Full Substitutability in Trading Networks*

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Abstract

Trading networks generalize and unify models of matching with bilateral contracts and indivisible goods exchange. We extend earlier models' canonical definitions of substitutability to the trading network context and show that all these definitions are equivalent. We also show that substitutability corresponds to submodularity of the indirect utility function, the single improvement property, and a no complementarities condition. We prove that substitutability is preserved under economically important transformations such as trade endowments, mergers, and limited liability. Finally, we show that substitutability implies monotonicity conditions called the Laws of Aggregate Supply and Demand.

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1 Introduction

Various forms of substitutability are essential for establishing the existence of equilibria in diverse settings such as matching, auctions, exchange economies with indivisible goods, and trading networks (Kelso and Crawford, 1982; Roth, 1984; Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Ausubel and Milgrom, 2006; Hatfield and Milgrom, 2005; Sun and Yang, 2006, 2009; Ostrovsky, 2008; Hatfield et al., 2013). Substitutability arises in a number of important applications, including matching with distributional constraints (Abdulkadiroğlu and Sönmez, 2003; Hafalir et al., 2013; Sönmez and Switzer, 2013; Sönmez, 2013; Westkamp, 2013; Ehlers et al., 2014; Echenique and Yenmez, 2014; Kominers and Sönmez, 2014; Kamada and Kojima, 2015), supply chains (Ostrovsky, 2008), and markets with horizontal subcontracting (Hatfield et al., 2013), as well as "swap" deals in exchange markets (Milgrom, 2009), and combinatorial auctions for bank securities (Klemperer, 2010; Baldwin and Klemperer, 2014).

The diversity of settings in which substitutability plays a role has led to a variety of different definitions of substitutability, and a number of restrictions on preferences that appear in some definitions but not in others.[1] In this paper, we show how the different definitions of substitutability are related to each other, while dispensing with some of the restrictions in the preceding literature. We use a general trading network model that allows us to embed the key substitutability concepts from the matching, auctions, and exchange economy literatures. Our main result shows that all the substitutability concepts are equivalent.

We also show that substitutability can be recast in terms of submodularity of the indirect utility function, the single improvement property, a "no complementarities" condition, and a condition from discrete convex analysis called $M^\natural$-concavity. We prove that substitutability is preserved under three economically important transformations: trade endowments, mergers, and limited liability. Finally, we show that substitutability implies two key monotonicity conditions: the Law of Aggregate Supply and the Law of Aggregate Demand. Our analysis explicitly incorporates technical issues (such as indifferences and unbounded utility functions) that were unaddressed in the preceding literature.

We adopt the general trading in networks framework of Hatfield et al. (2013) that subsumes models of both matching and indivisible goods exchange. In this framework, (bilateral) contracts specify the provision of a good or service from a seller to a buyer and a monetary transfer. Given any two agents, there may be multiple contracts between them, and an agent could be involved in some contracts as a buyer and in others as a seller. Agents'

[1] For instance, some definitions assume "free disposal"/"monotonicity," under which an agent is always weakly better off with a larger set of goods than with a smaller one, while other definitions do not; some definitions assume that all bundles of goods are feasible for the agent, while others do not; and so on.

preferences are defined by cardinal utility functions over sets of contracts and are quasilinear with respect to the numeraire. In order to allow for various feasibility constraints, agents' preferences are allowed to be unboundedly negative. Furthermore, unlike the literature on indivisible goods exchange, we do not require that agents necessarily place a higher value on larger (in the superset sense) bundles of goods and services.

We first focus on the widely used substitutability conditions from the literatures on matching, auctions, and indivisible goods exchange. The relationships between these conditions are not immediate: Matching models typically use single-valued choice functions and formulate substitutability as the condition that an expansion of the choice set cannot make a previously rejected object desirable. On the other hand, models of auctions and exchange economies with indivisible goods use demand correspondences and formulate substitutability as the condition that increasing the price of one object should not lead to a decrease in demand for some other object whose price has not changed. We generalize these canonical definitions of substitutability to our framework and show that they are both equivalent to a condition that we term *full substitutability*. Intuitively, preferences are fully substitutable if contracts are substitutes for each other in a generalized sense, i.e., whenever an agent gains a new purchase opportunity, he becomes both less willing to make other purchases and more willing to make sales, and whenever he gains a new sales opportunity, he becomes both less willing to make other sales and more willing to make purchases. We then extend earlier results to show several practically useful equivalents of substitutability: submodularity of the indirect utility function, the single improvement property of Gul and Stacchetti (1999), which suggests a polynomial time algorithm for computing utility-maximizing choices, the "no complementarities" condition of Gul and Stacchetti (1999), and $M^\natural$-concavity (Murota, 2003).[2] Next, we show that assigning trade endowments, merging agents, and incorporating limited liability into an agent's utility function all preserve full substitutability. Finally, we show that in our model, full substitutability implies the Laws of Aggregate Supply and Demand (Hatfield and Kominers, 2012), extending an analogous result for two-sided matching markets (Hatfield and Milgrom, 2005).

1.1 History and Related Literature

Kelso and Crawford (1982) introduced the gross substitutability condition, which simultaneously applies in both matching and exchange economy contexts. The Kelso and Crawford

[2]For some of these equivalences we develop new proof techniques that enable significantly shorter arguments than were used in the prior literature. For others, we translate existing proof strategies to our richer setting and verify that the equivalences continue to hold without the boundedness and monotonicity assumptions imposed in the prior literature.

(1982) gross substitutability condition was subsequently extended and generalized in each context, giving rise to two (mostly) independent literatures.

In matching models, (choice-theoretic) substitutability of match partners (or contracts with match partners) guarantees the existence of stable outcomes (Roth, 1984; Hatfield and Milgrom, 2005; Hatfield and Kominers, 2013). Ostrovsky (2008) generalized the classic substitutability conditions from two-sided matching models to the context of supply chain networks by introducing a pair of related assumptions: same-side substitutability and cross-side complementarity. These assumptions impose two constraints: First, when an agent's opportunity set on one side of the market expands, that agent does not choose any options previously rejected from that side of the market. Second, when an agent's opportunity set on one side of the market expands, that agent does not reject any options previously chosen from the other side of the market. Both Ostrovsky (2008) and Hatfield and Kominers (2012) showed that under same-side substitutability and cross-side complementarity, a stable outcome always exists if the contractual set has supply chain structure. Moreover, Hatfield and Kominers (2012) showed that same-side substitutability and cross-side complementarity are together equivalent to the assumption of quasisubmodularity of the indirect utility function—an adaptation of submodularity to the setting without transfers.

In exchange economies with indivisible goods, the gross substitutability condition is sufficient to guarantee the existence of core allocations (Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000). Ausubel and Milgrom (2002) offered a convenient alternative definition of gross substitutability for a setting with continuous prices, in which demand is not guaranteed to be single-valued, and showed that gross substitutability is equivalent to submodularity of the indirect utility function. Sun and Yang (2006) introduced the gross substitutability and complementarity condition for the setting of indivisible object allocation. The gross substitutability and complementarity condition, akin to same-side substitutability and cross-side complementarity, requires that objects can be divided into two groups such that objects in the same group are substitutes and objects in different groups are complements. Sun and Yang (2009) showed that like gross substitutability, the gross substitutability and complementarity condition is equivalent to submodularity of the indirect utility function.

