r/10}(z) \subset B_{3\sqrt{2}r/10}(p) \subset B_{r/2}(p)$. Since $\Sigma_j(p; r/2)$ contains the connected component of $\Sigma_j \cap B_{r/2}(p)$ containing p and $\Sigma_j(z; r/10)$ is connected we see that $z \in \Sigma_j(z; r/10) \subset \Sigma_j(p; r/2)$. This proves the claim. ☐

Since $\operatorname{area}(\Sigma_j(p_{\ell,j}; r/10)) \geq Cr^n$ we assumption (1) gives that L is uniformly bounded. Thus, by Corollary 28.4 we can pass to a subsequence and pass each $\Sigma_j(p_{\ell,j}; r/2)$ to the limit to obtain $\Sigma_\infty(p_\ell; r/2)$. Note that the minimal hypersurfaces $\Sigma_j(p_{\ell,j}; r/2) \cap B_{r/4}(p_\infty)$ are pairwise disjoint and have no boundary (in $B_{r/4}(p_\infty)$). Thus, the maximum principle implies that $\Sigma_\infty(p_\ell; r/2) \cap B_{r/4}(p_\infty)$ are either pairwise disjoint or else equal. As such,

$$\Sigma_\infty \cap B_{r/4}(p_\infty) = \cup_{\ell=1}^L \Sigma_\infty(p_\ell; r/2) \cap B_{r/2}(p_\infty)$$

is a smooth embedded minimal hypersurface. Since p_∞ was arbitrary, we see that $\Sigma_\infty \subset \Omega$ is a smooth embedded minimal hypersurface.

It remains to prove that Σ_j converges to Σ_∞ as claimed. Let P denote the nearest point projection to Σ_∞ and let $P_j = P|_{\Sigma_j}$. For a fixed compact set K, we have that P is smooth at least on $\Sigma_j \cap K$ for j sufficiently large. Moreover, we have that $(dP_j)_p = \operatorname{proj}_{T_{P(p)}\Sigma_\infty}|_{T_p\Sigma_j}$.

Note that[27]

$$\sup_{p\in\Sigma_j\cap K} |\nu_{\Sigma_j}(p) - \nu_{\Sigma_\infty}(P_j(p))| \to 0.$$

Thus, we find that $(dP_j)_p$ is invertible for all $p \in \Sigma_j \cap K$ as long as we take j sufficiently large (depending on K). The inverse function theorem thus shows that Σ_j is locally graphical over Σ_∞ (with multiplicity), i.e. for $p_\infty \in \Sigma_\infty \cap K$, taking j sufficiently large depending only on K, there's $w_{1,j} < \cdots < w_{M,j}$ defined on some $B_r^{\Sigma_\infty}(p_\infty)$ so that graph $w_{\ell,j} \subset \Sigma_j$. Covering $\Sigma_\infty \cap K$ by finitely many sets of this form, we can patch the graphs together to complete the proof. □

For example, if Σ_j is the blow-down sequence of catenoids, then the curvature and area are uniformly bounded on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. The previous theorem makes precise the notion of "convergence" of Σ_j to $\mathbb{R}^2 \setminus \{0\}$ with multiplicity 2.

29. Minimal surfaces with bounded area

Consider $\Sigma_j^n \subset \Omega \subset \mathbb{R}^{n+1}$ minimal hypersurfaces (without boundary in Ω).

Lemma 29.1. *Suppose there's $p_j \in \Sigma_i$ so that $p_j \to p_\infty \in \Omega$ and $|A_{\Sigma_j}|(p_j) \to \infty$. Then there's $q_j \in \Sigma_j$ with $q_j \to p_\infty$ and so that for $\lambda_j := |A_{\Sigma_i}|(q_j) \to \infty$ we have that $\tilde\Sigma_j := \lambda_j(\Sigma_j - q_j)$ has uniformly bounded curvature on compact subsets of $\mathbb{R}^{n+1}$ and $|A_{\tilde\Sigma_j}|(0) = 1$*

Proof. Let $r_j < d(p_j, \partial\Omega)$ tend to zero sufficiently slowly so that

$$|A_{\Sigma_j}|(p_j)d(p_j, \partial B_{r_j}(p_j)) \to \infty$$

and $\Sigma_j \pitchfork \partial B_{r_j}(p_j)$. Then by the point-picking argument used in the proof of ε-regularity for harmonic maps (Theorem 5.5) we can choose q_j achieving

$$\max_{q\in\Sigma_j\cap B_{r_j}(p_j)} |A_{\Sigma_j}|(q)d(q, \partial B_{r_j}(p_j)).$$

Since $|A_{\tilde\Sigma_j}|(x) = \lambda_j^{-1}|A_{\Sigma_j}|(q_j + \lambda_j x)$, the same calculation as in Theorem 5.5 applies to show that $\tilde\Sigma_j$ has uniformly bounded curvature on compact sets. □

We now assume that for all $K \Subset \Omega$ we have area$(\Sigma_j \cap K) \leq \alpha(K)$. Let μ_∞ be the limiting area measure. Note that for a.e. $r > 0$ we have that

$$\lim_{j\to\infty} \operatorname{area}(\Sigma_j \cap B_r(p)) = \mu_\infty(B_r(p)),$$

which implies (via the monotonicity formula, Theorem 9.3) that

$$r \mapsto \Theta_{\mu_\infty}(p, r) := \frac{\mu_\infty(B_r(p))}{\omega_n r^n}$$

[27]If this failed at $p_j \in \Sigma_j \cap K$ we can find a uniform graphical region $\Sigma_j(p_j, r/2)$ that converges to some part of Σ_∞ which implies that $T_{p_j}\Sigma_j$ converges to $T_{p_\infty}\Sigma_\infty$ and $P_j(p_j) \to p_\infty$; this is a contradiction

is non-decreasing. Thus we can set $\Theta_{\mu_\infty}(p) := \lim_{r\searrow 0}\Theta_{\mu_\infty}(p,r)$.

Lemma 29.2. *The blow-up surfaces $\tilde\Sigma_j$ from Lemma 29.1 have uniform area bounds on compact sets and thus pass to a subsequential limit in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$ to a minimal hypersurface $\tilde\Sigma_\infty \subset \mathbb{R}^{n+1}$ with $\partial\tilde\Sigma_\infty = \emptyset$, $|A_{\tilde\Sigma}| \leq |A_{\tilde\Sigma}|(0) = 1$, and*

$$\Theta_{\tilde\Sigma_\infty}(x,r) := \frac{|\tilde\Sigma_\infty \cap B_r(x)|}{\omega_n r^n} \leq \Theta_{\mu_\infty}(p_\infty)$$

for any $x \in \mathbb{R}^{n+1}$ and $r > 0$.

