Full Substitutability[*]

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Abstract

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, and exchange economies with indivisible goods. We extend earlier models' definitions of substitutability to settings in which each agent can be both a buyer in some transactions and a seller in others, and show that all these definitions are equivalent. We then introduce a new class of substitutable preferences that allows us to model intermediaries with production capacity. We also prove that substitutability is preserved under economically important transformations such as trade endowments, mergers, and limited liability.

JEL Classification: C62, C78, D44, D47

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1 Introduction

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, exchange economies with indivisible goods, and trading networks (Kelso and Crawford, 1982; Roth, 1984; Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Ausubel and Milgrom, 2006; Hatfield and Milgrom, 2005; Sun and Yang, 2006, 2009; Ostrovsky, 2008; Hatfield et al., 2013). Substitutability arises in a number of important applications, including matching with distributional constraints (Abdulkadiroğlu and Sönmez, 2003; Hafalir et al., 2013; Sönmez and Switzer, 2013; Sönmez, 2013; Westkamp, 2013; Ehlers et al., 2014; Echenique and Yenmez, 2015; Kominers and Sönmez, 2016; Kamada and Kojima, 2015), supply chains (Ostrovsky, 2008), markets with horizontal subcontracting (Hatfield et al., 2013), "swap" deals in exchange markets (Milgrom, 2009), and combinatorial auctions for bank securities (Klemperer, 2010; Baldwin and Klemperer, 2016).

The diversity of settings in which substitutability plays a role has led to a variety of different definitions of substitutability, and a number of restrictions on preferences that appear in some definitions but not in others.[1] In this paper, we show how the different definitions of substitutability are related to each other, while dispensing with some of the restrictions in the preceding literature. We consider agents who can simultaneously be buyers in some transactions and sellers in others; this allows us to embed the key substitutability concepts from the matching, auctions, and exchange economy literatures.[2] Our main result shows that all the substitutability concepts are equivalent. We call preferences satisfying these conditions *fully substitutable*.[3]

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[1] For instance, some definitions assume "free disposal"/"monotonicity," under which an agent is always weakly better off with a larger set of goods than with a smaller one, while other definitions do not; some definitions assume that all bundles of goods are feasible for each agent, while others do not; and so on.

[2] While all of the results in our paper consider the preferences of a single agent, and thus do not depend on the details of the agent's setting, for concreteness, notational simplicity, and continuity with prior literature, we state and prove these results in the general trading network setting of Hatfield et al. (2013).

[3] We use the modifier "fully" to highlight the possibility that under such preferences, an agent can be both a buyer in some transactions and a seller in others, whereas under the "gross substitutes" preferences of Kelso and Crawford (1982), an agent can be only a buyer or only a seller.

We introduce a rich new class of fully substitutable preferences that models the preferences of intermediaries with production capacity. To show that these preferences are fully substitutable, we rely on several properties of fully substitutable functions that we establish in this paper. Additionally, we make use of a novel proof technique of using “dummy layers.” This technique simplifies modeling the preferences of a firm with several alternative production technologies.

We also prove that full substitutability is preserved under several economically important transformations: trade endowments and obligations, mergers, and limited liability. We show that full substitutability can be recast in terms of submodularity of the indirect utility function, the single improvement property, a “no complementarities” condition, and a condition from discrete convex analysis called $M^{\natural}$-concavity. Finally, we prove that full substitutability implies two key monotonicity conditions: the Laws of Aggregate Supply and Demand.

All of our results explicitly incorporate economically important features that were not fully addressed in the earlier literature, such as indifferences, non-monotonicities, and unbounded utility functions. In particular, unbounded utility functions allow us to model firms with technological constraints under which some production plans are infeasible (and will therefore never be undertaken under any vector of prices).

1.1 History and Related Literature

For two-sided settings, Kelso and Crawford (1982) introduced the (demand-theoretic) gross substitutability condition, under which substitutability is expressed in terms of changes in an agent’s demand as prices change. Roth (1984) introduced a related (choice-theoretic) definition, under which substitutability is expressed in terms of changes in an agent’s choice as the set of available options changes. These conditions were subsequently extended and generalized, giving rise to two (mostly) independent literatures.

In two-sided matching models, (choice-theoretic) substitutability guarantees the existence of stable outcomes (Roth, 1984; Hatfield and Milgrom, 2005; Hatfield and Kominers, 2013).

Ostrovsky (2008) generalized the classic substitutability conditions to the context of supply chain networks by introducing a pair of related assumptions: same-side substitutability and cross-side complementarity. These assumptions impose two constraints: First, when an agent's opportunity set on one side of the market expands, that agent does not choose any options previously rejected from that side of the market. Second, when an agent's opportunity set on one side of the market expands, that agent does not reject any options previously chosen from the other side of the market. Both Ostrovsky (2008) and Hatfield and Kominers (2012) showed that under same-side substitutability and cross-side complementarity, a stable outcome always exists if the contractual set has a supply chain structure;[4] see also the recent work by Fleiner et al. (2017). Moreover, Hatfield, Kominers, and Jagadeesan (2017) showed that same-side substitutability and cross-side complementarity are together equivalent to the assumption of weak quasisubmodularity of the indirect utility function—an adaptation of submodularity to the setting without transfers.[5]

In exchange economies with indivisible goods, (demand-theoretic) gross substitutability guarantees the existence of core allocations and competitive equilibria (Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000). Ausubel and Milgrom (2002) offered a convenient alternative definition of gross substitutability for a setting with continuous prices, in which demand is not guaranteed to be single-valued, and showed that gross substitutability is equivalent to submodularity of the indirect utility function. Sun and Yang (2006) introduced the gross substitutability and complementarity condition for the setting of indivisible object allocation. The gross substitutability and complementarity condition, akin to same-side substitutability and cross-side complementarity, requires that objects can be divided into two groups such that objects in the same group are substitutes and objects in different groups are complements. Sun and Yang (2009) showed that like gross substitutability, the gross substitutability and complementarity condition is equivalent to submodularity of the indirect

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[4] By supply chain structure, we mean that there is a (partial) ordering of agents such that no agent sells to an agent above him in the ordering or buys from an agent below him in the ordering.

[5] This is a correction of a result of Hatfield and Kominers (2012).

utility function.

Subsequent to our work, Baldwin and Klemperer (2016) obtained additional insights on the underlying mathematical structure of fully substitutable preferences using the techniques of tropical geometry. Baldwin and Klemperer (2016) study the set of price vectors for which the demand correspondence is multi-valued, and associate them with convex-geometric objects called tropical hypersurfaces. Then, using the normal vectors that determine agents' tropical hypersurfaces, they distinguish among preferences that are strongly substitutable, are gross substitutable, or have complementarities.[6]

The discrete mathematics literature has explored several other concepts that are equivalent to substitutability in certain settings. We provide one point of connection to that literature in Section 6.5, where we establish the equivalence of full substitutability and $M^\natural$-concavity in our setting. In a recent working paper, Candogan et al. (2016) use this equivalence result to recast the problem of finding a competitive equilibrium in a network economy as a discrete concave optimization problem, which in turn allows them to construct computationally efficient algorithms for finding an equilibrium. Paes Leme (2014) provides a detailed survey that covers the discrete-mathematical substitutability concepts and their algorithmic properties.[7]

1.2 Structure of the Paper

The rest of the paper is organized as follows. In Section 2, we present our framework. In Section 3, we present three definitions of full substitutability, and show that they are all equivalent. In Section 4, we discuss what kinds of preference features are and are not allowed under full substitutability and present several economically important classes of fully substitutable preferences. In Section 5, we discuss transformations that preserve full substitutability. In Section 6, we provide several alternative characterizations of full

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[6] Our full substitutability concept corresponds to the "strong substitutes demand type" in Baldwin and Klemperer (2016).

[7] Unlike in our paper, the setting of Paes Leme (2014) assumes that all bundles of goods are feasible for the agent. Consequently, not all of the algorithmic results discussed by Paes Leme (2014) can be applied directly in our setting.

substitutability. In Section 7, we show that full substitutability implies a monotonicity condition that implies the Laws of Aggregate Supply and Demand. Section 8 concludes the main body of the paper.

In Appendix A, we present six additional definitions of full substitutability, which deal explicitly with indifferences in preferences. We discuss the connections of these definitions to those in the earlier literature and to the three definitions in Section 3.

All proofs are presented in the Online Appendix. In Online Appendix B, we prove that all six definitions of full substitutability in Appendix A are equivalent, and are also equivalent to the three main definitions in Section 3. Online Appendix C contains the proofs of the other results in the paper.

2 Model

The results we present consider the preferences of an individual agent, and thus do not depend on the environment in which that agent is located. However, for notational convenience and for continuity with the related literature, we present these results in the trading network setting of Hatfield et al. (2013).[8]

There is an economy with a finite set I of *agents* and a finite set Ω of *trades.* Each trade $\omega \in \Omega$ is associated with a *buyer* $b(\omega) \in I$ and a *seller* $s(\omega) \in I$, with $b(\omega) \neq s(\omega)$. We allow Ω to contain multiple trades associated to the same pair of agents, and allow for the possibility of trades $\omega \in \Omega$ and $\psi \in \Omega$ such that the seller of ω is the buyer of ψ, i.e., $s(\omega) = b(\psi)$, and the seller of ψ is the buyer of ω, i.e., $s(\psi) = b(\omega)$.