Subsequent to our work, Baldwin and Klemperer (2014) obtained additional insights on the underlying mathematical structure of fully substitutable preferences using the techniques of tropical geometry. Baldwin and Klemperer (2014) study the set of price vectors for which the demand correspondence is multi-valued, and associate them with convex-geometric objects called tropical hypersurfaces. Then, using the normal vectors that determine agents' tropical hypersurfaces, they distinguish among preferences that are strongly substitutable, are gross substitutable, or have complementarities.[3]

The discrete mathematics literature has explored several other concepts that are equivalent to substitutability in certain settings. We provide one point of connection to that literature in Section 4.5, where we establish the equivalence of full substitutability and $M^{\natural}$-concavity in our setting. Paes Leme (2014) provides an excellent survey that covers the discrete-mathematical substitutability concepts and their algorithmic properties in detail.[4]

2 Model

We use the general trading network model of Hatfield et al. (2013): We consider an economy with a finite set I of *agents* and a finite set Ω of *trades*. Each trade $\omega \in \Omega$ is associated with a *buyer* $b(\omega) \in I$ and a *seller* $s(\omega) \in I$, with $b(\omega) \neq s(\omega)$. Each trade $\omega \in \Omega$ specifies the nonpecuniary terms and conditions associated with the direct exchange of a single unit of an indivisible good or service between $s(\omega)$ and $b(\omega)$.[5] For concreteness, we may interpret the finite set of trades Ω as a subset of $I \times I \times O \times \mathbb{N} \times T$, where O denotes a set of object types, $\mathbb{N}$ specifies serial numbers differentiating each object of a given type $o \in O$, and T consists of possible terms of exchange. In this context, a trade $\omega \in \Omega$ is a 5-tuple, and two trades that differ in any single dimension are distinct.[6] However, we allow Ω to contain multiple trades associated to the same pair of agents, and allow the possibility of trades $\omega \in \Omega$ and $\psi \in \Omega$ such that the seller of ω is the buyer of ψ, i.e., $s(\omega) = b(\psi)$, and the seller of ψ is the buyer of ω, i.e., $s(\psi) = b(\omega)$.

We augment the set of trades by introducing a quasilinear numeraire. Formally, we let $X \equiv \Omega \times \mathbb{R}$ denote the set of *contracts* in the economy, where a contract $x \in X$ is a 2-tuple: $x = (\omega, p_\omega)$, with $\omega \in \Omega$ and $p_\omega \in \mathbb{R}$. For any contract $x = (\omega, p_\omega)$, p_ω is the *price* of trade ω paid by the *seller* $s(x) \equiv s(\omega)$ to the *buyer* $b(x) \equiv b(\omega)$. Since for each trade $\omega \in \Omega$ its price p_ω is allowed to vary freely, we can have infinitely many contracts associated to trade ω that differ only in prices.

A set of contracts Y is *feasible* if it does not contain two or more contracts for the same

[3]Our full substitutability concept corresponds to the *demand* $\mathcal{D}^n_{os}$ condition in Baldwin and Klemperer (2014).

[4]Unlike in our paper, the setting of Paes Leme (2014) assumes that all bundles of goods are feasible for the agent. Consequently, not all of the algorithmic results discussed by Paes Leme (2014) can be applied directly in our setting.

[5]To emphasise the absence of prices at this stage, we could also interpret the seller in a trade as being the "provider" of a good, while the buyer is an "acquirer."

[6]Thus, in particular, an exchange of unit $17 \in \mathbb{N}$ of object $o \in O$ from agent i to agent j is encoded as a different trade from exchange of unit $714 \in \mathbb{N}$ of object $o \in O$ from agent i to agent j. The finiteness of Ω requires that there are finitely many types of goods, each type is available in limited quantities, and the terms and conditions represent combinations of elements that can only take discrete values.

trade: formally, Y is feasible if $(\omega, p_\omega), (\omega, \hat{p}_\omega) \in Y$ implies that $p_\omega = \hat{p}_\omega$. We call a feasible set of contracts an *outcome*. An outcome specifies the trades that are executed, with their associated prices, but does not specify prices for trades that are not executed. An *arrangement* is a 2-tuple $[\Psi; p]$, with $\Psi \in \Omega$ and $p \in \mathbb{R}^\Omega$. An arrangement is similar to an outcome in that it specifies the trades that are executed; however, an arrangement specifies prices for *all* the trades in the economy. For any arrangement $[\Psi; p]$, we denote by $\kappa([\Psi; p]) \equiv \cup_{\psi \in \Psi} \{(\psi, p_\psi)\}$ the outcome induced by $[\Psi; p]$.

For each agent $i \in I$, we let $Y_{i \rightarrow} \equiv \{y \in Y : i = s(y)\}$ denote the set of contracts in Y in which i is the seller and let $Y_{\rightarrow i} \equiv \{y \in Y : i = b(y)\}$ denote the set of contracts in Y in which i is the buyer; we let $Y_i \equiv Y_{i \rightarrow} \cup Y_{\rightarrow i}$. We let $a(Y) \equiv \bigcup_{y \in Y} \{b(y), s(y)\}$ denote the set of agents involved in Y as either buyers or sellers; abusing notation slightly, we write $a(y) \equiv a(\{y\})$. We use analogous notation with regard to sets of trades $\Psi \subseteq \Omega$ (e.g., $a(\omega) = \{b(\omega), s(\omega)\}$, and $\Psi_i \equiv \{\omega \in \Psi : i \in a(\omega)\}$). For a set of contracts $Y \subseteq X$, we let $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\}$ denote the set of trades associated to contracts in Y.

2.1 Preferences

Each agent i has a *valuation* $u_i : 2^{\Omega_i} \rightarrow \mathbb{R} \cup \{-\infty\}$ over the sets of trades in which he is involved, with $u_i(\varnothing) \in \mathbb{R}$. Allowing the utility of an agent to equal $-\infty$ formalizes the idea that an agent, due to technological constraints, may only be able to produce or sell certain outputs contingent upon procuring appropriate inputs; for example, if $\psi, \omega \in \Omega$ with $b(\psi) = s(\omega) = i$ and agent i cannot sell ω unless he has procured ψ, then $u_i(\{\omega\}) = -\infty$.[7] The assumption that $u_i(\varnothing)$ is finite for each $i \in I$ implies that no agent is obligated to engage in market transactions—an agent can opt out from participating in the market at a finite cost $u_i(\varnothing) \in \mathbb{R}$.

We assume that agent i has quasilinear *preferences* over the set of trades and an (associated) transfer t induced by a *utility function*

$$U_i(\Psi, t) = u_i(\Psi) + t.$$

That is, agent i (weakly) prefers (Ψ, t) to (Φ, t') if $U^i(\Psi, t) \geq U^i(\Phi, t')$.

[7] In the classical exchange economy literature (Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999), the utility of an agent i is defined over bundles of objects Ω as $u_i : 2^{\Omega_i} \rightarrow \mathbb{R}$, and the utility is normalized such that $u_i(\varnothing) = 0$. While these assumptions are completely innocuous and natural in the context of exchange economies, they immediately rule out the kinds of technological constraints discussed above. Furthermore, these assumptions would require an agent who does not buy anything to sell every possible trade that he can sell.