Proof. The monotonicity formula implies that

$$\Theta_{\tilde\Sigma_j}(x,r) = \Theta_{\Sigma_j}(q_j + \lambda_j^{-1}x, \lambda_j^{-1}r) \leq \Theta_{\Sigma_j}(q_j + \lambda_j^{-1}x, s) \to \Theta_{\mu_\infty}(p_\infty, s)$$

for a.e. $s > 0$ and j sufficiently large so that $\lambda_j r \leq s$. This implies that $\tilde\Sigma_j$ has uniform area bounds on compact subsets of $\mathbb{R}^{n+1}$ so we can apply Theorem 28.5 to pass to a subsequential limit $\tilde\Sigma_\infty$ (with no boundary). The previous inequality passes to the limit for a.e. $r > 0$ giving

$$\Theta_{\tilde\Sigma_\infty}(x,r) \leq \Theta_{\mu_\infty}(p_\infty, s).$$

Sending $s \to 0$ completes the proof. □

In particular $\Theta_{\tilde\Sigma_\infty}(\infty) := \lim_{r\to\infty}\Theta_{\tilde\Sigma_j}(x,r)$ is $\leq \Theta_{\mu_\infty}(p_\infty)$.

Exercise 29.1. Show that $\Theta_{\tilde\Sigma_\infty}(\infty)$ is independent of the choice of x.

Example 29.3. Consider Σ_j the Costa–Hoffman–Meeks surface with genus $\to\infty$ so that Σ_j converges to catenoid $\cup$ plane. Note that $\Theta_{\mu_\infty}(p_\infty) = 2$ for p_∞ in the intersection circle. The blow-up procedure will produce the singly periodic Scherk's surface $\tilde\Sigma_\infty$ with $\Theta_{\tilde\Sigma_\infty}(\infty) = 2$.

One can also have strict inequality by a slight modification. Let $\lambda_j \to 0$ so that $\lambda_j\Sigma_j$ converges to a plane with multiplicity three, smoothly away from $p_\infty = 0$. In this case we have $\Theta_{\mu_\infty}(p_\infty) = 3$ but still $\Theta_{\tilde\Sigma_\infty}(\infty) = 2$.

30. White's easy Allard

Consider the setup in the previous section: $\Sigma_j \subset \Omega$ are minimal hypersurfaces with uniformly bounded area on compact sets and the curvature of Σ_j blows up at $p_\infty \in \Omega$. Let μ_∞ be the limiting area measure.

The following is a form of ε-regularity in this context.

Theorem 30.1 (White's Allard theorem [All72, Whi05])**.** *$\Theta_{\mu_\infty}(p_\infty) > 1$.*

Proof. Since we normalized by the second fundamental form we have $\tilde\Sigma_\infty$ is non-flat. Thus, we can apply the monotonicity formula at $x \in \tilde\Sigma_\infty$ to get

$$1 = \Theta_{\tilde\Sigma_\infty}(x) < \Theta_{\tilde\Sigma_\infty}(x,r) \to \Theta(\tilde\Sigma_\infty) \leq \Theta_{\mu_\infty}(p_\infty)$$

since equality cannot hold in the monotonicity formula. ☐

Note that this proof only applies to the limit of smooth objects, whereas Allard's result applies to arbitrary "weak solutions" (stationary varifolds). On the other hand, Allard's proof is much more involved.

One may improve this slightly using:

Theorem 30.2. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth non-flat minimal hypersurface with $|A_\Sigma| \leq |A_\Sigma|(0) = 1$ then $\Theta(\Sigma) \geq 1 + \varepsilon$ for $\varepsilon = \varepsilon(n)$.*

Proof. Assume there's Σ_j as in the theorem with $\Theta(\Sigma_j) \searrow 1$. They have uniformly bounded curvature and area on compact sets and thus we can pass to a subsequential limit Σ. Note that $|A_\Sigma|(0) = 1$ so Σ is non-flat. On the other hand, we have

$$\Theta_\Sigma(0, r) = \lim_{j\to\infty} \Theta_{\Sigma_j}(0, r) \leq \lim_{j\to\infty} \Theta(\Sigma_j) = 1$$

for a.e. $r > 0$. Thus $\Theta(\Sigma) = 1$ a contradiction. ☐

Open Question 6. What's the optimal value of $\varepsilon(n)$?

It's known that $1 + \varepsilon(2) = 2$ (e.g. attained by the catenoid) by geometric measure theory methods and that $1+\varepsilon(3) = \frac{\pi}{2} \approx 1.57$ (attained by smooth minimal hypersurface asymptotic to the quadratic cone $C_{2,2}$, cf. [Maz17]) by Marques–Neves's resolution of the Willmore conjecture [MN14] but there's no qualitative estimate for $\varepsilon(\geq 4)$ (see however [IW15, BW24]).

31. Bounded total curvature

Remark 31.1. Suppose that $\Sigma^2 \subset (M, g)$ is a closed minimal surface in a closed (or homogeneously regular) 3-manifold. The traced Gauss equations give $|A|^2 = R_g - 2\operatorname{Ric}_g(\nu, \nu) - 2K_\Sigma \leq C_g - 2K_\Sigma$ so

$$\int_\Sigma |A|^2 \leq C_g|\Sigma| - 4\pi\chi(\Sigma) \leq C(|\Sigma|, \chi(\Sigma)).$$

Motivated by this remark, we study the compactness of $\Sigma_j^n \subset \Omega \subset \mathbb{R}^{n+1}$ with

$$\int_{\Sigma_j} |A|^n \leq \Lambda \tag{31.1}$$

and area uniformly bounded on compact sets. (If we generalized the previous section to limits in Riemannian manifolds this discussion would apply to e.g. bounded area and genus minimal surfaces in a 3-manifold.)