A *contract* x is a pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$ that specifies a trade and an associated price. For a contract $x = (\omega, p_\omega)$, we denote by $b(x) \equiv b(\omega)$ and $s(x) \equiv s(\omega)$ the buyer and the seller associated with the trade ω of x. The set of possible contracts is $X \equiv \Omega \times \mathbb{R}$. A set of

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[8] In particular, presenting the results in the framework of Hatfield et al. (2013) allows us to apply the results of Hatfield et al. (2013) in the proof of Theorem 4 in Section 5.2 (on "mergers" of agents with fully substitutable preferences). In turn, Theorem 4 allows us to prove the full substitutability of preferences in the "intermediary with production capacity" preference class that we introduce in Section 4.2.

contracts $Y \subseteq X$ is *feasible* if it does not contain two or more contracts for the same trade: formally, Y is feasible if $(\omega, p_\omega), (\omega, \hat{p}_\omega) \in Y$ implies that $p_\omega = \hat{p}_\omega$. We call a feasible set of contracts an *outcome*. An outcome specifies a set of trades along with associated prices, but does not specify prices for trades that are not in that set. An *arrangement* is a pair $[\Psi; p]$, with $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^\Omega$. Note that an arrangement specifies prices for *all* the trades in the economy. For any arrangement $[\Psi; p]$, we denote by $\kappa([\Psi; p]) \equiv \cup_{\psi \in \Psi} \{(\psi, p_\psi)\} \subseteq X$ the outcome induced by $[\Psi; p]$.

For a set of contracts $Y \subseteq X$ and agent $i \in I$, we let $Y_{i\to} \equiv \{y \in Y : i = s(y)\}$ denote the set of *upstream* contracts in Y for i, i.e., the subset of contracts in Y for which i is the seller. Similarly, let $Y_{\to i} \equiv \{y \in Y : i = b(y)\}$ denote the set of *downstream* contracts in Y for i, i.e., the subset of contracts in Y for which i is the buyer; we let $Y_i \equiv Y_{i\to} \cup Y_{\to i}$. We use analogous notation with regard to sets of trades $\Psi \subseteq \Omega$. For a set of contracts $Y \subseteq X$, we let $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\} \subseteq \Omega$ denote the set of trades associated with contracts in Y.

2.1 Preferences

Each agent i has a *valuation* (or *preferences*) $u_i : \wp(\Omega_i) \to \mathbb{R} \cup \{-\infty\}$ over the sets of trades in which he is involved, with $u_i(\varnothing) \in \mathbb{R}$.[9,10] Allowing the utility of agent i to equal $-\infty$ formalizes the idea that i, due to technological constraints, may only be able to produce or sell certain outputs contingent upon procuring appropriate inputs; e.g., if $\psi, \omega \in \Omega$ with $b(\psi) = s(\omega) = i$ and agent i cannot sell ω unless he has procured ψ, then $u_i(\{\omega\}) = -\infty$.[11] The assumption that $u_i(\varnothing)$ is finite for each $i \in I$ implies that no agent is obligated to engage in market transactions at highly unfavorable prices; he can always choose a (finite) outside

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[9] We assume that trades in $\Omega \smallsetminus \Omega_i$ do not affect i, and abuse notation slightly by writing $u_i(\Psi) \equiv u_i(\Psi_i)$ for $\Psi \subseteq \Omega$.

[10] Here, $\wp(\cdot)$ denotes the power set.

[11] In the classical exchange economy literature (Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999), the valuation of an agent i is defined over bundles of objects Ω as $u_i : 2^{\Omega_i} \to \mathbb{R}$, and is normalized such that $u_i(\varnothing) = 0$. While these assumptions are completely innocuous and natural in the context of exchange economies, they immediately rule out the kinds of technological constraints discussed above.

option.

The valuation u_i over bundles of trades gives rise to a quasilinear utility function U_i over bundles of trades and associated transfers. Specifically, for any feasible set of contracts $Y \subseteq X$, we define

$$U_i(Y) \equiv u_i(\tau(Y)) + \sum_{(\omega,p_\omega)\in Y_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in Y_{\to i}} p_\omega,$$

and, slightly abusing notation, for any arrangement $[\Psi; p]$, we define

$$U_i([\Psi; p]) \equiv u_i(\Psi) + \sum_{\psi\in\Psi_{i\to}} p_\psi - \sum_{\psi\in\Psi_{\to i}} p_\psi.$$

Note that by construction, $U_i([\Psi; p]) = U_i(\kappa([\Psi; p]))$.

The *choice correspondence* of agent i from the set of contracts $Y \subseteq X$ is defined by

$$C_i(Y) \equiv \underset{Z\subseteq Y_i;\ Z \text{ is feasible}}{\arg\max} \{U_i(Z)\}$$

and the *demand correspondence* of agent i, given a price vector $p \in \mathbb{R}^\Omega$, is defined by

$$D_i(p) \equiv \underset{\Psi\subseteq\Omega_i}{\arg\max}\{U_i([\Psi; p])\}.$$

Note that both choice and demand correspondences can be multi-valued. Also, the choice correspondence may be empty-valued (e.g., if Y is the set of all contracts with prices strictly between 0 and 1 for a particular trade, i.e., $Y = \{(\omega, p_\omega) : p_\omega \in (0, 1)\}$, and $u^i(\{\omega\}) = 1$), while the demand correspondence always contains at least one element. When the set Y is finite, the choice correspondence is also guaranteed to contain at least one element.

3 Substitutability Concepts

We now introduce three substitutability concepts that generalize the existing definitions from matching, auctions, and exchange economies with indivisible goods. In the matching literature, it is standard to formulate substitutability in terms of (single-valued) choice functions and to consider expansions of the set of available contracts on one side. In the literature on economies with indivisible goods, it is standard to formulate substitutability in terms of (multi-valued) demand functions and to consider disadvantageous price changes, i.e., increases in input prices or decreases in output prices. Finally, in auction theory it is standard to formulate substitutability in terms of demand functions and to consider an increase (or decrease) of the entire price vector. In this section, for the of ease the exposition and to allow for a more direct comparison of the different substitutability notions, we follow the approach of Ausubel and Milgrom (2002) and restrict attention to opportunity sets and vectors of prices for which choices and demands are single-valued. In Appendix A, we introduce additional definitions that explicitly deal with indifferences and multi-valued correspondences, and prove that those definitions are equivalent to each other and to the definitions given in this section.

3.1 Choice-Language Full Substitutability

First, we define full substitutability in the language of sets and choices, adapting and merging the Ostrovsky (2008) same-side substitutability and cross-side complementarity conditions. In choice language, we say that a choice correspondence C_i is fully substitutable if, when attention is restricted to sets of contracts for which C_i is single-valued, whenever the set of options available to i on one side expands, i rejects a larger set of contracts on that side (same-side substitutability) and selects a larger set of contracts on the other side (cross-side complementarity).

Definition 1. The preferences of agent i are *choice-language fully substitutable (CFS)* if:

1. for all sets of contracts $Y, Z \subseteq X$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;
2. for all sets of contracts $Y, Z \subseteq X$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{i\to} \smallsetminus Y^*_{i\to} \subseteq Z_{i\to} \smallsetminus Z^*_{i\to}$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.

3.2 Demand-Language Full Substitutability

Our second definition uses the language of prices and demands, adapting the gross substitutes and complements condition (GSC) of Sun and Yang (2006).[12] We say that a demand correspondence D_i is fully substitutable if, when attention is restricted to prices for which demands are single-valued, a decrease in the price of some inputs for agent i leads to a decrease in his demand for other inputs and to an increase in his supply of outputs, and an increase in the price of some outputs leads to a decrease in his supply of other outputs and an increase in his demand for inputs.

Definition 2. The preferences of agent i are *demand-language fully substitutable (DFS)* if:

1. for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \subseteq \Psi_{\rightarrow i}$ and $\Psi_{i\rightarrow} \subseteq \Psi'_{i\rightarrow}$;
2. for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{i\rightarrow} : p_\omega = p'_\omega\} \subseteq \Psi_{i\rightarrow}$ and $\Psi_{\rightarrow i} \subseteq \Psi'_{\rightarrow i}$.

3.3 Indicator-Language Full Substitutability

Our third definition is essentially a reformulation of Definition 2, using a convenient vector notation due to Hatfield and Kominers (2012). For each agent i, for any set of trades $\Psi \subseteq \Omega_i$, define the *(generalized) indicator function* $e_i(\Psi) \in \{-1, 0, 1\}^{\Omega_i}$ to be the vector with component $e_{i,\omega}(\Psi) = 1$ for each upstream trade $\omega \in \Psi_{\rightarrow i}$, $e_{i,\omega}(\Psi) = -1$ for each downstream trade $\omega \in \Psi_{i\rightarrow}$, and $e_{i,\omega}(\Psi) = 0$ for each trade $\omega \notin \Psi$. The interpretation of $e_i(\Psi)$ is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in Ψ, and "buys" a strictly negative amount if he is the seller of such a trade.

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[12] The definition of full substitutability that corresponds directly to GSC is Definition A.4, demand-language contraction full substitutability DCFS. See Appendix A for a detailed discussion of the connection between GSC and DCFS.