Specifically, for any feasible set of contracts $Y \subseteq X$, we set

$$U_i(Y) \equiv U_i\left(\tau(Y), \sum_{(\omega,p_\omega)\in Y_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in Y_{\to i}} p_\omega\right) = u_i(\tau(Y)) + \sum_{(\omega,p_\omega)\in Y_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in Y_{\to i}} p_\omega.$$

and for any arrangement $[\Psi; p]$, we set

$$U_i([\Psi; p]) \equiv U_i\left(\Psi, \sum_{\psi\in\Psi_{i\to}} p_\psi - \sum_{\psi\in\Psi_{\to i}} p_\psi\right) = u_i(\Psi) + \sum_{\psi\in\Psi_{i\to}} p_\psi - \sum_{\psi\in\Psi_{\to i}} p_\psi.$$

The *choice correspondence* of agent i from the set of contracts $Y \subseteq X$ is defined by

$$C_i(Y) \equiv \{Z \subseteq Y_i : Z \text{ is feasible}, \forall \text{ feasible } Z' \subseteq Y_i,\ U_i(Z) \geq U_i(Z')\}$$

and the *demand correspondence* of agent i, given a price vector $p \in \mathbb{R}^\Omega$ is defined by

$$D_i(p) \equiv \{\Psi \subseteq \Omega_i : \forall\ \Phi \subseteq \Omega_i,\ U_i([\Psi; p]) \geq U_i([\Phi; p])\}.$$

3 Substitutability Concepts

We now introduce three substitutability concepts that generalize the existing definitions from matching, auctions, and exchange economies with indivisible goods. For convenience, in this section, we use the approach of Ausubel and Milgrom (2002) and restrict attention to opportunity sets and vectors of prices for which choices and demands are single-valued. In Appendices A and B, we introduce additional definitions that explicitly deal with indifferences and multi-valued correspondences, and prove that those definitions are equivalent to each other and to the definitions given in this section.

3.1 Choice-Language Full Substitutability

First, we define full substitutability in the language of sets and choices, adapting and merging the Ostrovsky (2008) same-side substitutability and cross-side complementarity conditions. In choice language, we say that a choice correspondence C_i is fully substitutable if, when attention is restricted to sets of contracts for which C_i is single-valued, whenever the set of options available to i on one side expands, i rejects a larger set of contracts on that side (same-side substitutability), and selects a larger set of contracts on the other side (cross-side complementarity).

Definition 1. *The preferences of agent i are* choice-language fully substitutable (CFS) *if:*

1. *for all sets of contracts* $Y, Z \subseteq X_i$ *such that* $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{i\to} = Z_{i\to}$, *and* $Y_{\to i} \subseteq Z_{\to i}$, *for the unique* $Y^* \in C_i(Y)$ *and* $Z^* \in C_i(Z)$, *we have* $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ *and* $Y^*_{i\to} \subseteq Z^*_{i\to}$;
2. *for all sets of contracts* $Y, Z \subseteq X_i$ *such that* $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{\to i} = Z_{\to i}$, *and* $Y_{i\to} \subseteq Z_{i\to}$, *for the unique* $Y^* \in C_i(Y)$ *and* $Z^* \in C_i(Z)$, *we have* $Y_{i\to} \smallsetminus Y^*_{i\to} \subseteq Z_{i\to} \smallsetminus Z^*_{i\to}$ *and* $Y^*_{\to i} \subseteq Z^*_{\to i}$.

3.2 Demand-Language Full Substitutability

Our second definition uses the language of prices and demands, generalizing the gross substitutes and complements condition (GSC) of Sun and Yang (2006). We say that a demand correspondence D_i is fully substitutable if, when attention is restricted to prices for which demands are single-valued, a decrease in the price of some inputs for agent i leads to a decrease in his demand for other inputs and to an increase in his supply of outputs, and an increase in the price of some outputs leads to the decrease in his supply of other outputs and an increase in his demand for inputs.

Definition 2. *The preferences of agent* i *are* demand-language fully substitutable (DFS) *if:*

1. *for all price vectors* $p, p' \in \mathbb{R}^\Omega$ *such that* $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{i\to}$, *and* $p_\omega \geq p'_\omega$ *for all* $\omega \in \Omega_{\to i}$, *for the unique* $\Psi \in D_i(p)$ *and* $\Psi' \in D_i(p')$, *we have* $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ *and* $\Psi_{i\to} \subseteq \Psi'_{i\to}$;
2. *for all price vectors* $p, p' \in \mathbb{R}^\Omega$ *such that* $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{\to i}$, *and* $p_\omega \leq p'_\omega$ *for all* $\omega \in \Omega_{i\to}$, *for the unique* $\Psi \in D_i(p)$ *and* $\Psi' \in D_i(p')$, *we have* $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ *and* $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

3.3 Indicator-Language Full Substitutability

Our third definition is essentially a reformulation of Definition 2, using a convenient vector notation due to Hatfield and Kominers (2012). For each agent i, for any set of trades $\Psi \subseteq \Omega_i$, define the *(generalized) indicator function* $e_i(\Psi) \in \{-1, 0, 1\}^{\Omega_i}$ to be the vector with component $e_{i,\omega}(\Psi) = 1$ for each upstream trade $\omega \in \Psi_{\to i}$, $e_{i,\omega}(\Psi) = -1$ for each downstream trade $\omega \in \Psi_{i\to}$, and $e_{i,\omega}(\Psi) = 0$ for each trade $\omega \notin \Psi$. The interpretation of $e_i(\Psi)$ is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in Ψ, and "buys" a strictly negative amount if he is the seller of such a trade.

Definition 3. *The preferences of agent* i *are* indicator-language fully substitutable (IFS) *if for all price vectors* $p, p' \in \mathbb{R}^\Omega$ *such that* $|D_i(p)| = |D_i(p')| = 1$ *and* $p \leq p'$, *for the unique* $\Psi \in D_i(p)$ *and* $\Psi' \in D_i(p')$, *we have* $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ *for each* $\omega \in \Omega_i$ *such that* $p_\omega = p'_\omega$.

Definition 3 clarifies the reason for the term "full substitutability:" an agent is more willing to "demand" a trade (i.e., keep an object that he could potentially sell, or buy an object that he does not initially own) if prices of other trades increase.

3.4 Equivalence of the Definitions

The main result of this section is that the three definitions of full substitutability presented are all equivalent. Subsequently, we use the term *full substitutability* to refer to all our substitutability concepts.

Theorem 1. *Choice-language full substitutability (CFS), demand-language full substitutability (DFS), and indicator-language full substitutability (IFS) are all equivalent.*

Theorem 1 follows from the more general equivalence result (Theorem B.1) that we prove in Appendix B.