Let μ_∞ denote the limiting area measure and consider $p_\infty \in \operatorname{supp} \mu_\infty := \Sigma_\infty$ so that the curvature blows up at p_∞. Lemma 29.2 gives a blow-up $\tilde{\Sigma}_\infty$ at p_∞ with $\Theta(\tilde{\Sigma}_\infty) \leq \Theta_{\mu_\infty}(p_\infty)$

and $|A_{\tilde{\Sigma}_\infty}|(0) = 1$. Since (31.1) is scale invariant and $\tilde{\Sigma}_\infty$ is non-flat we have

$$0 < \int_{\tilde{\Sigma}_\infty} |A_{\tilde{\Sigma}_\infty}|^n \leq \limsup_{r\to 0} \limsup_{j\to\infty} \int_{\Sigma_j \cap B_r(p_\infty)} |A_{\Sigma_j}|^n$$

(which is $\leq \Lambda$).

We can make this uniform as follows:

Lemma 31.2. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is complete with $|A_\Sigma| \leq |A_\Sigma|(0) = 1$ then $\int_\Sigma |A|^n \geq \varepsilon(n) > 0$.*

Proof. Suppose there's Σ_j as in the theorem with $\int_{\Sigma_j} |A|^n \to 0$. Theres a uniform $r > 0$ so that the graphical region $\Sigma_j(0, r)$ exists and $\Sigma_j(0, r/2)$ converges as graphs to some $\Sigma_\infty(0, r/2)$. Note this limit cannot be flat since we normalized $|A_{\Sigma_j}|(0) = 1$. Thus

$$\int_{\Sigma_\infty(0,r/2)} |A|^n = \lim_{j\to\infty} \int_{\Sigma_j(0,r/2)} |A|^n \leq \lim_{j\to 0} \int_{\Sigma_j} |A|^n = 0.$$

This is a contradiction. □

As such, we find that

$$0 < \varepsilon(n) \leq \int_{\tilde{\Sigma}_\infty} |A_{\tilde{\Sigma}_\infty}|^n \leq \limsup_{r\to 0} \limsup_{j\to\infty} \int_{\Sigma_j \cap B_r(p_\infty)} |A_{\Sigma_j}|^n.$$

This can only happen at $\leq \frac{\Lambda}{\varepsilon}$ points. Let $\mathcal{B}$ be the set of such points. For $p \in \mathcal{B}$ and $r_0 = \frac{1}{2} d(p, \partial\Omega)$ we note that $\mu_\infty(B_{r_0}(p)) \leq C$ since μ_∞ is a Radon measure. Since μ_∞ satisfies the monotonicity of area ratios, we see that $\mu_\infty(B_r(p)) \leq C r^n$ for $r < r_0$.

In sum, we obtain:

Theorem 31.3. *There's a finite set of points $\mathcal{B}$ with $|\mathcal{B}| \leq \frac{\Lambda}{\varepsilon}$ so that after passing to a subsequence Σ_j converges smoothly with finite multiplicity on compact subsets of $\Omega \setminus \mathcal{B}$ to $\Sigma_\infty \subset \Omega \setminus \mathcal{B}$ a smooth minimal hypersurface with bounded area on compact subsets, $\int_{\Sigma_\infty} |A_{\Sigma_\infty}|^n < \infty$, and $|\Sigma_\infty \cap B_r(p)| \leq C r^n$ for $p \in \mathcal{B}$ and $r \leq \frac{1}{2} d(p, \partial\Omega)$.*

In the next section we show that Σ_∞ is smooth across $\mathcal{B}$ even if the convergence is not.

One should keep in mind the examples of the Catenoid or Hoffman–Meeks deformation family that converge away from a finite set of points to a plane with multiplicity. On the other hand, the Costa–Hoffman–Meeks surfaces with unbounded genus can converge to a minimal surface with a circle of singularities.

We conclude this section with the following important observation. We assume that $\mathcal{B}$ has the property that Σ_j does *not* converge smoothly near any point in $\mathcal{B}$ (we can just discard points in $\mathcal{B}$ where there's smooth convergence).

Lemma 31.4. *Suppose that there's a component Σ of Σ_∞ with $\Sigma \cap \mathcal{B} \neq \emptyset$. Then Σ occurs with multiplicity > 1.*

Proof. Suppose that Σ occurs with multiplicity 1 but there's $p \in \Sigma \cap \mathcal{B}$. Then, writing μ_∞ for the limit of the area measures, we have $\Theta_{\mu_\infty}(p) = 1$. Thus by the converse of the Allard regularity theorem (Theorem 30.1) we get that the curvature cannot blowup at p. This is a contradiction. □

32. Removable singularities

Suppose that $\Sigma^n \subset B \setminus \{0\}$ is a minimal hypersurface with bounded area on compact sets, $|\Sigma \cap B_r| \leq Cr^n$ for $r < 1$ and $\int_\Sigma |A|^n < \infty$. Let $\lambda_j \to \infty$ and set $\Sigma_j := \lambda_j \Sigma_j$ be any blow-up sequence. Note that for $K \Subset \mathbb{R}^{n+1} \setminus \{0\}$ we have

$$\int_{\Sigma_j \cap K} |A|^n \to 0$$

and $|\Sigma_j \cap K| \leq \alpha(K)$ by the area growth bound for Σ. Thus by the compactness theorem just proven (Theorem 31.3) we can pass to a subsequential limit $\Sigma_\infty \subset \mathbb{R}^{n+1} \setminus \{0\}$. Note that there are no curvature concentration points since for $j \gg 0$ the total curvature will be $< \varepsilon$. We have that

$$\int_{\Sigma_\infty} |A|^n = 0$$

so Σ_∞ is flat. Assuming that $n \geq 2$ this implies that Σ_∞ is a finite union of parallel hyperplanes. Let f be the restriction of $|x|$ to $\Sigma \cap B_1$.

Lemma 32.1. *For $r \ll 1$, any critical point of f on $\Sigma \cap B_r$ is a non-denenerate local minimum and in particular f is Morse on Σ.*

Proof. Consider $p_j \in \Sigma$ with $p_j \to 0$ and $(df)_{p_j} = 0$. Let $\lambda_j = |p_j|^{-1}$ and take $\Sigma_j = \lambda_j \Sigma$ as above. Pass to a subsequence so that Σ_j converges in $C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\})$ to Σ_∞ a union of parallel hyperplanes and $\lambda_j p_j \to p_\infty \in \Sigma_\infty \cap \partial B_1$. Let f_∞ be the restriction of $|x|$ to Σ_∞. Since the p_j were critical points of f, p_∞ must be a critical point of f_∞. Thus Σ_∞ must contain the plane through p_∞ that's tangent to ∂B_1 at p_∞. Thus p_∞ is a non-degenerate local minimum of f_∞. This implies that for j large, p_j was a non-degenerate local minimum of f. This proves the asseriton. □

Without loss of generality (dilate Σ) we can assume that this holds for all $r \leq 1$ and that $\Sigma \pitchfork \partial B_1$.