Definition 3. The preferences of agent i are *indicator-language fully substitutable (IFS)* if for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

Definition 3 clarifies the reason for the term "full substitutability"—an agent is more willing to "demand" a trade (i.e., keep an object that he could potentially sell, or buy an object that he does not initially own) if prices of other trades increase.

3.4 Equivalence of the Definitions

The main result of this section is that the three definitions of full substitutability presented are all equivalent. Subsequently, we use the term *full substitutability* to refer to all our substitutability concepts.

Theorem 1. *Choice-language full substitutability (CFS), demand-language full substitutability (DFS), and indicator-language full substitutability (IFS) are all equivalent.*

In Appendix A, we introduce six additional definitions of full substitutability, which explicitly deal with indifferences in preferences and with expanding, contracting, or both expanding and contracting sets of available "options"; we discuss in detail how those definitions relate to various definitions of substitutability considered in the literatures on matching, auctions, and exchange economies. We then prove that the six definitions introduced in Appendix A, as well as CFS, DFS, and IFS, are all equivalent (Theorem A.1). Theorem 1 thus follows immediately from Theorem A.1.

4 Classes of Fully Substitutable Preferences

Full substitutability is a natural condition, but it does rule out certain classes of preferences. For instance, full substitutability rules out situations in which an agent would only be willing to sell multiple units of a good at a particular price, but not one unit (at that same price).

Similarly, situations in which one agent's multiple inputs are complements[13] are also precluded. More generally, both economies of scale and complementarities in production or consumption are ruled out. Full substitutability also places more subtle restrictions: in Section 4.2, we provide an example which shows that the full substitutability assumption can also rule out the case of preferences representing an agent who is capacity constrained and requires different types of inputs to produce different types of outputs.

At the same time, full substitutability also allows for many rich types of preferences. By construction, fully substitutable preferences include, as a special case, "one-sided" preferences that satisfy the gross substitutability condition of Kelso and Crawford (1982). Gross substitutability has been extensively studied in the literatures on matching, competitive equilibrium, and discrete concave optimization, and a variety of examples and classes of preferences satisfying the gross substitutability condition have been presented.[14] Preferences satisfying the gross subsitutability condition arise in a wide variety of applications: For instance, Klemperer (2010) makes use of grossly substitutable preferences in the design of the "Product-Mix" auction which has been and continues to be used by the Bank of England to allocate funds to banks via securitized loans. Similarly, preferences in electricity markets can often be expressed substituably via assignment messages (Milgrom, 2009).

Beyond one-sided preferences, it is easy to see that the full substitutability condition allows for environments with homogeneous goods in which agents have increasing marginal costs of production and diminishing marginal utilities of consumption. It also allows for richer classes of "two-sided" preferences that involve complementarities between the contracts an agent can form as a buyer and those that he can form as a seller. In Sections 4.1 and 4.2 below, we discuss two such classes of preferences.

We start with "intermediary" preferences, under which an intermediary is trying to maximize his profit from matching some of his inputs to some of the requests that he receives.

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[13] See, e.g., the discussion in Milgrom (2000) on the presence of complementarities in spectrum auctions.

[14] See, e.g., Kelso and Crawford (1982), Hatfield and Milgrom (2005), Milgrom (2009), Milgrom and Strulovici (2009), Ostrovsky and Paes Leme (2015), and Paes Leme (2014).

The full substitutability of this class of preferences has been established in the earlier literature (see Hatfield et al. (2013), as well as related earlier work by Shapley (1962) and Sun and Yang (2006)). However, the machinery developed in the current paper allows us to give a new and more direct proof of this result.

We then introduce a novel class of preferences, "intermediary with production capacity," under which an intermediary has access to some limited production capacity and needs this capacity to transform inputs into outputs. To prove that preferences in this class are fully substitutable, we use the properties of fully substitutable preferences established in the current paper. Specifically, the proof proceeds by representing an "intermediary with production capacity" as an outcome of a merger of several "elementary" agents for whom full substitutability is immediate, and then invoking Theorem 4 of Section 5.2 on the preservation of full substitutability under the "merger" operation.

4.1 Preferences of Intermediaries

We start with *intermediary* preferences, introduced by Hatfield et al. (2013) in the context of used car dealers, but applicable more generally.[15]

The preferences of an *intermediary* i are represented as follows: There are a number of heterogeneous inputs for i (e.g., used cars, raw diamonds, and temporary workers), formally represented as a set of upstream contracts $Y_{\to i}$. Each element $(\varphi, p_\varphi) \in Y_{\to i}$ specifies the characteristics of the particular input and the price at which this input is available to

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[15] A closely related class of preferences was introduced by Sun and Yang (2006, Section 4) in the context of two-sided markets in which agents on one side (firms) have preferences over agents and objects on the other side (workers and machines) that are determined by the productivity of each worker on each machine. Sun and Yang (2006) showed that such preferences satisfy the gross substitutes and complements (GSC) condition, with workers being substitutes for one another, with machines being substitutes for one another, and with workers and machines being complements. The (GSC) properties of Sun and Yang's preferences, in turn, are closely related to the complementarity and substitutability properties in the optimal assignment problem, established by Shapley (1962). An important substantive difference between our intermediary preferences and the classes of preferences discussed by Shapley (1962) and Sun and Yang (2006) is that we incorporate the possibility that some pairs of "inputs" and "outputs" are physically incompatible, so that they are never "matched" under any vector of prices. By contrast, in the settings of Shapley (1962) and Sun and Yang (2006), for any given worker and any given machine there exists a vector of prices such that that worker and machine will be matched.

intermediary i. The intermediary also has a set of requests (e.g., for used cars, for engagement rings, and for temp services), represented as a set of downstream contracts $Y_{i\to}$. Each element $(\psi, p_\psi) \in Y_{i\to}$ specifies the characteristics required by the contract's customer and the price that customer is willing to pay.

Some inputs φ and requests ψ are compatible with each other, while others are not.[16] For every compatible input–request pair (φ, ψ), there is also a cost $c_{\varphi,\psi}$ of preparing the input φ for resale to satisfy the compatible request ψ.[17,18] Intermediary i's objective is to match some of the inputs in $Y_{\to i}$ to some of the requests in $Y_{i\to}$ in a way that maximizes his profit, $\sum_{(\varphi,\psi)\in\mu}(p_\psi - p_\varphi - c_{\varphi,\psi})$, where μ denotes the set of compatible input–request pairs that the intermediary selects.

Formally, following Hatfield et al. (2013), define a *matching*, μ, as a set of pairs of trades (φ, ψ) such that φ is an element of $\Omega_{\to i}$ (i.e., an input available to intermediary i), ψ is an element in $\Omega_{i\to}$ (i.e., a request received by i), φ and ψ are compatible, and each trade in Ω_i belongs to at most one pair in μ. Slightly abusing notation, let the cost of matching μ, $c(\mu)$, be equal to the sum of the costs of pairs involved in μ (i.e., $c(\mu) = \sum_{(\varphi,\psi)\in\mu} c_{\varphi,\psi}$).

For a set of trades $\Xi \subseteq \Omega_i$, let $\mathcal{M}(\Xi)$ denote the set of matchings μ of elements of Ξ such that every element of Ξ belongs to exactly one pair in μ.[19] Then the valuation of intermediary i over sets of trades $\Xi \subseteq \Omega_i$ is given by:

$$u_i(\Xi) = \begin{cases} -\min_{\mu\in\mathcal{M}(\Xi)}\{c(\mu)\} & \text{if } \mathcal{M}(\Xi) \neq \varnothing \\ -\infty & \text{if } \mathcal{M}(\Xi) = \varnothing, \end{cases}$$

i.e., $u_i(\Xi)$ is equal to the cost of the cheapest way of matching all requests and inputs in Ξ if

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[16] For instance, a given raw diamond can only be turned into polished diamonds of certain grades, and thus can only be used for some engagement rings but not others. A particular temp worker is only qualified to perform certain types of jobs.

[17] For example, the cost of repairing a car, turning a diamond into an engagement ring, or training a worker to perform a specific set of tasks.

[18] Note that we could formally allow all pairs of inputs and requests to be compatible, and encode incompatibilities by saying that for some pairs (φ, ψ), the cost $c_{\varphi,\psi}$ is infinite.

[19] Of course, $\mathcal{M}(\Xi)$ can be empty; e.g., it is empty if the number of inputs in Ξ is not equal to the number of requests, or if there are some requests in Ξ that are not compatible with any input in Ξ.

such a matching is possible, and is equal to $-\infty$ otherwise.[20] (Note that $u_i(\varnothing) = 0$.) The utility function of i over feasible sets of contracts is induced by valuation u_i in the standard way formalized in Section 2.1.

Proposition 1. *Intermediary preferences are fully substitutable.*

Hatfield et al. (2013) present a rather involved proof of Proposition 1. Sun and Yang (2006) also present an elaborate proof of an analogous result for the two-sided setting (Theorem 4.1 in their paper, with the proof on pages 1397–1401). The results of the current paper allow us to construct a much simpler and shorter proof, presented in Online Appendix C. Proposition 1 follows as a special case of Proposition 2, which shows the full substitutability of the new class of preferences that we introduce in the next section, the class of *intermediaries with production capacity* preferences. Proposition 2, in turn, follows directly from our result on mergers of agents with fully substitutable preferences (Theorem 4 of Section 5.2).