4 Properties Equivalent to Full Substitutability

4.1 Submodularity of the Indirect Utility Function

A classical approach (see, e.g., the work of Gul and Stacchetti (1999) and Ausubel and Milgrom (2002)) relates substitutability of the utility function to submodularity of the indirect utility function. In particular, every (grossly) substitutable utility function corresponds to a submodular indirect utility function and vice versa.[8]

For price vectors $p, \bar{p} \in \mathbb{R}^\Omega$, let the *join* of p and $\bar{p}$, denoted $p \vee \bar{p}$, be the pointwise maximum of p and $\bar{p}$; let the *meet* of p and $\bar{p}$, denoted $p \wedge \bar{p}$, be the pointwise minimum of p and $\bar{p}$.

Definition 4. *The* indirect utility function *of agent i,*

$$V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} \{U_i([\Psi; p])\},$$

is submodular *if, for all price vectors $p, \bar{p} \in \mathbb{R}^\Omega$, we have that*

$$V_i(p \wedge \bar{p}) + V_i(p \vee \bar{p}) \leq V_i(p) + V_i(\bar{p}).$$

[8]Similar correspondences hold in markets without transferable utility: In many-to-many matching with contracts markets without transfers, every substitutable choice function can be represented by a submodular indirect utility function, and every submodular indirect utility function corresponds to a substitutable choice function (Hatfield and Kominers, 2013). In trading networks without transferable utility, every indirect utility function representing a fully substitutable choice function is quasi-submodular (Hatfield and Kominers, 2012).

Theorem 2. *The preferences of an agent are fully substitutable if and only if they induce a submodular indirect utility function.*

4.2 The Single Improvement Property

Gul and Stacchetti (1999) first observed (in the setting of exchange economies) that substitutability is equivalent to the *single improvement property*—an agent's preferences are substitutable if and only if, when an agent does not have an optimal bundle, that agent can make himself better off by adding a single item, dropping a single item, or both. Sun and Yang (2009) extended this result to their setting. Baldwin and Klemperer (2014) showed that in their setting the single improvement property is equivalent to requiring that agents have *complete* preferences.

Definition 5. *The preferences of agent i have the* single improvement property *if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that*

1. $U_i([\Psi, p]) < U_i([\Phi, p])$,
2. *there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi)$, and*
3. *there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi)$.*

The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by

1. obtaining one item not currently held (either by making a new purchase, i.e., adding a trade in $\Omega_{\to i} \smallsetminus \Psi$, or by canceling a sale, i.e., removing a trade in $\Psi_{i\to}$),
2. relinquishing one item currently held (either by canceling a purchase, i.e., removing a trade in $\Psi_{\to i}$, or by making a new sale, i.e., adding a trade in $\Omega_{i\to} \smallsetminus \Psi$), or
3. both obtaining one item not currently held and relinquishing one item currently held.

For instance, an agent may buy one more input and commit to provide one additional output as a "single improvement."

Moreover, when the preferences of agent i satisfy the single improvement property, it is easy to find an optimal bundle since, at any non-optimal bundle, a local adjustment can strictly increase the utility of i.

We now generalize the earlier results of Gul and Stacchetti (1999) and Sun and Yang (2009) to our setting.

Theorem 3. *The preferences of an agent are fully substitutable if and only if they have the single improvement property.*

4.3 Object-Language Substitutability

An intuitive way of thinking about trades in our setting is to consider each trade as representing the transfer of an underlying object. Under this interpretation, an agent's preferences over trades are fully substitutable if and only if that agent's preferences over objects have the standard Kelso and Crawford (1982) property of gross substitutability. This interpretation allows us to rewrite indicator-language full substitutability to more naturally correspond to the intuitive explanation of the concept given in Section 3.

Formally, we consider each trade $\omega \in \Omega$ as transferring an underlying object from $s(\omega)$ to $b(\omega)$; we denote this underlying object as $\mathfrak{o}(\omega)$. We call the set of all underlying objects $\boldsymbol{\Omega}$. Hence, for agent i, we can think of a set of trades Ψ as leaving that agent with a set of objects corresponding to trades in Ψ where i is a buyer and trades not in Ψ where i is a seller. We define the set of objects held by agent i after executing the set of trades Ψ as

$$\mathfrak{o}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \Omega_i \smallsetminus \Psi_{i\to}\}.$$

Conversely, we define the trade associated with an object $\boldsymbol{\omega}$ as $\mathfrak{t}(\boldsymbol{\omega})$; note that $\mathfrak{t}(\mathfrak{o}(\omega)) = \omega$. We also define the set of trades executed by i for a given set of held objects $\boldsymbol{\Psi} \subseteq \boldsymbol{\Omega}_i \equiv \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : i \in \{b(\mathfrak{t}(\boldsymbol{\omega})), s(\mathfrak{t}(\boldsymbol{\omega}))\}$ as

$$\mathfrak{t}_i(\boldsymbol{\Psi}) = \{\omega \in \Omega_{\to i} : \mathfrak{o}(\omega) \in \boldsymbol{\Psi}\} \cup \{\omega \in \Omega_{i\to} : \mathfrak{o}(\omega) \in \boldsymbol{\Omega}_i \smallsetminus \boldsymbol{\Psi}\}.$$

Hence, for a partition of objects $\{\boldsymbol{\Psi}^i\}_{i\in I}$, the set of trades that implements this partition is given by

$$\bigcup_{i\in I} \mathfrak{t}_i(\boldsymbol{\Psi}^i).$$

For notational simplicity, for a set of objects $\boldsymbol{\Psi}$, we let $u_i(\boldsymbol{\Psi}) \equiv u_i(\mathfrak{t}_i(\boldsymbol{\Psi}))$.

Using object language, we can also reformulate indicator-language full substitutability to *object-language full substitutability.*

Definition 6. *The preferences of agent i are* object-language fully substitutable (OFS) *if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, if $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi)$, then $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi')$ for each $\boldsymbol{\omega} \in \boldsymbol{\Omega}_i$ such that $p_{\mathfrak{t}(\boldsymbol{\omega})} = p'_{\mathfrak{t}(\boldsymbol{\omega})}$.*

Under object-language full substitutability, increases in the prices of objects $\boldsymbol{\psi} \neq \boldsymbol{\omega}$ cannot decrease the agent's demand for objects $\boldsymbol{\omega}$ whose prices do not change. That is, the agent's preferences over objects are grossly substitutable, in the sense of Kelso and Crawford (1982).

We can now understand the indicator vector $e_{i,\psi}(\Psi)$ as encoding whether the object $\boldsymbol{\psi} = \mathfrak{o}(\psi)$ is transferred under Ψ:

- If $\psi \in \Psi_{\to i}$, then $\boldsymbol{\psi} \in \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = 1$, i.e., i obtains the object associated with ψ.
- If $\psi \in \Psi_{i\to}$, then $\boldsymbol{\psi} \notin \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = -1$, i.e., i gives up the object associated with ψ.
- Finally, if $\psi \notin \Psi$, then $e_{i,\psi}(\Psi) = 0$, i.e., i neither obtains nor gives up the object associated with ψ.

Additionally, object-language full substitutability helps us define a "no complementarities condition," equivalent to full substitutability, in the next section. Also, it is useful in our proof fully substitutable preferences satisfy the Laws of Aggregate Supply and Demand (under quasilinear utility).

We can reformulate the definition of the single improvement property in terms of objects.