Lemma 32.2. *Let Σ' be a connected component of Σ. If f has any critical points on Σ' then Σ' is a smooth minimal surface in B_1 with $0 \notin \Sigma'$.*

Proof. Consider $\Sigma'_t := \Sigma \cap f^{-1}(t)$. Passing a critical point corresponds to attaching a n-disk and thus exactly component of Σ'_t disappears at each critical point as t decreases from 1 towards 0. Since Σ' is connected, the only possibility is that Σ' has exactly one critical point after which $\Sigma'_t = \emptyset$. Thus f is bounded below on Σ'. □

Lemma 32.3. *There's $r_0 \in (0,1]$ so that if Σ' is a component of Σ with $\Sigma' \cap B_{r_0} \neq \emptyset$ then f has no critical points on Σ'.*

Proof. If not, there's Σ'_j a component of Σ with a critical point of f so that $\Sigma'_j \cap B_{j^{-1}} \neq \emptyset$. Lemma 32.2 gives that they're each smooth minimal hypersurfaces in B that avoid 0. Thus, we can pass to a subsequence all of the Σ'_j are pairwise disjoint. As such, the monotonicity formula gives $|\Sigma'_j \cap B_{1/2}| \geq c$. This contradicts the assumption that $|\Sigma \cap B_{1/2}| \leq C(1/2)^n$. □

We can thus dilate Σ so as to assume that f has no critical points and $\Sigma \pitchfork \partial B_1$.

Proposition 32.4. *Let Σ' be a connected component of Σ and let $\lambda_j \to \infty$. Let $\Sigma'_j = \lambda_j \Sigma'$. Up to passing to a subsequence, Σ'_j converges to Σ'_∞ a hyperplane through the origin with multiplicity one.*

Proof. If Σ'_∞ has a component that does not pass through the origin then repeating the proof of Lemma 32.1 we can see that f has a critical point on Σ', contradiction. Thus, it remains to prove that the multiplicity is one. Note that $\Sigma' \pitchfork \partial B_t$ for all $t \in (0,1]$ and the Morse theoretic argument as in Lemma 32.2 imply that distinct components of $\Sigma' \cap \partial B_t$ remain disconnected for all t. Thus, we have that Σ' is disconnected, a contradiction. □

Without loss of generality, we can thus replace Σ by one component and show that this component extends across $\{0\}$. (Note that *a posteriori* there's only one component of Σ: any two components would have 0 in their extension and thus would necessarily agree by the maximum principle.)

Theorem 32.5. *Σ extends across $\{0\}$*

(Compare with uniqueness of the limit in removable singularities for harmonic maps Theorem 5.7.) There are many proofs of this result. This proof is based on [Whi18] (cf. [BS18]).

Proof. Up to a rotation we can assume that for some $\lambda_j \to \infty$, $\lambda_j \Sigma$ converges to $\Pi_0 := \mathbb{R}^n \times \{0\}$. Our first goal is to prove that all blow-up limits are Π_0 (not some rotation).

By definition of convergence, there's $\varphi_j \in C^\infty(\partial B_1 \cap \Pi_0)$ so that the graph of φ_j agrees with $(\lambda_j \Sigma_j) \cap ((\partial B_1 \cap \Pi_0) \times \mathbb{R})$ and $\varphi_j \to 0$ in C^∞. Let v_j solve the minimal surface equation on $B_1 \cap \Pi_0$ with boundary data φ_j. Note that v_j can be found via the implicit function theorem since the linearization of the minimal surface equation at $u = 0$ is $d\mathcal{M}|_0 : w \mapsto \Delta w$. The implicit function theorem proof also gives that $v_j \to 0$ in $C^\infty(B_1 \cap \Pi_0)$. Let $h_j \in \mathbb{R}$ be chosen so that $0 \in \operatorname{graph}(v_j + h_j)$. Note that $h_j \to 0$.

Without loss of generality we can assume that $h_j \geq 0$. Let s_j be the infimum of $s \geq h_j$ so that $\operatorname{graph}(v_j + s)$ lies above Σ in $(B_1 \cap \Pi_0) \times \mathbb{R}$. Since $h_j \geq 0$, $\operatorname{graph}(v_j + s)$ cannot make contact with Σ at the boundary when $s > h_j$. Similarly, by definition of h_j, $\operatorname{graph}(v_j + s)$ lies above the origin. Thus, if $s_j > h_j$ then we would have interior one-sided contact, so

$\Sigma = \operatorname{graph}(v_j + s_j) \setminus \{0\}$, a contradiction since $s_j > h_j$ so the surfaces are disjoint at the boundary. Thus we see that $s_j = h_j$ so Σ lies below $\Gamma_j := \operatorname{graph}(v_j + h_j)$.

Now if $\tilde{\lambda}_j \to \infty$ has $\tilde{\lambda}_j\Sigma$ converging to some other plane Π_1, we can pass to a subsequence so that $\tilde{\lambda}_j = \lambda_j\mu_j$ with $\mu_j \to \infty$. By the previous paragraph, we know that $\tilde{\lambda}_j\Sigma := \mu_j(\lambda_j\Sigma)$ lies below $\mu_j\Gamma_j$ in $(B_1 \cap \Pi_0) \times \mathbb{R}$. Note that $\mu_j\Gamma_j$ limits to the limit of the tangent planes to Γ_j at 0, which is Π_0 since $v_j \to 0$ in C^1. On the other hand, $\tilde{\lambda}_j\Sigma$ limits to $\Pi_1 \neq \Pi_0$. This is a contradiction.

Thus we find that the unit normal ν to Σ limits to e_{n+1} as we approach the origin. Thus $\lambda_j\Sigma \cup \{0\}$ is the graph over $B_1 \cap \Pi_0$ of a C^1-function w_j. As in Proposition 19.3 there's a second order elliptic operator L_j so that $L_j(v_j - w_j) = 0$. Examining the proof, we see that the coefficients of L_j depend on the C^1-norm of v_j and w_j and the second derivatives of *one of them* (which we can choose to be w_j). Thus, the coefficients of L_j are C^0 which suffices to apply the strong maximum principle to conclude that $v_j = w_j$. Thus Σ extends smoothly across $\{0\}$, completing the proof. □

33. Multiplicity and stability

In this section we prove:

Theorem 33.1. *Suppose that $\Sigma_j \subset \Omega \subset \mathbb{R}^{n+1}$ with uniformly bounded area and total curvature* (31.1) *converge to Σ_∞ in $C^\infty_{\mathrm{loc}}(\Sigma \setminus \mathcal{B})$. If a two-sided component of Σ_∞ has multiplicity > 1 then this component is stable.*

We begin with a warmup calculation.