4.2 Preferences of Intermediaries with Production Capacity

For the intermediary preferences considered in Section 4.1, the intermediary either does not need to use any of his own resources to facilitate the matches between inputs and requests, or when he does, those resources could be expressed in monetary terms: there was a cost $c_{\varphi,\psi}$ of "preparing" input φ for request ψ. In some settings, however, we may want to consider intermediaries who need to rely on specific physical resources that they have in order to turn inputs into outputs, and it is more appropriate to think of these resources as fixed. For example, a manufacturer may have a fixed set of machines, and needs to assign a set of workers to those machines and at the same time needs to decide which outputs to produce on the machines. An agricultural firm may have a fixed set of land lots, and needs to hire workers to work on these lots, and at the same time needs to decide which outputs to produce.

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[20] Under this valuation function, any set chosen by intermediary i will contain an equal number of offers and requests. In principle, we could consider a more general (yet still fully substitutable) valuation function in which an intermediary has utility for an input that he does not resell. In that case, the intermediary may end up choosing more offers than requests.

A steel manufacturer has access to a variety of inputs (different sources of iron ore and scrap metal) and can produce a variety of outputs (different grades and types of steel products), and needs to assign these inputs and outputs to the fixed number of steel plants that it has.

In general, the preferences of an agent with capacity constraints may not be fully substitutable. For example, consider a firm that has exactly one machine, can hire workers Ann and Bob, and has requests for outputs α and β. Suppose Ann can use the machine to produce output α (but not β), while Bob can use the machine to produce output β (but not α). In this case, the preferences of the firm are not fully substitutable: reducing a price of an input (say, Ann) may lead to the firm choosing to drop an output (β), violating Part 1 of demand language full substitutability, Definition 2. In this section, however, we identify a rich class of preferences that are fully substitutable despite the presence of capacity constraints.

Specifically, consider an intermediary i who has access to a number of inputs, formally represented as a set of upstream contracts $Y_{\to i}$. Each element $(\varphi, p_\varphi) \in Y_{\to i}$ specifies the characteristics of the particular input and the price at which this input is available to intermediary i. The intermediary also has a set of requests, represented as a set of downstream contracts $Y_{i\to}$. Each element $(\psi, p_\psi) \in Y_{i\to}$ specifies the characteristics required by the contract's customer and the price that customer is willing to pay. Finally, the intermediary has a set M of *machines*; each machine $m \in M$ can be used to prepare one input for one output.

For each input φ and machine m, there is a cost $c_{\varphi,m} \in \mathbb{R} \cup \{+\infty\}$ of preparing the input to work with the machine (e.g., the cost of training a particular worker, or the cost of transporting iron ore from its source). For each machine m and each request ψ, there is a cost $c_{m,\psi} \in \mathbb{R} \cup \{+\infty\}$ of using this machine to produce the requested output (e.g., the cost of water required to produce a particular agricultural crop on a particular land lot, or the cost of transporting a batch of steel to its destination). Note that we allow both costs to take the value $+\infty$, to enable the possibility that a particular input is not compatible with a particular machine, or a particular machine is not compatible with a particular request. The total cost of preparing input φ for request ψ using machine m is thus $c_{\varphi,m} + c_{m,\psi}$. The objective of intermediary i is to match some of the inputs in $Y_{\to i}$ to some of the requests in $Y_{i\to}$, via some of the machines, in a way that maximizes his profit, $\sum_{(\varphi,m,\psi)\in\mu}(p_\psi - p_\varphi - c_{\varphi,m} - c_{m,\psi})$, where μ denotes the set of input–machine–request triples that the intermediary selects.

Formally, define a *matching*, μ, as a set of triples (φ, m, ψ) such that

1. φ is an element of $\Omega_{\to i}$,
2. m is a machine available to intermediary i,
3. ψ is an element of $\Omega_{i\to}$, and
4. each φ belongs to at most one triple in μ, each m belongs to at most one triple in μ, and each ψ belongs to at most one triple in μ.

Slightly abusing notation, let the cost of matching μ, $c(\mu)$, be equal to the sum of the costs of triples involved in μ, i.e., $c(\mu) = \sum_{(\varphi,m,\psi)\in\mu}(c_{\varphi,m} + c_{m,\psi})$.

For a set of trades $\Xi \subseteq \Omega_i$, let $\mathcal{M}(\Xi)$ denote the set of matchings μ of elements of Ξ and machines available to the intermediary, such that every element of Ξ belongs to exactly one triple in μ. Then the valuation of intermediary i over sets of trades $\Xi \subseteq \Omega_i$ is given by:

$$u_i(\Xi) = \begin{cases} -\min_{\mu\in\mathcal{M}(\Xi)}\{c(\mu)\} & \text{if } \mathcal{M}(\Xi) \neq \varnothing \\ -\infty & \text{if } \mathcal{M}(\Xi) = \varnothing, \end{cases}$$

i.e., $u_i(\Xi)$ is equal to the cost of the cheapest way of satisfying all requests in Ξ using all of the inputs in Ξ and some of the machines, if such a production plan is possible; and is equal to $-\infty$ otherwise. The utility function of intermediary i over feasible sets of contracts is induced by valuation u_i in the usual way.

Proposition 2. *Intermediary with production capacity preferences are fully substitutable.*

The intuition behind the proof of Proposition 2 is as follows: First, if an intermediary i has only one machine, then his preferences are fully substitutable. Next, if the intermediary has multiple machines (say, a set M of machines), he can be, in essence, viewed as a "merger" of $|M|$ single-machine agents. However, we can not "merge" the $|M|$ single-machine agents directly, as we must account for the constraints that a given input can be used by at most one machine (and, similarly, the constraints that a given request can be satisfied by at most one machine). To address these issues, we introduce a novel proof strategy: We ensure that each input and each output is only used at most once by the merged firm by adding a layer of "input dummy" firms and a layer of "request dummy" firms. Each input dummy firm enforces the constraint that the input corresponding to that dummy firm is used by at most one machine within the merged firm. Similarly, each request dummy firm enforces the constraint that the request corresponding to that dummy firm is fulfilled by at most one machine. The merger operation then combines these dummy firms with the single-machine firms. By Theorem 4 of Section 5.2, the preferences of this merged firm are fully substitutable, and it is clear that the preferences merged firm reflect the valuation u^i.

Note that while we use the "dummy firm layers and mergers" construction for the specific purpose of proving the full substitutability of intermediary with production capacity preferences, this technique may be useful more generally for incorporating various restrictions (say, incompatibility of some input trades) in agents' preferences while maintaining full substitutability, both in trading network and two-sided settings.

5 Transformations

In this section, we show that fully substitutable preferences can be transformed and combined in several economically interesting ways that preserve full substitutability. We first consider the possibility that an agent is endowed with the right to execute any trades in a given set and the possibility that an agent has an obligation to execute all trades in a given set. We also examine mergers, where the valuation function of the merged entity is constructed as the convolution of the valuation functions of the merging parties.[21] Finally, we consider a form of limited liability, where an agent may back out of some agreed-upon trades in exchange for paying an exogenously-fixed penalty.

5.1 Trade Endowments and Obligations

Suppose an agent i is endowed with the right (but not the obligation) to execute trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ. Let

$$\hat{u}_i^{(\Phi,p_\Phi)}(\Psi) \equiv \max_{\Xi \subseteq \Phi} \left\{ u_i(\Psi \cup \Xi) + \sum_{\xi \in \Xi_{i\to}} p_\xi - \sum_{\xi \in \Xi_{\to i}} p_\xi \right\}$$

be a valuation over trades in $\Omega \smallsetminus \Phi$; $\hat{u}_i^{(\Phi,p_\Phi)}$ represents agent i having a valuation over trades in $\Omega \smallsetminus \Phi$ consistent with u_i while being endowed with the option of executing any trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ.

Theorem 2. *If the preferences of agent i are fully substitutable, then the preferences induced by the valuation function $\hat{u}_i^{(\Phi,p_\Phi)}$ are fully substitutable for any $\Phi \subseteq \Omega_i$ and $p_\Phi \in \mathbb{R}^\Phi$.*

Intuitively, when we endow agent i with access to the trades in Φ at prices p_Φ, we are effectively restricting (1) the set of prices that may change and (2) the set of trades that are required to be substitutes in the demand-theoretic definition of full substitutability (Definition 2). Naturally, this process cannot *create* complementarities among trades in $\Omega \smallsetminus \Phi$, given that under u_i these trades already are substitutes for each other *and* for the trades in Φ. Hence, $\hat{u}_i^{(\Phi,p_\Phi)}$ induces fully substitutable preferences over trades in $\Omega \smallsetminus \Phi$.