Definition 7. *The preferences of agent i have the* single improvement property *if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that*

1. $U^i([\Psi, p]) < U^i([\Phi, p])$,
2. *there exists at most one object $\boldsymbol{\omega} \in \mathfrak{o}_i(\Phi) \smallsetminus \mathfrak{o}_i(\Psi)$, and*
3. *there exists at most one object $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi) \smallsetminus \mathfrak{o}_i(\Phi)$.*

Using object language, we obtain a definition of the single improvement property that exactly matches the intuition provided on page 10. The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by

1. obtaining one object $\boldsymbol{\omega}$ not currently held, i.e., $\boldsymbol{\omega} \notin \mathfrak{o}_i(\Psi)$,
2. relinquishing one object $\boldsymbol{\omega}$ currently held, i.e., $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi)$, or
3. both obtaining one object and relinquishing one object.

4.4 The No Complementarities Condition

Gul and Stacchetti (1999) observed that substitutability is equivalent to the *no complementarities condition*; we extend this observation here.

Definition 8. *The preferences of agent i satisfy the* no complementarities condition *if, for every price vector p, for any $\Phi, \Psi \in D_i(p)$, and for any $\bar{\boldsymbol{\Psi}} \subseteq \mathfrak{o}_i(\Psi)$, there exists $\bar{\boldsymbol{\Phi}} \subseteq \mathfrak{o}_i(\Phi)$ such that $\mathfrak{t}_i((\boldsymbol{\Psi} \smallsetminus \bar{\boldsymbol{\Psi}}) \cup \bar{\boldsymbol{\Phi}}) \in D_i(p)$.*

The no complementarities condition requires that for any pair of optimal bundles of objects, $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, and for any $\bar{\boldsymbol{\Psi}} \subseteq \boldsymbol{\Psi}$, there exists a set of objects $\bar{\boldsymbol{\Phi}} \subseteq \boldsymbol{\Phi}$ that "substitute" for the objects in $\bar{\boldsymbol{\Psi}}$, in the sense that $(\boldsymbol{\Psi} \smallsetminus \bar{\boldsymbol{\Psi}}) \cup \bar{\boldsymbol{\Phi}}$ is optimal.

Theorem 4. *The preferences of an agent are fully substitutable if and only if they satisfy the no complementarities condition.*

The proof of Theorem 4 is an adaptation of the proof of Theorem 1 of Gul and Stacchetti (1999). Gul and Stacchetti (1999) assume that valuation functions are monotone and bounded from below; thus, in our proof of Theorem 4, we must be careful to ensure that non-monotonicities and unboundedness do not invalidate the Gul and Stacchetti (1999) proof strategy.

4.5 $M^\natural$-Concavity over Objects

Reijnierse et al. (2002) and Fujishige and Yang (2003) independently observed that gross substitutability in the Kelso and Crawford (1982) model is equivalent to a classical condition from discrete optimization theory, *$M^\natural$-concavity.* In our object-language notation, the condition can be stated as follows.

Definition 9. *The valuation u_i is $M^\natural$-*concave over objects *if for all $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \boldsymbol{\Omega}_i$, for any $\boldsymbol{\psi} \in \boldsymbol{\Psi}$,*

$$u_i(\boldsymbol{\Psi}) + u_i(\boldsymbol{\Phi}) \leq \max\Big\{u_i(\boldsymbol{\Psi} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\boldsymbol{\Phi} \cup \{\boldsymbol{\psi}\}), \\ \max_{\boldsymbol{\varphi} \in \boldsymbol{\Phi}}\{u_i(\boldsymbol{\Psi} \cup \{\boldsymbol{\varphi}\} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\boldsymbol{\Phi} \cup \{\boldsymbol{\psi}\} \smallsetminus \{\boldsymbol{\varphi}\})\}\Big\}.$$

A valuation function is $M^\natural$-concave if, for any sets of objects $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$, the average valuation between $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ can be weakly increased by either moving one object from $\boldsymbol{\Psi}$ to $\boldsymbol{\Phi}$ or swapping one object in $\boldsymbol{\Psi}$ for one object in $\boldsymbol{\Phi}$.

Theorem 5. *The preferences of an agent are fully substitutable if and only if the associated valuation function is $M^\natural$-concave over objects.*

This equivalence result follows from Theorem 7 of Murota and Tamura (2003), which shows that $M^\natural$-concavity is equivalent to the single improvement property—and which in turn, by our Theorem 3, implies the equivalence between full substitutability and $M^\natural$-concavity.

5 Transformations

We now consider several economically-motivated valuation function transformations. We first consider the possibility that an agent is endowed with the right to execute any trades in a given set. We also examine mergers, where the valuation function of the merged entity is constructed as the convolution of the valuation functions of the merging parties.[9] Finally, we consider a form of limited liability, where an agent may back out of some agreed-upon trades in exchange for paying an exogenously-fixed penalty. We show that all of these transformations preserve substitutability.

5.1 Trade Endowments

Suppose an agent i is endowed with the right to execute trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ. Let

$$\hat{u}_i^{(\Phi,p_\Phi)}(\Psi) \equiv \max_{\Xi \subseteq \Phi} \left\{ u_i(\Psi \cup \Xi) + \sum_{\xi \in \Xi_{\rightarrow i}} p_\xi - \sum_{\xi \in \Xi_{\rightarrow i}} p_\xi \right\} \tag{1}$$

be a valuation over trades in $\Omega \smallsetminus \Phi$; $\hat{u}_i^{(\Phi,p_\Phi)}$ represents agent i having a valuation over trades in $\Omega \smallsetminus \Phi$ consistent with u_i, while being endowed with the option of executing any trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ.

Theorem 6. *If the preferences of agent i are fully substitutable, then the preferences induced by the valuation function $\hat{u}_i^{(\Phi,p_\Phi)}$ (defined in* (1)*) are fully substitutable for any $\Phi \subseteq \Omega_i$ and $p_\Phi \in \mathbb{R}^\Phi$.*

When we endow i with access to the trades in Φ at prices p_Φ, we are effectively restricting (1) the set of prices that may change and (2) the set of trades that are required to be

[9] In two-sided matching settings, the operations of "endowment" and "merger" were used by Hatfield and Milgrom (2005) to construct the class of endowed assignment valuations, starting with singleton preferences and iteratively applying these operations. Hatfield and Milgrom (2005) show that these operations preserve substitutability (Theorems 13 and 14), and thus show that all endowed assignment valuation preferences are substitutable. Ostrovsky and Paes Leme (2014) show that there exist substitutable preferences that cannot be represented as an endowed assignment valuation, and introduce the class of matroid-based valuations, which is obtained by iteratively applying the "endowment" and "merger" operations to weighted-matroid valuations. Since every weighted-matroid valuation is substitutable (Murota, 1996; Murota and Shioura, 1999; Fujishige and Yang, 2003), every matroid-based valuation is also substitutable. It is an open question whether every substitutable valuation is a matroid-based valuation.

substitutes in the demand-theoretic definition of full substitutability (Definition 2). Naturally, this process cannot *create* complementarities among trades in $\Omega \smallsetminus \Phi$, given that under u_i these trades already are substitutes for each other *and* for the trades in Φ. Hence, $\hat{u}_i^{(\Phi,p_\Phi)}$ induces fully substitutable preferences over trades in $\Omega \smallsetminus \Phi$.