Lemma 33.2. *Suppose that $w_{1,k} < w_{2,k}$ are smooth functions on the unit ball $B \subset \mathbb{R}^n$ solving the minimal surface equation and with $w_{1,k}, w_{2,k} \to 0$ in C^∞. Then $v_k := \frac{w_{2,k} - w_{1,k}}{(w_{2,k} - w_{1,k})(0)}$ converges subsequently in C^∞_{loc} to a positive harmonic function v on B with $v(0) = 1$.*

Proof. In the proof of Proposition 19.3 we saw that for $a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}$ we can write

$$0 = \sum_{i,j=1}^{n} a_{ij}(Dw_{2,k}) D^2_{ij} v_k + \sum_{\ell,i,j=1}^{n} \left(\int_1^2 D_\ell a_{ij}(Dw_{t,k}) dt \right) D_\ell v_k \tag{33.1}$$

for $w_{t,k} = w_{1,k} + (t-1)(w_{2,k} - w_{1,k})$. Since $w_{1,k}, w_{2,k}$ converge smoothly to zero, this is a strictly elliptic PDE of the form

$$0 = L_k v_k = \sum_{i,j}^{n} a^{(k)}_{ij} D^2_{ij} v_k + \sum_{\ell=1}^{n} b^{(k)}_\ell D_\ell v_k$$

where $a^{(k)}_{ij} \to \delta_{ij}$ and $b^{(k)}_\ell \to 0$ in C^∞. Thus, the Harnack inequality (B.1) implies[28] that for any $B' \Subset B$ we have $\sup_{B'} v_k \leq C$ and $\inf_{B'} v_k \geq C^{-1}$. As such, we can apply Schauder

[28] Since the coefficients are converging smoothly, we can rewrite the equation for v_k in divergence form with controlled coefficients.

estimates (A.2) and Arzelà–Ascoli to pass v_k to a subsequential limit in C^∞_{loc}. The limit v will be harmonic and will have $v(0) = 1$. □

This remains true for graphs over non-flat minimal surfaces in the following form:

Proposition 33.3. *Suppose that $\Sigma \subset \Omega \subset \mathbb{R}^{n+1}$ is a connected, two-sided, minimal hypersurface. Suppose that $W_1 \Subset W_2 \Subset \Sigma$ is an exhaustion by compact sets so that there's $w_{1,j} < w_{2,j}$ defined on W_j with $\operatorname{graph}_{W_j} w_{\ell,j}$ is a minimal surface and $w_{\ell,j} \to 0$ in $C^\infty_{\text{loc}}(\Sigma)$. Fixing $p \in W_1$, define $v_k = \frac{w_{2,k}-w_{1,k}}{(w_{2,k}-w_{1,k})(p)}$. Then after passing to a subsequence, v_k converges in $C^\infty_{\text{loc}}(\Sigma)$ to $v > 0$ solving $L_\Sigma v = 0$ for $L_\Sigma = \Delta_\Sigma + |A|^2$ the second variation operator.*

This remains true in an ambient Riemannian manifold except $L = \Delta_\Sigma + |A|^2 + \operatorname{Ric}_g(\nu, \nu)$.

Proof. Observe that when we derived (33.1) above what we really did was a Taylor expansion of differential operators. Write $H(u)$ for the mean curvature of the graph of u over Σ. We have

$$0 = H(w_{2,k}) - H(w_{1,k}) = \int_1^2 \partial_t(H(w_{t,k}))dt.$$

Note that

$$\partial_t(H(w_{t,k})) = \frac{d}{ds}\Big|_{s=0} H(w_{t,k} + s(w_{2,k} - w_{1,k}) = L_k(w_{2,k} - w_{1,k})$$

is simply the *linearization* (derivative) of $H(\cdot)$ at $w_{t,k}$. We can argue that L_k is uniformly elliptic on compact sets (similarly as above) and thus after normalizing the graphs, we get a solution to $Lv = 0$ where L_k converges to L, the linearization of $H(\cdot)$ at 0).

To determine L we thus need to compute the directional derivative of H at 0. For $w, \varphi \in C^\infty_c(W)$, $W \Subset \Sigma$ let $\Gamma_{s,t} := \operatorname{graph}_W(sw + t\varphi)$. For (s,t) sufficiently close to $(0,0)$ this will be a smooth hypersurface. Write $\Gamma_s = \Gamma_{s,0}$ and S, T for the velocity of this family with respect to s, t. The first variation formula gives

$$\frac{\partial}{\partial t}\Big|_{t=0} \operatorname{area}(\Gamma_{s,t}) = \int_{\Gamma_s} H_{\Gamma_s} \langle T, \nu_{\Gamma_s}\rangle \, d\mu_{\Gamma_s}$$

Differentiate this at $s = 0$ with respect to s. Since $H_{\Gamma_0} = H_\Sigma = 0$ we get

$$\frac{\partial^2}{\partial t \partial s}\Big|_{s=t=0} \operatorname{area}(\Gamma_{s,t}) = \int_{\Gamma_s} \partial_s H_{\Gamma_s}|_{s=0} \varphi d\mu_\Sigma.$$

Set $Q(u,u) = \int_\Sigma |\nabla u|^2 - |A|^2 u$ (the second variation operator for normal variations). Note that Q is a bilinear form. Thus, we have

$$\begin{aligned} 2\frac{\partial^2}{\partial t \partial s}\Big|_{s=t=0} &= \left[\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^2 - \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right]_{s=t=0} \operatorname{area}(\Gamma_{s,t}) \\ &= Q(w+\varphi, w+\varphi) - Q(w,w) - Q(\varphi,\varphi) \\ &= 2Q(w,\varphi) \end{aligned}$$

$$= \int_\Sigma \langle \nabla w, \nabla \varphi \rangle - |A|^2 w \varphi d\mu_\Sigma$$
$$= - \int_\Sigma (\Delta w + |A|^2 w) \varphi d\mu_\Sigma.$$