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[21] In two-sided matching settings, the operations of *endowment* and *merger* were used by Hatfield and Milgrom (2005) to construct the class of *endowed assignment valuations*, starting with singleton preferences and iteratively applying these operations. Hatfield and Milgrom (2005) showed that the endowment and merger operations preserve substitutability (Theorems 13 and 14 of Hatfield and Milgrom (2005)) in their context, and thus showed that all endowed assignment valuation preferences are substitutable. Ostrovsky and Paes Leme (2015) showed that there exist substitutable preferences that cannot be represented as using endowed assignment valuations, and introduced the class of *matroid-based valuations*, which is obtained by iteratively applying the endowment and merger operations to weighted-matroid valuations. As every weighted-matroid valuation is substitutable (Murota, 1996; Murota and Shioura, 1999; Fujishige and Yang, 2003), every matroid-based valuation is also substitutable. It is an open question whether every substitutable valuation is a matroid-based valuation.

Apart from endowments, agents may have obligations, that is, an agent i may be obliged to execute trades in some set $\Phi \subseteq \Omega_i$ at *fixed* prices p_Φ. We now show that if an agent's preferences are initially fully substitutable, they remain fully substitutable when an obligation arises to execute some trades at pre-specified prices. Suppose agent i is obliged to execute trades in $\Phi \subseteq \Omega_i$ at prices p_Φ and that Φ is technologically feasible in the sense that $u_i(\Phi) \neq -\infty$. Let

$$\tilde{u}_i^{(\Phi,p_\Phi)}(\Psi) \equiv u_i(\Psi \cup \Phi) + \sum_{\varphi \in \Phi_{i\rightarrow}} p_\varphi - \sum_{\varphi \in \Phi_{\rightarrow i}} p_\varphi$$

be a valuation over trades in $\Omega \smallsetminus \Phi$; $\tilde{u}_i^{(\Phi,p_\Phi)}$ represents agent i having a valuation over trades in $\Omega \smallsetminus \Phi$ consistent with u_i while being obliged to execute all trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ.

Theorem 3. *If the preferences of agent i are fully substitutable, then the preferences induced by the valuation function $\tilde{u}_i^{(\Phi,p_\Phi)}$ are fully substitutable for any $\Phi \subseteq \Omega_i$ and $p_\Phi \in \mathbb{R}^\Phi$ such that $u_i(\Phi) \neq -\infty$.*

The idea of the proof is to note that the demand correspondence of agent i with valuation $\tilde{u}_i^{(\Phi,p_\Phi)}$ does not depend on prices p_Φ—changing these prices simply leads to a shift in the agent's utility function by a fixed amount. Thus, we can assume that the trades that the agent is obliged to *buy* have negative and very large (in absolute magnitude) prices, while the trades that the agent is obliged to *sell* have positive and very large prices. Under those assumptions, "obligations" become "endowments" (because the agent would voluntarily want to execute all of these trades), and thus Theorem 3 follows from Theorem 2.

Combining Theorems 2 and 3, we see that if the preferences of agent i are fully substitutable, then they remain fully substitutable when i is endowed with some trades and obliged to execute others (assuming that the obligation is technologically feasible).

5.2 Mergers

The second transformation we consider is the case when several agents merge. Given a set of agents J, we denote the set of trades that involve only agents in J as $\Omega^J \equiv \{\omega \in \Omega : \{b(\omega), s(\omega)\} \subseteq J\}$. We let the *convolution* of the valuation functions $\{u_j\}_{j \in J}$ be defined as

$$u_J(\Psi) \equiv \max_{\Phi \subseteq \Omega^J} \left\{ \sum_{j \in J} u_j(\Psi \cup \Phi) \right\} \tag{1}$$

for sets of trades $\Psi \subseteq \Omega \smallsetminus \Omega^J$. The convolution u_J represents a "merger" of the agents in J, as it treats the agents in J as able to execute any within-J trades costlessly.

Theorem 4. *For any set of agents $J \subseteq I$, if the preferences of each $j \in J$ are fully substitutable, then the preferences induced by the convolution u_J (defined in* (1)*) are fully substitutable.*

While Theorem 4 is of independent interest, note that we also use it in the proof of Proposition 2, where we show the full substitutability of "intermediary with production capacity" preferences.

Note that substitutability is not preserved following *dissolution/de-mergers*. For example, if agents i and j only trade with each other (i.e., $\Omega_i = \Omega_j$), then the preferences induced by the convolution valuation $u_{\{i,j\}}$ are trivially fully substitutable, even if the preferences of i and j are not.

Note also that while merging agents preserves substitutability, the same cannot be said about merging trades between two agents. For example, consider a simple economy with agents i and j and four trades: set Ω consists of trades χ, φ, ψ, and ω. Agent i is the buyer in all of these trades, and agent j is the seller. The valuation of agent i is as follows:

$$u_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \\ 1 & |\Psi_i| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The preferences of i are clearly fully substitutable. But now consider merging the trades χ and φ into a single trade ξ. The resulting valuation function of i over the subsets of $\tilde{\Omega} \equiv (\Omega \smallsetminus \{\chi, \varphi\}) \cup \{\xi\}$ is given by

$$\tilde{u}_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \text{ or } \xi \in \Psi \\ 1 & |\Psi_i| = 1 \text{ and } \xi \notin \Psi \\ 0 & \text{otherwise.} \end{cases}$$

Valuation function $\tilde{u}_i$ is not fully substitutable. To see this, note that for price vector $p = (p_\xi, p_\psi, p_\omega) = (1.7, 0.8, 0.8)$, the unique optimal demand of agent i is $\{\psi, \omega\}$, but for price vector $p' = (p'_\xi, p'_\psi, p'_\omega) = (1.7, 1, 0.8)$, the unique optimal demand of agent i is $\{\xi\}$. That is, under price vector p', agent i no longer demands the trade ω, even though its price remains unchanged while the price of ψ increases and the price of ξ remains unchanged.

5.3 Limited Liability

The final transformation we consider is "limited liability." Specifically, suppose that after agreeing to a trade, an agent is allowed to renege on that trade in exchange for paying a fixed penalty. We show that this transformation preserves substitutability. In addition to being economically interesting, the preservation of substitutability under limited liability is also useful technically; indeed, it enables us to transform unbounded utility functions into bounded ones while preserving substitutability. (The fact that this transformation preserves substitutability simplifies analysis in a number of settings; see, e.g., the proof of Theorem 1 in Hatfield et al. (2013).)

Formally, consider a fully substitutable valuation function u_i for agent i. Take an arbitrary set of trades $\Phi \subseteq \Omega_i$, and for every trade $\varphi \in \Phi$, pick $\Pi_\varphi \in \mathbb{R}$—the penalty for reneging on trade φ. (For mathematical completeness, we allow Π_φ to be negative.) Define the modified valuation function $\hat{u}_i$ as

$$\hat{u}_i(\Psi) \equiv \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}. \tag{2}$$

That is, under valuation $\hat{u}_i$, agent i can "buy out" some of the trades to which he has committed (provided these trades are in the set Φ of trades the agent may renege on), and pay the corresponding penalty for each trade he buys out.

Theorem 5. *For any $\Phi \subseteq \Omega_i$ and $\Pi_\Phi \in \mathbb{R}^\Phi$, if agent i has fully substitutable preferences, then the valuation function $\hat{u}_i$ with limited liability (as defined in (2)) induces fully substitutable preferences.*

A common assumption in the earlier literature on two-sided matching and exchange economies (e.g., Kelso and Crawford (1982) and Gul and Stacchetti (1999)) is that buyers' valuation functions are *monotonic*.[22] Intuitively, monotonicity corresponds to the special case of our setting in which an agent has free disposal, in the sense that he can renege on any trade at no cost. More formally, if u_i is fully substitutable, then Theorem 5 implies that we can obtain a fully substitutable and monotonic valuation function $\hat{u}_i$ by allowing the agent to renege on any trade in Ω_i at a per-trade cost of $\Pi_\varphi = 0$, for all $\varphi \in \Omega_i$.

6 Properties Equivalent to Full Substitutability

In this section, we discuss several interesting properties of valuation functions that turn out to be equivalent to full substitutability. While these results are of independent interest, some of them are also useful in applications. For example: The submodularity equivalence we prove in Section 6.1 is used in our proof that substitutability is preserved under trade endowments (Theorem 2). The object-language formulation of full substitutability we develop in Section 6.3 is used in showing that substitutability implies monotone–substitutability, which requires that full substitutability and the Laws of Aggregate Supply and Demand hold

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[22] Monotonicity of the valuation function u_i requires that, for all Ξ and Ψ such that $\Xi \subseteq \Psi \subseteq \Omega_i$, $u_i(\Psi) \geq u_i(\Xi)$.

jointly (Theorem 10); a closely related transformation is used in the proof of the main result of Hatfield et al. (2013). The single-improvement property, which we introduce in Section 6.2, and $M^\natural$-concavity, discussed in Section 6.5, are useful for efficiently computing the choice function of an agent with fully substitutable preferences, because they imply that local search for an optimal bundle eventually reaches a global optimum (Paes Leme, 2014).

6.1 Submodularity of the Indirect Utility Function

A classical approach (see, e.g., the work of Gul and Stacchetti (1999) and Ausubel and Milgrom (2002)) relates substitutability of the utility function to submodularity of the indirect utility function. In particular, every (grossly) substitutable utility function corresponds to a submodular indirect utility function and vice versa.[23]

For price vectors $p, \bar{p} \in \mathbb{R}^\Omega$, let the *join* of p and $\bar{p}$, denoted $p \vee \bar{p}$, be the pointwise maximum of p and $\bar{p}$; let the *meet* of p and $\bar{p}$, denoted $p \wedge \bar{p}$, be the pointwise minimum of p and $\bar{p}$.