5.2 Mergers

The second transformation we consider is the case when several agents act in concert, e.g., following a merger. Given a set of agents J, we denote the set of trades that involve only agents in J as $\Omega^J \equiv \{\omega \in \Omega : \{b(\omega), s(\omega)\} \subseteq J\}$. We let the *convolution* of the valuation functions $\{u_j\}_{j\in J}$ be defined as

$$u_J(\Psi) \equiv \max_{\Phi \subseteq \Omega^J} \left\{ \sum_{j \in J} u_j(\Psi \cup \Phi) \right\} \tag{2}$$

for sets of trades $\Psi \subseteq \Omega \smallsetminus \Omega^J$. The convolution u_J represents a "merger" of the agents in J, as it treats the agents in J as able to execute any within-J trades costlessly.

Theorem 7. *For any set of agents $J \subseteq I$, if the preferences of each $j \in J$ are fully substitutable, then the preferences induced by the convolution u_J (defined in* (2)*) are fully substitutable.*

When the preferences of each agent $j \in J$ are fully substitutable, for any given price vector for trades in $\Omega \smallsetminus \Omega^J$, it is utility-maximizing for the merged entity J to choose as if it were still composed of individual agents. Hence, the merger has no effect on the demand of agents in J for trades with agents not in J; consequently, the merged entity exhibits substitutable preferences.

Note that substitutability is not preserved following *dissolution/de-mergers*. For example, if agents i and j only trade with each other (i.e., $\Omega_i = \Omega_j$), then the preferences induced by the convolution valuation $u_{\{i,j\}}$ are trivially fully substitutable, even if the preferences of i and j are not.

Note also that while merging agents preserves substitutability, the same cannot be said about merging trades between two agents. For example, consider a simple economy with agents i and j and four trades: set Ω consists of trades χ, φ, ψ, and ω. Agent i is the buyer in all of these trades, and agent j is the seller. The valuation of agent i is as follows:

$$u_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \\ 1 & |\Psi_i| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The preferences of i are clearly fully substitutable. But now consider merging the trades χ and φ into a single trade ξ. The resulting valuation function of i over the subsets of $\tilde{\Omega} \equiv (\Omega \smallsetminus \{\chi, \varphi\}) \cup \{\xi\}$ is given by

$$\tilde{u}_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \text{ or } \xi \in \Psi \\ 1 & |\Psi_i| = 1 \text{ and } \xi \notin \Psi \\ 0 & \text{otherwise.} \end{cases}$$

Valuation function $\tilde{u}_i$ is not fully substitutable. To see this, note that for price vector $p = (p_\xi, p_\psi, p_\omega) = (1.7, 0.8, 0.8)$, the unique optimal demand of agent i is $\{\psi, \omega\}$, but for price vector $p' = (p'_\xi, p'_\psi, p'_\omega) = (1.7, 1, 0.8)$, the unique optimal demand of agent i is $\{\xi\}$. That is, under price vector p', agent i no longer demands the trade ω, even though its price remains unchanged while the price of ψ increases and the price of ξ remains unchanged.

5.3 Limited Liability

The final transformation we consider is "limited liability." Specifically, suppose that after agreeing to a trade, an agent is allowed to renege on that trade in exchange for paying a fixed penalty. We show that this transformation preserves substitutability. In addition to being economically interesting, the preservation of substitutability under limited liability is also useful technically; indeed, it enables us to transform unbounded utility functions into bounded ones while preserving substitutability. (The fact that this transformation preserves substitutability simplifies the technical analysis in a number of settings; see, e.g., the proof of Theorem 1 in Hatfield et al. (2013).)

Formally, consider a fully substitutable valuation function u_i for agent i. Take an arbitrary set of trades $\Phi \subseteq \Omega_i$, and for every trade $\varphi \in \Phi$, pick $\Pi_\varphi \in \mathbb{R}$—the penalty for reneging on trade φ. (For mathematical completeness, we allow Π_φ to be negative.) Define the modified valuation function $\hat{u}_i$ as

$$\hat{u}_i(\Psi) \equiv \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}. \tag{3}$$

That is, under valuation $\hat{u}_i$, agent i can "buy out" some of the trades to which he has committed (provided these trades are in the set Φ of trades the agent may renege on), and pay the corresponding penalty for each trade he buys out.

Theorem 8. *For any $\Phi \subseteq \Omega_i$ and $\Pi_\Phi \in \mathbb{R}^\Phi$, if agent i has fully substitutable preferences, then the valuation function $\hat{u}_i$ with limited liability (as defined in* (3)*) induces fully substitutable preferences.*

6 Implications of Full Substitutability

6.1 Stable Outcomes and Competitive Equilibria

In other work, we show that when all agents' preferences are fully substitutable, outcomes that are *stable* (in the sense of matching theory) exist for any underlying network structure (Hatfield et al., 2013, Theorems 1 and 5). Furthermore, full substitutability of preferences guarantees both that the set of stable outcomes is essentially equivalent to the set of competitive equilibria with personalized prices (Hatfield et al., 2013, Theorems 5 and 6) and that all stable outcomes are in the core and are efficient (Hatfield et al., 2013, Theorem 9).[10] Full substitutability also delineates a maximal domain for the existence of stable outcomes in our framework: for any domain of preferences strictly larger than that of full substitutability, the existence of stable outcomes and competitive equilibria cannot be guaranteed (Hatfield et al., 2013, Theorem 7). These results all build upon methods and insights from the prior literature on markets with fully substitutable preferences—especially the work of Crawford and Knoer (1981), Kelso and Crawford (1982), Gul and Stacchetti (1999, 2000), and Sun and Yang (2006, 2009).

6.2 Laws of Aggregate Supply and Demand

In two-sided matching markets with transfers and quasilinear utility, all fully substitutable preferences satisfy a monotonicity condition called the Law of Aggregate Demand (Hatfield and Milgrom, 2005).[11] The analogues of this condition for the current setting are the *Laws of Aggregate Supply and Demand* for trading networks, first introduced by Hatfield and Kominers (2012).

[10]Moreover, when all agents' preferences are fully substitutable, the set of stable outcomes is equivalent to the set of *chain stable* outcomes (Hatfield et al., 2015).

[11]In the context of two-sided matching with contracts, the Law of Aggregate Demand is essential for "rural hospitals" and strategy-proofness results (see Hatfield and Milgrom (2005); Hatfield and Kominers (2013)).

Definition 10. *The preferences of agent i satisfy the* Law of Aggregate Demand *if for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{\to i}| - |Y^*_{\to i}| \geq |Z^*_{i\to}| - |Y^*_{i\to}|$.*

The preferences of agent i satisfy the Law of Aggregate Supply *if for all finite sets of contracts Y and Z such that $Y_{i\to} \subseteq Z_{i\to}$ and $Y_{\to i} = Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{i\to}| - |Y^*_{i\to}| \geq |Z^*_{\to i}| - |Y^*_{\to i}|$.*

In our network setting with quasilinear utilities and continuous transfers, preference full substitutability implies the Laws of Aggregate Supply and Demand.