Since φ was arbitrary, this gives that

$$\partial_s H_{\Gamma_s}|_{s=0} = -L_\Sigma w.$$

Thus, if we repeat the Taylor's theorem argument as in Lemma 33.2 we see that v_k solves $L_k v_k = 0$ where the coefficients of L_k limit to L_Σ. The proof can then be completed as before. □

Proposition 33.4 (Barta [Bar37])**.** *Suppose that there's a positive function $v \in C^\infty(\Sigma)$ with $L_\Sigma v = 0$. Then Σ is stable.*

Proof. Let $w = \log v$. We compute $\nabla w = \frac{\nabla v}{v}$ and

$$\Delta w = \frac{\Delta v}{v} - |\nabla w|^2 \leq -|A|^2 - |\nabla w|^2.$$

For $\varphi \in C_c^\infty(\Sigma)$ we thus have

$$\int_\Sigma |A|^2 \varphi^2 \leq \int_\Sigma (-\Delta w - |\nabla w|^2) \varphi^2 \leq \int_\Sigma 2|\varphi||\nabla \varphi||\nabla w| - |\nabla w|^2 \varphi^2 \leq \int_\Sigma |\nabla \varphi|^2.$$

This completes the proof. □

In the context of Theorem 33.1, this implies that any component $\Sigma \subset \Sigma_\infty$ that occurs with multiplicity > 1 has $\Sigma \setminus \mathcal{B}$ stable. Now, the assertion follows from:

Lemma 33.5. *If $\Sigma \setminus \{p\}$ is stable then Σ is stable.*

Proof. The log-cutoff trick (cf. Lemma 5.8) lets us approximate any $\varphi \in C_c^\infty(\Sigma)$ with $\varphi_j \in C_c^\infty(\Sigma \setminus \{p\})$ with $\int_\Sigma |\nabla \varphi_j|^2 \to \int_\Sigma |\nabla \varphi|^2$ and $\varphi_j \to \varphi$ pointwise. □

34. Choi–Schoen compactness

Recall (see Remark 31.1) that Gauss–Bonnet gives that for $\Sigma^2 \subset (M^3, g)$ minimal, we have $\int_\Sigma |A|^2 \leq C(|\Sigma|, \text{genus}(\Sigma))$. As such, an appropriate generalization of the previous sections to account for the background Riemannian metric proves:

Theorem 34.1. *Suppose that $\Sigma_j \subset (M^3, g)$ is a sequence of closed minimal surfaces in a closed 3-manifold with uniformly bounded area and genus. Then, theres a closed minimal surface $\Sigma_\infty \subset (M, g)$ and finite set of points $\mathcal{B}$ so that Σ_j converges in $C^\infty_{\text{loc}}(M \setminus \mathcal{B})$ to Σ_∞, possibly with multiplicity > 1 on some components. Any two-sided component that intersects $\mathcal{B}$ is stable.*

Since two-sided stable minimal surfaces do not exist in 3-manifolds with Ric >0 we see that multiplicity and non-smooth convergence cannot happen for (two-sided) limits of minimal surfaces of bounded genus and area in positive Ricci curvature.

We now improve this result by showing that if $\Sigma \subset (M^3, g)$ is a minimal surface in a 3-manifold with Ric >0 then $|\Sigma| \leq C(\text{genus}(\Sigma))$, i.e. the area bound follows automatically from the genus bound. The proof is surprisingly indirect.

34.1. **Choi–Wang's eigenvalue bound.** Consider (Ω, g) a compact Riemannian manifold with smooth boundary Σ. (In practice, Ω will be the closure of some component of $M \setminus \Sigma$.) The Bochner formula gives

$$\frac{1}{2}\Delta|\nabla f|^2 - g(\nabla\Delta f, \nabla f) = |D^2 f| + \text{Ric}(\nabla f, \nabla f)$$

Integrating this over Ω (terms with no subscript are ambient terms and terms with respect to the induced metric on Σ will have a Σ subscript; the outwards pointing unit normal will be ν) we get

$$\begin{aligned}
\int_\Omega |D^2 f|^2 + \text{Ric}(\nabla f, \nabla f) &= \int_\Omega \frac{1}{2}\Delta|\nabla f|^2 - g(\nabla\Delta f, \nabla f) \\
&= \int_\Omega (\Delta f)^2 + \int_\Sigma \frac{1}{2}\nabla_\nu |\nabla f|^2 - (\Delta f)\nabla_\nu f \\
&= \int_\Omega (\Delta f)^2 + \int_\Sigma D^2 f(\nabla f, \nu) - (\Delta f)\nabla_\nu f \\
&= \int_\Omega (\Delta f)^2 + \int_\Sigma D^2 f(\nabla_\Sigma f, \nu) - (\Delta f - D^2 f(\nu,\nu))\nabla_\nu f.
\end{aligned}$$

For $X \in T\Sigma$ we have

$$X g(\nabla f, \nu) = D^2 f(X, \nu) + g(\nabla f, D_X \nu) = D^2 f(X,\nu) + A(\nabla_\Sigma f, X)$$

so

$$D^2 f(\nabla_\Sigma f, \nu) = g(\nabla_\Sigma f, \nabla_\Sigma(\nabla_\nu f)) - A(\nabla_\Sigma f, \nabla_\Sigma f).$$

We also have that if $e_1, \dots, e_n$ is an orthonormal frame for Σ then

$$\begin{aligned}
\Delta f &= \text{tr}\, D^2 f \\
&= D^2 f(\nu,\nu) + \sum_{i=1}^n g(D_{e_i}\nabla f, e_i) \\
&= D^2 f(\nu,\nu) + \sum_{i=1}^n g(D_{e_i}\nabla_\Sigma f, e_i) + \sum_{i=1}^n g(D_{e_i}(\nabla_\nu f \nu), e_i) \\
&= D^2 f(\nu,\nu) + \sum_{i=1}^n g(\nabla_{e_i}\nabla_\Sigma f, e_i) + \nabla_\nu f \sum_{i=1}^n g(D_{e_i}\nu, e_i)
\end{aligned}$$

$$= D^2 f(\nu,\nu) + \Delta_\Sigma f + H\nabla_\nu f.$$

Remark 34.2. This is a useful formula. Note that it generalizes the well-known expression for the Laplacian in spherical coordinates since the sphere of radius r in $\mathbb{R}^{n+1}$ has mean cuvature $\frac{n}{r}$ (with the outwards pointing unit normal) so we get $\Delta f = \partial_r^2 f + \frac{n}{r}\partial_r f + \frac{1}{r^2}\Delta_{\mathbb{S}^n} f$.