Definition 4. The *indirect utility function* of agent i,

$$V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} \{U_i([\Psi; p])\},$$

is *submodular* if, for all price vectors $p, \bar{p} \in \mathbb{R}^\Omega$, we have that

$$V_i(p \wedge \bar{p}) + V_i(p \vee \bar{p}) \leq V_i(p) + V_i(\bar{p}).$$

Theorem 6. *The preferences of an agent are fully substitutable if and only if they induce a submodular indirect utility function.*

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[23] Similar correspondences hold in markets without transferable utility: In many-to-many matching with contracts markets without transfers, every substitutable choice function can be represented by a submodular indirect utility function, and every submodular indirect utility function corresponds to a substitutable choice function (Hatfield and Kominers, 2013). In trading networks without transferable utility, every indirect utility function representing a fully substitutable choice function is weakly quasisubmodular (Hatfield et al., 2017).

6.2 The Single Improvement Property

Gul and Stacchetti (1999) first observed (in the setting of exchange economies) that substitutability is equivalent to the *single improvement property*—an agent's preferences are substitutable if and only if, when an agent does not have an optimal bundle, that agent can make himself better off by adding a single item, dropping a single item, or doing both. Sun and Yang (2009) extended this result to their setting. Baldwin and Klemperer (2016) showed that in their setting the single improvement property is equivalent to requiring that agents have *complete* preferences.

Definition 5. The preferences of agent i have the *single improvement property* if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that

1. $U_i([\Psi, p]) < U_i([\Phi, p])$,
2. there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi)$, and
3. there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi)$.[24]

The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by

1. obtaining one item not currently held (either by making a new purchase, i.e., adding a trade in $\Omega_{\rightarrow i} \smallsetminus \Psi$, or by canceling a sale, i.e., removing a trade in $\Psi_{i\rightarrow}$),
2. relinquishing one item currently held (either by canceling a purchase, i.e., removing a trade in $\Psi_{\rightarrow i}$, or by making a new sale, i.e., adding a trade in $\Omega_{i\rightarrow} \smallsetminus \Psi$), or
3. both obtaining one item not currently held and relinquishing one item currently held.

For instance, an agent may buy one more input and commit to provide one additional output as a "single improvement."

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[24] Recall that the definition of the generalized indicator function e_i is given in Section 3.3.

Moreover, when the preferences of agent i satisfy the single improvement property, it is easy to find an optimal bundle since, at any non-optimal bundle, a local adjustment can strictly increase the utility of i.

We now generalize the earlier results of Gul and Stacchetti (1999) and Sun and Yang (2009) to our setting.

Theorem 7. *The preferences of an agent are fully substitutable if and only if they have the single improvement property.*

6.3 Object-Language Substitutability

An alternative way of thinking about trades in our setting is to consider each trade as representing the transfer of an underlying object. Under this interpretation, an agent's preferences over trades are fully substitutable if and only if that agent's preferences over objects have the standard Kelso and Crawford (1982) property of gross substitutability. This interpretation allows us to rewrite indicator-language full substitutability to more naturally correspond to the intuitive explanation of the concept given in Section 3.

Formally, we consider each trade $\omega \in \Omega$ as transferring an underlying object from $s(\omega)$ to $b(\omega)$; we denote this underlying object as $\mathfrak{o}(\omega)$. We call the set of all underlying objects $\boldsymbol{\Omega}$. Hence, after executing the set of trades $\Psi \subseteq \Omega_i$, agent i is left with both the set of objects corresponding to the trades in Ψ where i is a buyer and the set of objects corresponding to trades in $\Omega_i \smallsetminus \Psi$ where i is a seller. We define the set of objects held by agent i after executing the set of trades Ψ as

$$\mathfrak{o}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \Omega_{i\to} \smallsetminus \Psi_{i\to}\}.$$

Conversely, we define the trade associated with an object $\boldsymbol{\omega}$ as $\mathfrak{t}(\boldsymbol{\omega})$; note that $\mathfrak{t}(\mathfrak{o}(\omega)) = \omega$. We also define the set of trades executed by i for a given set of objects $\boldsymbol{\Psi} \subseteq \boldsymbol{\Omega}_i \equiv \{\boldsymbol{\omega} \in \boldsymbol{\Omega} :$

$i \in \{b(\mathfrak{t}(\boldsymbol{\omega})), s(\mathfrak{t}(\boldsymbol{\omega}))\}$ as

$$\mathfrak{t}_i(\boldsymbol{\Psi}) = \{\omega \in \Omega_{\to i} : \mathfrak{o}(\omega) \in \boldsymbol{\Psi}\} \cup \{\omega \in \Omega_{i\to} : \mathfrak{o}(\omega) \in \boldsymbol{\Omega}_i \smallsetminus \boldsymbol{\Psi}\}.$$

Hence, for a partition of objects $\{\boldsymbol{\Psi}^i\}_{i\in I}$, the set of trades that implements this partition is given by

$$\bigcup_{i\in I} \mathfrak{t}_i(\boldsymbol{\Psi}^i).$$

For a set of objects $\boldsymbol{\Psi}$, we let

$$u_i(\boldsymbol{\Psi}) \equiv u_i(\mathfrak{t}_i(\boldsymbol{\Psi})) = u_i([\mathfrak{t}_i(\boldsymbol{\Psi})]_{\to i} \cup [\Omega_i \smallsetminus \mathfrak{t}_i(\boldsymbol{\Psi})]_{i\to}).$$

Using object language, we can also reformulate indicator-language full substitutability (Definition 3) to *object-language full substitutability.*

Definition 6. The preferences of agent i are *object-language fully substitutable (OFS)* if for all price vectors $p, p' \in \mathbb{R}^{\Omega}$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \le p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, if $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi)$, then $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi')$ for each $\boldsymbol{\omega} \in \boldsymbol{\Omega}_i$ such that $p_{\mathfrak{t}(\boldsymbol{\omega})} = p'_{\mathfrak{t}(\boldsymbol{\omega})}$.

Under object-language full substitutability, an increase in the price of object $\boldsymbol{\psi}$ cannot decrease the agent's demand for any object $\boldsymbol{\omega} \neq \boldsymbol{\psi}$. That is, the agent's preferences over objects are grossly substitutable, in the sense of Kelso and Crawford (1982).

We can now interpret the indicator vector $e_{i,\psi}(\Psi)$ as encoding whether the object $\boldsymbol{\psi} = \mathfrak{o}(\psi)$ is transferred under Ψ:

- If $\psi \in \Psi_{\to i}$, then $\boldsymbol{\psi} \in \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = 1$, i.e., i obtains the object associated with ψ.
- If $\psi \in \Psi_{i\to}$, then $\boldsymbol{\psi} \notin \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = -1$, i.e., i gives up the object associated with ψ.
- Finally, if $\psi \notin \Psi$, then $e_{i,\psi}(\Psi) = 0$, i.e., i neither obtains nor gives up the object associated with ψ.

Additionally, object-language full substitutability helps us define a "no complementarities condition," equivalent to full substitutability, in the next section. It is also useful in our proof that fully substitutable preferences satisfy the Laws of Aggregate Supply and Demand (under quasilinear utility).

We can reformulate the definition of the single improvement property in terms of objects.

Definition 7. The preferences of agent i have the *single improvement property* if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that

1. $U^i([\Psi, p]) < U^i([\Phi, p])$,
2. there exists at most one object $\boldsymbol{\omega} \in \mathfrak{o}_i(\Phi) \smallsetminus \mathfrak{o}_i(\Psi)$, and
3. there exists at most one object $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi) \smallsetminus \mathfrak{o}_i(\Phi)$.

Using object language, we obtain a definition of the single improvement property that exactly matches the intuition provided on page 26. The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by

1. obtaining one object $\boldsymbol{\omega}$ not currently held, i.e., $\boldsymbol{\omega} \notin \mathfrak{o}_i(\Psi)$,
2. relinquishing one object $\boldsymbol{\omega}$ currently held, i.e., $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi)$, or
3. both obtaining one object and relinquishing one object.

When substitutability is expressed in terms of preferences over trades, it is necessary to treat relationships between "same-side" and "cross-side" contracts differently. Both Sun and Yang (2006) and Ostrovsky (2008) introduced a concept of cross-side complementarity, which requires that agents treat buy-side contracts as complementary with sell-side contracts (as in our Definitions 1 and 2), which might suggest that there is something fundamentally different between how contracts on one side are interdependent with each other versus how contracts on different sides are interdependent. The representation of preferences in the language of object-language substitutability uncovers that cross-side complementarity is not really a complementarity condition *per se*: rather, it corresponds to an underlying substitutability condition over objects—the same one as in the case of same-side substitutability.

The formalization of substitutability in terms of preferences over objects (Definition 6) thus provides a very simple and compact interpretation of full substitutability that does not require treating two sides differently: it simply says that when an agent's object opportunity set shrinks, the agent does not reduce demand for any object that remains in his opportunity set. In particular, in settings with transferable utility, when prices increase, an agent's object opportunity set shrinks; hence, substitutability requires that the agent (weakly) increase his demand for objects whose prices do not rise.