Theorem 9. *If the preferences of agent i are fully substitutable and quasilinear in the numeraire, then they satisfy the Laws of Aggregate Supply and Demand.*

Theorem 9 generalizes Theorem 7 of Hatfield and Milgrom (2005), who showed the analogous result in the special case when agent i acts only as a buyer. The proof essentially follows from applying the Hatfield and Milgrom (2005) result to the agent's preferences over objects.

7 Conclusion

Economists have recently recognized a number of structural similarities in models of matching, auctions, and exchange economies (see, e.g., Kelso and Crawford (1982), Gul and Stacchetti (1999), Hatfield and Milgrom (2005), and Sun and Yang (2006)). The generality of our framework allows us to further unify matching, auctions, and exchange by showing that the key substitutability conditions from these literatures are all equivalent; establishing that there is essentially one "full substitutability" condition may enable economists to share insights across fields.

We show that substitutability has several economically important implications: When preferences are substitutable, they satisfy both a "no complementarities" condition as well as the single improvement property; consequently, it is easy, i.e., computationally tractable, for agents with substitutable preferences to find optimal demands given prices. We further show that mergers, limited liability and trade endowments are all substitutability-preserving transformations; these results may be useful, for instance, in the analysis of the used-car market (where mergers among dealers are common) and in the design of auctions (where limited liability is both technically useful and also a real constraint).

Expressing substitutability in terms of preferences over trades is not straightforward as, when an individual agent can be both a buyer and a seller, substitutability requires treating relationships between "same-side" and "cross-side" contracts differently. Both Sun and Yang (2006) and Ostrovsky (2008) introduced a concept of cross-side complementarity, which requires that agents treat buy-side contracts as complementary with sell-side contracts. Our work uncovers that cross-side complementarity is not really a complementarity condition *per se*: rather, it corresponds to an underlying substitutability condition over objects.

Meanwhile, our interpretation of substitutability in terms of preferences over objects (Definition 6) is straightforward: When an agent's object opportunity set shrinks, substitutability requires that the agent not reduce demand for any object that remains in his opportunity set. In particular, in settings with transferable utility, when prices increase, an agent's object opportunity set shrinks; hence, substitutability requires that the agent (weakly) increase his demand for objects, both his and others', whose prices do not rise.

Appendix

A Full Substitutability Definitions with Indifferences

In this Appendix, we introduce six alternative definitions of full substitutability, as follows:

- Definitions A.1 and A.2 are analogues of our choice-language definition (Definition 1),
- Definitions A.3 and A.4 are analogues of our demand-language definition (Definition 2), and
- Definitions A.5 and A.6 are analogues of our indicator-language definition (Definition 3).

In contrast to Definitions 1, 2 and 3, which consider single-valued choices and demands, Definitions A.1–A.6 explicitly consider multi-valued correspondences and deal directly with indifferences. By explicitly accounting for indifferences and multi-valued correspondences, we directly generalize the original gross substitutability condition of Kelso and Crawford (1982) to our setting. Moreover, the conditions that explicitly account for indifferences turn out to be useful for proving various results on trading networks, as we discuss below.

Definition A.1, stated in the language of choice functions, and Definition A.3, stated in the language of demand functions, are conceptually related in that in both definitions the set of "options" available on one side expands, while the set of options on the other side remains unchanged.[12] The idea of expanding options on one side originated in the matching literature, where it is natural to consider an expansion in the set of available trades, which in turn induces an expansion in the set of available contracts (see Ostrovsky (2008), Westkamp (2010), Hatfield and Kominers (2012), and Hatfield et al. (2013)). Definition A.1 is the full substitutability concept used by Hatfield et al. (2015) to prove the equivalence of stability and chain stability in trading networks.[13] The equivalence of Definition A.3 (DEFS) to other definitions of full substitutability is used in the proof of Theorem 6 of Hatfield et al. (2013) on the equivalence of stability and competitive equilibrium.

Definition A.2, stated in the language of choice functions, and Definition A.4, stated in the language of demand functions, are related in that in both definitions the set of "options"

[12] In choice-language, the "options" are the contracts available to choose from. In demand-language, the expansion of the set of "options" corresponds to prices of trades moving in the direction advantageous for the agent: trades in which he is the buyer become cheaper, and trades in which he is the seller become more expensive.

[13] Hatfield et al. (2015) do not assume the quasilinearity of preferences or the continuity of transfers, and thus our equivalence results do not apply to the most general version of their setting.

available on one side contracts, while the set of options on the other side remains unchanged.[14] Definition A.4 (DCFS) is the full substitutability definition that corresponds most directly to the original definition of gross substitutability of Kelso and Crawford (1982) and the definition of Gul and Stacchetti (1999, 2000): When an agent is not a seller in any trade in the economy, the (DCFS) condition directly reduces to those definitions of gross substitutability. It is also the definition that corresponds to the gross substitutes and complements condition of Sun and Yang (2006, 2009). The equivalence of the (DCFS) condition to other full substitutability conditions (in particular, to the (IFS) and (DFS) conditions that only consider single-valued demands) is used in the proof of Theorem 1 of Hatfield et al. (2013) on the existence of competitive equilibria, in the step of the proof that "transforms" a trading network economy to a Kelso-Crawford two-sided, one-to-many matching market. The equivalence of the (DCFS) condition to the "single-valued" substitutability conditions implies that agents' preferences in the "transformed" market satisfy the gross substitutes condition of Kelso and Crawford (1982), making it possible to apply the results of Kelso and Crawford (1982) to the "transformed" market.

In contrast to Definitions A.1–A.4, which consider a change in the set of available options on one side while keeping the options on the other side unchanged, Definitions A.5 and A.6 consider changes in the set of options available on both sides simultaneously (i.e., the set of options on one side expands while the set of options on the other side contracts). This idea is in line with the auction literature, where it is standard to consider the effects of a weak increase (or decrease) of the entire price vector (see, e.g., Ausubel and Milgrom (2006) and Ausubel (2006)). We use Definitions A.5 and A.6 in the proof of Theorem 3 on the equivalence of full substitutability and the single-improvement property.

A.1 Choice-Language Full Substitutability

Our next two definitions are analogues of Definition 1.

Definition A.1. *The preferences of agent i are* choice-language expansion fully substitutable (CEFS) *if:*

1. *for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*

[14] In demand-language, the contraction of the set of "options" corresponds to prices of trades moving in the direction disadvantageous for the agent: trades in which he is the buyer become more expensive, and trades in which he is the seller become cheaper.

2. *for all finite sets of contracts* $Y, Z \subseteq X_i$ *such that* $Y_{\to i} = Z_{\to i}$ *and* $Y_{i\to} \subseteq Z_{i\to}$, *for every* $Y^* \in C_i(Y)$, *there exists* $Z^* \in C_i(Z)$ *such that* $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$ *and* $Y^*_{\to i} \subseteq Z^*_{\to i}$.