Exercise 34.1. If $\Sigma^n \subset \mathbb{S}^{n+1}$ prove that n is an eigenvalue of the Laplacian on Σ (with eigenfunctions given by restrictions of coordinate functions from $\mathbb{R}^{n+2}$).

Thus, we can rearrange the expression above to read

$$\begin{aligned}\int_\Omega (\Delta f)^2 - |D^2 f|^2 &= \int_\Omega \operatorname{Ric}(\nabla f,\nabla f)\\ &\quad + \int_\Sigma -g(\nabla_\Sigma f, \nabla_\Sigma(\nabla_\nu f)) + A(\nabla_\Sigma f,\nabla_\Sigma f) + (\Delta_\Sigma f + H\nabla_\nu f)\nabla_\nu f\\ &= \int_\Omega \operatorname{Ric}(\nabla f,\nabla f)\\ &\quad + \int_\Sigma (2\Delta_\Sigma f + H\nabla_\nu f)\nabla_\nu f + A(\nabla_\Sigma f,\nabla_\Sigma f).\end{aligned}$$

This is the *Reilly formula* [Rei77].

Exercise 34.2. Prove that if Ω is compact with $\operatorname{Ric} > 0$ and $\partial\Omega$ has $H = 0$ then $\partial\Omega$ must be connected. (Compare with Exercise 20.2).

Theorem 34.3 (Choi–Wang [CW83]). *Suppose that (M^{n+1}, g) is orientable and has $\operatorname{Ric} \geq k$. Then if $\Sigma^n \subset (M,g)$ is a two-sided minimal hypersurface then*

$$\lambda_1(\Delta_\Sigma) \geq \frac{k}{2}$$

where $\lambda_1(\Delta_\Sigma) > 0$ is the lowest non-zero eigenvalue of the Laplacian.

Proof. Since $H_n(M) = 0$ (see Corollary 24.1) we see that Σ separates M into two compact manifolds $\Omega, \tilde\Omega$ with $\partial\Omega = \partial\tilde\Omega = \Sigma$.

Let $\Delta_\Sigma\varphi + \lambda\varphi = 0$ be a non-trivial eigenfunction on Σ. Choose the unit normal ν so that that $\int_\Sigma A(\nabla_\Sigma\varphi, \nabla_\Sigma\varphi) \geq 0$ and then adjust the labeling so that ν points out of Ω. Solve $\Delta f = 0$ on Ω with $f|_{\partial\Omega} = \varphi$. Then the Reilly formula gives

$$\begin{aligned}0 &\geq k\int_\Omega |\nabla f|^2 + \int_\Sigma 2(\Delta_\Sigma\varphi)\nabla_\nu f\\ &\geq k\int_\Omega |\nabla f|^2 - 2\lambda\int_\Sigma f\nabla_\nu f\\ &= (k-2\lambda)\int_\Omega |\nabla f|^2.\end{aligned}$$

If $\int_\Omega |\nabla f|^2 = 0$ then f and thus φ is constant. This cannot occur by assumption so we thus have $k \leq 2\lambda$. This completes the proof. □

Corollary 34.4. *If $\Sigma^n \subset \mathbb{S}^{n+1}$ is an embedded minimal hypersurface then $\lambda_1(\Delta_\Sigma) \geq \frac{n}{2}$.*

Proof. The round sphere $\mathbb{S}^{n+1}$ has $\mathrm{Ric} = n$. □

The following is a famous open problem in the area. It looks innocuous but would have many important applications to classification problems. See [LY82, Bre13b].

Open Question 7 (Yau's conjecture on the eigenvalue)**.** If $\Sigma^n \subset \mathbb{S}^{n+1}$ is an embedded minimal hypersurface then is it true that $\lambda_1(\Delta_\Sigma) \geq n$. (This would be sharp by Exercise 34.1.)

34.2. **Yang–Yau eigenvalue bound.** For $\Sigma^2 \subset (M^3, g)$ a minimal surface in an ambient manifold of $\mathrm{Ric} > 2$, we have that $\lambda_1(\Delta_\Sigma) \geq 1$. The following estimate lets us convert this into a bound for the area of Σ.

Theorem 34.5 (Yang–Yau [YY80])**.** *Let (Σ, g) be a closed surface of genus γ. If g is any Riemannian metric on Σ then*

$$\lambda_1(\Delta_\Sigma)|\Sigma| \leq 8\pi \left\lfloor \frac{\gamma+3}{2} \right\rfloor$$

Proof. The variational characterization of eigenvalues gives

$$\lambda_1(\Delta_\Sigma) = \inf \left\{ \frac{\int_\Sigma |\nabla \varphi|^2}{\int_\Sigma \varphi^2} : \varphi \in C^\infty(\Sigma) \setminus \{0\}, \int_\Sigma \varphi = 0 \right\}.$$

Consider $\Phi : \Sigma \to \mathbb{S}^2 \subset \mathbb{R}^3$ coming from a meromorphic function on Σ (choose the Riemann surface structure compatible with g). Using a fixed point argument, we can compose Φ with a conformal automorphsim $\mathbb{S}^2 \to \mathbb{S}^2$ so as to assume that

$$\int_\Sigma \Phi_i = 0$$

for $i = 1, 2, 3$. Indeed, we can parametrize[29] the conformal group of S^2 by points in the ball $g : B \to PGL(2; \mathbb{C})$ so that $\lim_{g \to \partial B} g = \mathrm{Id}_{\partial B}$. Then $B \ni y \mapsto ⨍_\Sigma \Phi \circ g_y$ is a map $B \to B$ that extends to the identity $\partial B \to \partial B$

Since Φ is conformal and orientation preserving we find that

$$\sum_{i=1}^{3} \int_\Sigma |\nabla \Phi_i|^2 = 2\mathcal{E}(\Phi) = 2\mathcal{A}(\Phi) = 8\pi \deg \Phi$$

Note that the area of Φ is proportional to the degree since Φ is always orientation preserving and thus covers a.e. point exactly $\deg \Phi$ times.

On the other hand, since $\int \Phi_i = 0$, it's a valid test function in $\lambda_1(\Delta_\Sigma)$. Thus we have

$$\lambda_1(\Delta_\Sigma) \int_\Sigma \Phi_i^2 \leq \int_\Sigma |\nabla \Phi_i|^2.$$

[29]Explicitly take $g_y(x) = \left(\frac{1-|y|^2}{|x+y|^2}(x+y) + y \right)^*$ for $z^* = \frac{z}{|z|^2}$.