6.4 The No Complementarities Condition

Gul and Stacchetti (1999) proved that substitutability is equivalent to a *no complementarities condition*; we extend this observation here.

Definition 8. The preferences of agent i satisfy the *no complementarities condition* if, for every price vector p, for any $\Phi, \Psi \in D_i(p)$, and for any $\bar{\boldsymbol{\Psi}} \subseteq \mathfrak{o}_i(\Psi)$, there exists $\bar{\boldsymbol{\Phi}} \subseteq \mathfrak{o}_i(\Phi)$ such that $\mathfrak{t}_i((\boldsymbol{\Psi} \smallsetminus \bar{\boldsymbol{\Psi}}) \cup \bar{\boldsymbol{\Phi}}) \in D_i(p)$.

The no complementarities condition requires that for any pair of optimal bundles of objects, $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, and for any $\bar{\boldsymbol{\Psi}} \subseteq \boldsymbol{\Psi}$, there exists a set of objects $\bar{\boldsymbol{\Phi}} \subseteq \boldsymbol{\Phi}$ that "perfectly substitute" for the objects in $\bar{\boldsymbol{\Psi}}$, in the sense that $(\boldsymbol{\Psi} \smallsetminus \bar{\boldsymbol{\Psi}}) \cup \bar{\boldsymbol{\Phi}}$ is optimal.

Theorem 8. *The preferences of an agent are fully substitutable if and only if they satisfy the no complementarities condition.*

The proof of Theorem 8 is an adaptation of the proof of Theorem 1 of Gul and Stacchetti (1999). Gul and Stacchetti (1999) assume that valuation functions are monotone and bounded from below; thus, in our proof of Theorem 8, we must be careful to ensure that non-monotonicities and unboundedness do not invalidate the Gul and Stacchetti (1999) proof strategy.

6.5 $M^\natural$-Concavity over Objects

Reijnierse et al. (2002) and Fujishige and Yang (2003) independently observed that gross substitutability in the Kelso and Crawford (1982) model is equivalent to a classical condition from discrete optimization theory, $M^\natural$-*concavity* (Murota, 2003). In our object-language notation, the condition can be stated as follows.

Definition 9. The valuation u_i is $M^\natural$-*concave over objects* if for all $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \boldsymbol{\Omega}_i$, for any $\boldsymbol{\psi} \in \boldsymbol{\Psi}$,

$$u_i(\boldsymbol{\Psi}) + u_i(\boldsymbol{\Phi}) \leq \max\Big\{u_i(\boldsymbol{\Psi} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\boldsymbol{\Phi} \cup \{\boldsymbol{\psi}\}), \\ \max_{\boldsymbol{\varphi} \in \boldsymbol{\Phi}}\{u_i(\boldsymbol{\Psi} \cup \{\boldsymbol{\varphi}\} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\boldsymbol{\Phi} \cup \{\boldsymbol{\psi}\} \smallsetminus \{\boldsymbol{\varphi}\})\}\Big\}.$$

A valuation function is $M^\natural$-concave if, for any sets of objects $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, the sum of $u_i(\boldsymbol{\Psi})$ and $u_i(\boldsymbol{\Phi})$ is weakly increased when either we move a given object $\boldsymbol{\psi}$ from $\boldsymbol{\Psi}$ to $\boldsymbol{\Phi}$ or we swap $\boldsymbol{\psi}$ for some other object $\boldsymbol{\varphi} \in \boldsymbol{\Phi}$.

Theorem 9. *The preferences of an agent are fully substitutable if and only if the associated valuation function is* $M^\natural$*-concave over objects.*

This equivalence result follows from Theorem 7 of Murota and Tamura (2003), which shows that $M^\natural$-concavity is equivalent to the single improvement property—and which in turn, by our Theorem 7, implies the equivalence between full substitutability and $M^\natural$-concavity.

The equivalence result in Theorem 9 has recently been used by Candogan et al. (2016) to recast the problem of characterizing a competitive equilibrium in a network economy in terms of an equivalent discrete concave optimization problem. The Candogan et al. (2016) representation, in turn, allows Candogan et al. to apply the techniques of discrete concave optimization to the analysis of network economies and in particular, allows them to construct computationally efficient algorithms for finding competitive equilibria and blocking chains.

7 Monotone–Substitutability and the Laws of Aggregate Supply and Demand

In two-sided matching markets with transfers and quasilinear utility, all fully substitutable preferences satisfy a monotonicity condition called the Law of Aggregate Demand (Hatfield and Milgrom, 2005).[25] The analogues of this condition for the current setting are the *Laws of Aggregate Supply and Demand* for trading networks, first introduced by Hatfield and Kominers (2012).

Definition 10. The preferences of agent i satisfy the *Law of Aggregate Demand* if for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$.

The preferences of agent i satisfy the *Law of Aggregate Supply* if for all finite sets of contracts Y and Z such that $Y_{i\to} \subseteq Z_{i\to}$ and $Y_{\to i} = Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{i\to}| - |Z^*_{\to i}| \geq |Y^*_{i\to}| - |Y^*_{\to i}|$.

Intuitively, the choice correspondence C_i satisfies the Law of Aggregate Demand if, whenever the set of options available to i as a buyer expands, the net demand (i.e., the difference between the number of buy-side contracts chosen and the number of sell-side contracts chosen) increases. Similarly, the choice correspondence C_i satisfies the Law of Aggregate Supply if, whenever the set of options available to i as a seller expands, the net supply (i.e., the difference between the number of sell-side contracts chosen and the number of buy-side contracts chosen) increases. The conditions stated in Definition 10 extend the Hatfield and Milgrom (2005) Law of Aggregate Demand (see also Alkan and Gale (2003)) to the current setting, in which each agent can be both a buyer in some trades and a seller in others.

One subtle technical issue arises because choice correspondences are not necessarily

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[25] In the context of two-sided matching with contracts, the Law of Aggregate Demand is essential for "rural hospitals" and strategy-proofness results (see Hatfield and Milgrom (2005) and Hatfield and Kominers (2013)).

single-valued in our setting. Under fully substitutable preferences, for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, if $Y^* \in C_i(Y)$ then there exists a $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$. Meanwhile, when the Law of Aggregate Demand is satisfied, for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, if $Y^* \in C_i(Y)$ then there exists a $Z^* \in C_i(Z)$ such that $|Z^*_{\to i}| - |Y^*_{\to i}| \geq |Z^*_{i\to}| - |Y^*_{i\to}|$. However, in principle, it may be the case that there is no Z^* that *simultaneously* satisfies the conditions for full substitutability and the Law of Aggregate Demand. Yet in some applications, it is important to have a single Z^* that simultaneously satisfies both conditions (see, e.g., Hatfield et al. (2015)). Thus we introduce the following stronger condition, called *monotone–substitutability*.

Definition 11. The preferences of agent i are *monotone–substitutable* if:

1. for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$, $Y^*_{i\to} \subseteq Z^*_{i\to}$, and $|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|$;
2. for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{\to i} = Z_{\to i}$ and $Y_{i\to} \subseteq Z_{i\to}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$, $Y^*_{\to i} \subseteq Z^*_{\to i}$, and $|Z^*_{i\to}| - |Z^*_{\to i}| \geq |Y^*_{i\to}| - |Y^*_{\to i}|$.

In our setting, full substitutability implies monotone–substitutability, which in turn implies the Laws of Aggregate Supply and Demand.

Theorem 10. *If the preferences of agent i are fully substitutable, then they are monotone–substitutable.*

Corollary 1. *If the preferences of agent i are fully substitutable, then they satisfy the Laws of Aggregate Supply and Demand.*

Corollary 1 generalizes Theorem 7 of Hatfield and Milgrom (2005), who showed the analogous result in the special case when agent i acts only as a buyer.

8 Conclusion

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, and exchange economies with indivisible goods. We extended earlier models' canonical definitions of substitutability to a setting in which an agent can be both a buyer in some transactions and a seller in others, and showed that all these definitions are equivalent. We introduced a new class of substitutable preferences that allows us to model intermediaries with production capacity. We proved that substitutability is preserved under economically important transformations such as trade endowments and obligations, mergers, and limited liability. We also showed that substitutability corresponds to submodularity of the indirect utility function, the single improvement property, gross substitutability under a suitable transformation (object-language full substitutability), a no complementarities condition, and $M^\natural$-concavity. Finally, we showed that substitutability implies monotone–substitutability, which in turn implies the Laws of Aggregate Supply and Demand. All of our results explicitly incorporate economically important features such as indifferences, non-monotonicities, and unbounded utility functions that were not fully addressed in prior work.

In the current paper, we focused on the full substitutability of the preferences of an *individual* agent. In related work, we have explored the properties of *economies* with multiple agents whose preferences are fully substitutable. That work shows that when all agents' preferences are fully substitutable, outcomes that are *stable* (in the sense of matching theory) exist for any underlying network structure (Hatfield et al., 2013, Theorems 1 and 5). Furthermore, full substitutability of preferences guarantees that the set of stable outcomes is essentially equivalent to the set of competitive equilibria with personalized prices (Hatfield et al., 2013, Theorems 5 and 6) and to the set of *chain stable* outcomes (Hatfield et al., 2015, Theorem 1 and Corollary 1), and that all stable outcomes are in the core and are efficient (Hatfield et al., 2013, Theorem 9). Full substitutability also delineates a maximal domain for the existence of equilibria (Hatfield et al., 2013, Theorem 7): for any domain of preferences strictly larger than that of full substitutability, the existence of stable outcomes and competitive equilibria cannot be guaranteed.