Summing $i = 1, 2, 3$ and using $|\Phi|^2 = 1$ we have

$$\lambda_1(\Delta_\Sigma)|\Sigma| \leq 8\pi \deg \Phi.$$

Finally, we need to use Riemann surface theory to find a (nonconstant) meromorphic function on Σ with at most $d(\gamma)$ poles (this bounds the degree of the corresponding map $\Sigma \to \mathbb{S}^2$. Riemann–Roch implies[30] that we can take $d(\gamma) = \gamma + 1$. Improved estimates from "Brill–Noether theory" give the asserted bound (the exact form not relevant for us here) see [ESI84, Kar19]. □

34.3. **Compactness.** Combining the Choi–Wang eigenvalue bound (Theorem 34.3) and the Yang–Yau eigenvalue bound (Theorem 34.5) we find that if $\Sigma \subset (M, g)$ is a minimal surface in a 3-manifold with $\text{Ric} \geq k > 0$ then its area is bounded in terms of its genus

$$\frac{k}{2}|\Sigma| \leq \lambda_1(\Delta_\Sigma)|\Sigma| \leq C(\gamma).$$

Combined with the compactness result (Theorem 34.1) we thus conclude:

Theorem 34.6 (Choi–Schoen [CS85], cf. [Whi87])**.** *Suppose that* (M^3, g) *has* Ric > 0 *and does not contain any embedded one-sided surfaces. Then if* Σ_j *is a sequence of minimal surfaces with uniformly bounded genus and area a subsequence converges to a minimal surface* Σ_∞ *smoothly with multiplicity one.*

This applies to any metric on S^3 with Ric > 0, e.g. the round metric.

34.4. **Applications to the moduli space of minimal surfaces.** Let $\mathcal{M}_\gamma$ denote the set of embedded minimal surfaces in the round $\mathbb{S}^3$ of genus γ (modulo ambient isometries). Theorem 34.6 implies that $\mathcal{M}_\gamma$ is compact for each γ (with respect to C^∞ convergence with multiplicity one).

Almgren proved that (Theorem 26.3) $\mathcal{M}_0$ has one element (the equatorial S^2). As discussed in Remark 26.3, Brendle proved [Bre13a] that $\mathcal{M}_1$ has one element (the Clifford torus) and Lawson proved [Law70] that $\mathcal{M}_\gamma \neq \emptyset$ for all γ. There are many well-known questions about $\mathcal{M}_\gamma$.

Open Question 8. What is $\mathcal{M}_2$? Is $\mathcal{M}_\gamma$ always equal to a finite set? Can there exist a non-trivial 1-parameter family of embedded minimal surfaces $\Sigma_t \subset \mathbb{S}^3$? If $\mathcal{M}_\gamma$ is not a finite set, is it a smooth manifold?

For a "generic" Riemannian manifold (M^3, g) with Ric > 0 more is known. Combining Theorem 34.6 with White's "bumpy metric" theorem [Whi91], for a generic (in the Baire sense) metric g, there's at most finitely many minimal surfaces of each genus.

[30]For a divisor D with degree $\gamma + 1$, Riemann–Roch gives that $\ell(D) \geq \deg(D) - \gamma + 1 = 2$ where $\ell(D)$ is the set of meromorphic functions with poles only in D (counting order). Thus, we can find a non-constant merormorphic function with at most $\gamma + 1$ poles.

Appendices and references

Appendix A. Elliptic estimates

Recall the Sobolev norm $\|u\|_{W^{k,p}} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p}$. Clearly $\|\Delta u\|_{W^{k,p}} \leq C\|u\|_{W^{k+2,p}}$. Elliptic regularity says that this can (almost) be reversed. We recall the following three estimates for $\Omega' \Subset \Omega \subset \mathbb{R}^n$.

The first is $W^{k,p}$-elliptic regularity [GT01, Theorem 9.11]. For $k \in \mathbb{Z}_{\geq 0}$ and $p \in (1,\infty)$:

$$\|u\|_{W^{k+2,p}(\Omega')} \lesssim \|u\|_{L^p(\Omega)} + \|\Delta u\|_{W^{k,p}(\Omega)} \tag{A.1}$$

These estimates hold for more general operators but we will not need them in the text.

Next we have Schauder estimates [GT01, Problem 6.1]. For $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0,1)$:

$$\|u\|_{C^{k+2,\alpha}(\Omega')} \lesssim \|u\|_{C^0(\Omega)} + \|\Delta u\|_{C^{k,\alpha}(\Omega)} \tag{A.2}$$

Schauder estimates hold for a general non-divergence form operator

$$L = \sum_{i,j=1}^n a_{ij} D^2_{ij} + \sum_{j=1}^n b_j D_j + c$$

in place of Δ where the constant depends on estimates for $(a_{ij}) \geq \lambda \operatorname{Id}$ and $a, b, c \in C^\alpha$.

Appendix B. Harnack inequality

We now recall the Harnack inequality [GT01, Theorem 8.2] for elliptic operators. Suppose that $u \in W^{1,2}(\Omega)$ is a weak solution to a divergence form equation

$$\sum_{i,j=1}^n D_i(a_{ij} D_j u) + \sum_{i=1}^n b_i D_i u + cu = 0$$

where $(a_{ij}) \geq \lambda \operatorname{Id}$, and $a, b, c \in L^\infty$. Then for $B_R \subset \Omega$ we have:

$$\sup_{B_R} u \leq C \inf_{B_R} u \tag{B.1}$$

where C depends on λ, the L^∞ bounds, and R.

Appendix C. Sobolev inequalities

We recall the Morrey–Sobolev inequality[31] [GT01, Theorem 7.26]. For $p \in [n,\infty)$ and $k \in \mathbb{Z}_{\geq 0}$:

$$\|u\|_{C^{k,1-\frac{n}{p}}(\Omega')} \lesssim \|u\|_{W^{k+1,p}(\Omega)} \tag{C.1}$$

Similarly, we have the Sobolev inequality [GT01, Theorem 7.26]. For $p \in [1,n)$ and $k \geq \mathbb{Z}_{\geq 0}$:

$$\|u\|_{W^{k,\frac{n-p}{np}}(\Omega')} \lesssim \|u\|_{W^{k+1,p}(\Omega)}. \tag{C.2}$$

[31] This actually holds up to the boundary assuming $\partial\Omega$ is sufficiently regular.

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