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A Definitions of Full Substitutability That Consider Multi-Valued Choices and Demands

In this Appendix, we introduce six alternative definitions of full substitutability, as follows:

- Definitions A.1 and A.2 are analogues of choice-language full substitutability (Definition 1);
- Definitions A.3 and A.4 are analogues of demand-language full substitutability (Definition 2); and
- Definitions A.5 and A.6 are analogues of indicator-language full substitutability (Definition 3).

Finally, in Appendix A.4, we show Definitions A.1–A.6 are all equivalent to each other and to CFS, DFS, and IFS.

In contrast to Definitions 1, 2 and 3, which consider single-valued choices and demands, Definitions A.1–A.6 explicitly consider multi-valued correspondences and deal directly with indifferences. By explicitly accounting for indifferences and multi-valued correspondences, we directly generalize the original gross substitutability condition of Kelso and Crawford (1982) to our setting. Moreover, the conditions that explicitly account for indifferences turn out to be useful for proving various results on trading networks, as we discuss below.

Definition A.1, stated in the language of choice functions, and Definition A.3, stated in the language of demand functions, are conceptually related in that in both definitions the set of "options" available on one side expands, while the set of options on the other side remains unchanged.[26] The idea of expanding options on one side originated in the matching literature, where it is natural to consider an expansion in the set of available trades, which in

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[26] In choice-language, the "options" are the contracts available to choose from. In demand-language, the expansion of the set of "options" corresponds to prices of trades moving in the direction advantageous for the agent: trades in which he is the buyer become cheaper, and trades in which he is the seller become more expensive.

turn induces an expansion in the set of available contracts (see Ostrovsky (2008), Westkamp (2010), Hatfield and Kominers (2012), and Hatfield et al. (2013)). Definition A.1 is the full substitutability concept used by Hatfield et al. (2015) to prove the equivalence of stability and chain stability in trading networks.[27] The equivalence of Definition A.3 to other definitions of full substitutability is used in the proof of Theorem 6 of Hatfield et al. (2013) on the equivalence of stability and competitive equilibrium.

Definition A.2, stated in the language of choice functions, and Definition A.4, stated in the language of demand functions, are related in that in both definitions the set of "options" available on one side contracts, while the set of options on the other side remains unchanged.[28] Definition A.4, demand-language contraction full substitutability (DCFS), is the full substitutability definition that corresponds most directly to the original definition of gross substitutability of Kelso and Crawford (1982) and the definition of Gul and Stacchetti (1999, 2000): When an agent is not a seller in any trade in the economy, the DCFS condition directly reduces to those definitions of gross substitutability. It is also the definition that corresponds to the gross substitutes and complements condition of Sun and Yang (2006, 2009).[29] The equivalence of the DCFS condition to other full substitutability conditions (in

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[27] Hatfield et al. (2015) do not assume the quasilinearity of preferences or the continuity of transfers, and thus our equivalence results do not apply to the most general version of their setting.

[28] In demand-language, the contraction of the set of "options" corresponds to prices of trades moving in the direction disadvantageous for the agent: trades in which he is the buyer become more expensive, and trades in which he is the seller become cheaper.

[29] Sun and Yang (2006, 2009) studied exchange economies in which agents' preferences satisfy the gross substitutes and complements (GSC) condition. This condition requires that the set of objects can be partitioned into two sets S^1 and S^2 in such a way that whenever the price of one particular object in S^1 increases, each agent's demand for other objects in S^1 increases and each agent's demand for other objects in S^2 decreases; there is a symmetric requirement for the case where the price of an object in S^2 increases.

As discussed in Section IV.B of Hatfield et al. (2013), the framework of Sun and Yang (2006, 2009) can be embedded into the trading networks framework by viewing agents as intermediaries that (1) "buy inputs" from a set of artificial agents who each own one of the objects in S^1 and only care about the price received, and (2) "sell outputs" to a set of artificial agents who each can acquire only one particular object in S^2 and otherwise only care about the price charged. With this embedding, the (GSC) condition for exchange economies maps to the DCFS condition of our paper, and thus by our results is also equivalent to other definitions of full substitutability.

Note that while the framework of Sun and Yang (2006, 2009) can be embedded into the trading network framework of Hatfield et al. (2013) (as described above), the reverse is not true, because in the trading network framework, it will usually not be possible to partition the set of trades Ω into two sets Ω^1 and Ω^2 such that all agents' preferences simultaneously satisfy the (GSC) condition with respect to that partition (see Section IV.C of Hatfield et al. (2013) for details). Hence, in the presence of intermediaries, the trading particular, to the IFS and DFS conditions that only consider single-valued demands) is used in the proof of Theorem 1 of Hatfield et al. (2013) on the existence of competitive equilibria, in the step of the proof that "transforms" a trading network economy to a Kelso-Crawford two-sided, "many-to-one" matching market. The equivalence of the DCFS condition to the "single-valued" substitutability conditions implies that agents' preferences in the "transformed" market satisfy the gross substitutes condition of Kelso and Crawford (1982), making it possible to apply the results of Kelso and Crawford (1982) to the "transformed" market.

In contrast to Definitions A.1–A.4, which consider a change in the set of available options on one side while keeping the options on the other side unchanged, Definitions A.5 and A.6 consider changes in the set of options available on both sides simultaneously (i.e., the set of options on one side expands while the set of options on the other side contracts). This is similar to definitions used in the auction literature, where it is standard to consider the effects of a weak increase (or decrease) of the entire price vector (see, e.g., Ausubel and Milgrom (2006) and Ausubel (2006)). We use Definitions A.5 and A.6 in the proof of Theorem 7 on the equivalence of full substitutability and the single-improvement property.

A.1 Choice-Language Full Substitutability

Our next two definitions are analogues of Definition 1.

Definition A.1. The preferences of agent i are *choice-language expansion fully substitutable (CEFS)* if:

1. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;

2. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{\to i} = Z_{\to i}$ and $Y_{i\to} \subseteq Z_{i\to}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$ and

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network framework with agents' preferences satisfying the DCFS condition is more general than the exchange economy setting with agents' preferences satisfying the (GSC) condition.

$Y^*_{\to i} \subseteq Z^*_{\to i}$.

Definition A.2. The preferences of agent i are *choice-language contraction fully substitutable (CCFS)* if:

1. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Z^* \in C_i(Z)$, there exists $Y^* \in C_i(Y)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;
2. for all finite sets of contracts $Y, Z \subseteq X$ such that $Y_{\to i} = Z_{\to i}$ and $Y_{i\to} \subseteq Z_{i\to}$, for every $Z^* \in C_i(Z)$, there exists $Y^* \in C_i(Y)$ such that $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.

Note that we use Y as the "starting set" in CEFS and Z as the "starting set" in CCFS to make the two notions more easily comparable. Furthermore, note that in Case 1 of CEFS and CCFS, requiring $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ is equivalent to requiring that $Z^* \cap Y_{\to i} \subseteq Y^*$, and similarly, in Case 2, requiring $Y_{i\to} \smallsetminus Y^*_{i\to} \subseteq Z_{i\to} \smallsetminus Z^*_{i\to}$ is equivalent to requiring that $Z^* \cap Y_{i\to} \subseteq Y^*$.

A.2 Demand-Language Full Substitutability

Our next two definitions are analogues of Definition 2.

Definition A.3. The preferences of agent i are *demand-language expansion fully substitutable (DEFS)* if:

1. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;
2. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

Definition A.4. The preferences of agent i are *demand-language contraction fully substitutable (DCFS)* if:

1. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;
2. for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

Note that we use p as the "starting price vector" in DEFS and p' as the "starting price vector" in DCFS. Also, in Case 1 of DEFS and DCFS, requiring $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ is equivalent to requiring that $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'$, and similarly, in Case 2, requiring $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ is equivalent to requiring that $\{\omega \in (\Omega_{i\to} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{i\to} \smallsetminus \Psi'$.

A.3 Indicator-Language Full Substitutability

Our next two definitions are analogues of Definition 3.

Definition A.5. The preferences of agent i are *indicator-language increasing-price fully substitutable (IIFS)* if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \leq p'$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$, such that $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

Definition A.6. The preferences of agent i are *indicator-language decreasing-price fully substitutable (IDFS)* if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \leq p'$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$, such that $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

Defintion A.5 considers what happens as prices rise from p to p', requiring that a trade whose price does not change that is bought/not sold by i under p is still bought/not sold by i under p'. By contrast, Defintion A.6 considers what happens as prices fall from p' to p, requiring that a trade whose price does not change that is sold/not bought by i under p' is still sold/not bought by i under p.

A.4 Equivalence Result

We now present our most general equivalence, showing that our three main substitutability concepts and the six generalizatons introduced in this appendix are all equivalent.

Theorem A.1. *The CFS, DFS, IFS, CEFS, CCFS, DEFS, DCFS, IIFS, and IDFS conditions are all equivalent.*

In particular, Theorem A.1 implies Theorem 1.

We prove Theorem A.1 in Online Appendix B.