Definition A.2. *The preferences of agent* i *are* choice-language contraction fully substitutable (CCFS) *if:*

1. *for all finite sets of contracts* $Y, Z \subseteq X_i$ *such that* $Y_{i\to} = Z_{i\to}$ *and* $Y_{\to i} \subseteq Z_{\to i}$, *for every* $Z^* \in C_i(Z)$, *there exists* $Y^* \in C_i(Y)$ *such that* $(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i})$ *and* $Y^*_{i\to} \subseteq Z^*_{i\to}$;
2. *for all finite sets of contracts* $Y, Z \subseteq X_i$ *such that* $Y_{\to i} = Z_{\to i}$ *and* $Y_{i\to} \subseteq Z_{i\to}$, *for every* $Z^* \in C_i(Z)$, *there exists* $Y^* \in C_i(Y)$ *such that* $(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to})$ *and* $Y^*_{\to i} \subseteq Z^*_{\to i}$.

Note that we use Y as the "starting set" in (CEFS) and Z as the "starting set" in (CCFS) to make the two notions more easily comparable. Furthermore, note that in Case 1 of (CEFS) and (CCFS), requiring $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ is equivalent to requiring that $Z^* \cap Y_{\to i} \subseteq Y^*$, and similarly, in Case 2, requiring $Y_{i\to} \smallsetminus Y^*_{i\to} \subseteq Z_{i\to} \smallsetminus Z^*_{i\to}$ is equivalent to requiring that $Z^* \cap Y_{i\to} \subseteq Y^*$.

A.2 Demand-Language Full Substitutability

Our next two definitions are analogues of Definition 2.

Definition A.3. *The preferences of agent* i *are* demand-language expansion fully substitutable (DEFS) *if:*

1. *for all price vectors* $p, p' \in \mathbb{R}^\Omega$ *such that* $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{i\to}$ *and* $p_\omega \geq p'_\omega$ *for all* $\omega \in \Omega_{\to i}$, *for every* $\Psi \in D_i(p)$ *there exists* $\Psi' \in D_i(p')$ *such that* $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ *and* $\Psi_{i\to} \subseteq \Psi'_{i\to}$;
2. *for all price vectors* $p, p' \in \mathbb{R}^\Omega$ *such that* $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{\to i}$ *and* $p_\omega \leq p'_\omega$ *for all* $\omega \in \Omega_{i\to}$, *for every* $\Psi \in D_i(p)$ *there exists* $\Psi' \in D_i(p')$ *such that* $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ *and* $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

Definition A.4. *The preferences of agent* i *are* demand-language contraction fully substitutable (DCFS) *if:*

1. *for all price vectors* $p, p' \in \mathbb{R}^\Omega$ *such that* $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{i\to}$ *and* $p_\omega \geq p'_\omega$ *for all* $\omega \in \Omega_{\to i}$, *for every* $\Psi' \in D_i(p')$ *there exists* $\Psi \in D_i(p)$ *such that* $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ *and* $\Psi_{i\to} \subseteq \Psi'_{i\to}$;

2. *for all price vectors* $p, p' \in \mathbb{R}^{\Omega}$ *such that* $p_\omega = p'_\omega$ *for all* $\omega \in \Omega_{\to i}$ *and* $p_\omega \le p'_\omega$ *for all* $\omega \in \Omega_{i\to}$, *for every* $\Psi' \in D_i(p')$ *there exists* $\Psi \in D_i(p)$ *such that* $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ *and* $\Psi_{\to i} \subseteq \Psi'_{\to i}$.

Note that we use p as the "starting price vector" in (DEFS) and p' as the "starting price vector" in (DCFS). Also, in Case 1 of (DEFS) and (DCFS), requiring $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ is equivalent to requiring that $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'$, and similarly, in Case 2, requiring $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ is equivalent to requiring that $\{\omega \in (\Omega_{i\to} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{i\to} \smallsetminus \Psi'$.

A.3 Indicator-Language Full Substitutability

Our next two definitions are analogues of Definition 3.

Definition A.5. *The preferences of agent* i *are* indicator-language increasing-price fully substitutable (IIFS) *if for all price vectors* $p, p' \in \mathbb{R}^{\Omega}$ *such that* $p \le p'$, *for every* $\Psi \in D_i(p)$ *there exists* $\Psi' \in D_i(p')$, *such that* $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ *for each* $\omega \in \Omega_i$ *such that* $p_\omega = p'_\omega$.

Definition A.6. *The preferences of agent* i *are* indicator-language decreasing-price fully substitutable (IDFS) *if for all price vectors* $p, p' \in \mathbb{R}^{\Omega}$ *such that* $p \le p'$, *for every* $\Psi' \in D_i(p')$ *there exists* $\Psi \in D_i(p)$, *such that* $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ *for each* $\omega \in \Omega_i$ *such that* $p_\omega = p'_\omega$.

Note that we use p as the "starting price vector" in (IIFS) and $p' \ge p$ as the "starting price vector" in (IDFS).

B Equivalence of Full Substitutability Definitions

In this Appendix, we show that the three definitions in Section 3 and the six definitions in Appendix A are all equivalent. In particular, this implies Theorem 1.

Theorem B.1. *The (CFS), (DFS), (IFS), (CEFS), (CCFS), (DEFS), (DCFS), (IIFS), and (IDFS) conditions are all equivalent.*

Proof. It is immediate that (CEFS) and (CCFS) each imply (CFS), and (IIFS) and (IDFS) both imply (IFS). Below we establish the remaining equivalences by showing that (CFS) ⇒ (DFS), (DFS) ⇒ (DEFS), (DFS) ⇒ (DCFS), (DEFS) ⇒ (CEFS), (DCFS) ⇒ (CCFS), (DEFS) + (DCFS) ⇒ (IDFS) + (IIFS), and (IFS) ⇒ (DFS).

(CFS) ⇒ (DFS) We first show that Case 1 of (CFS) implies Case 1 of (DFS). For any agent i and price vector $p \in \mathbb{R}^\Omega$, let $X_i(p) \equiv \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{\to i},\ \hat{p}_\omega \geq p_\omega\} \cup \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{i\to},\ \hat{p}_\omega \leq p_\omega\}$, in essence denoting the set of contracts available to agent i under prices p.

Let price vectors $p, p' \in \mathbb{R}^\Omega$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$. Let $Y = X_i(p)$ and $Z = X_i(p')$. Clearly, $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Furthermore, it is immediate that $\Psi \in D_i(p)$ if and only if $\kappa([\Psi; p]) \in C_i(Y)$, and similarly, $\Psi' \in D_i(p')$ if and only if $\kappa([\Psi'; p']) \in C_i(Z)$. In particular, we have $|C_i(Y)| = |C_i(Z)| = 1$ and can thus apply (CFS) to the sets Y and Z.

Take the unique $\Psi \in D_i(p)$, let $Y^* = \kappa([\Psi, p])$, and note that $Y^* \in C_i(Y)$. By (CFS), the unique $Z^* \in C_i(Z)$ satisfies $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$. Let $\Psi' = \tau(Z^*)$ and note that $\Psi' \in D_i(p')$. We show that Ψ' satisfies the conditions in Case 1 of Definition 2.

Note that $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ implies $\{\omega \in \Omega_{\to i} \smallsetminus \Psi_{\to i} : p_\omega = p'_\omega\} \subseteq \tau(Y_{\to i}) \smallsetminus \tau(Y^*_{\to i}) \subseteq \tau(Z_{\to i}) \smallsetminus \tau(Z^*_{\to i}) \subseteq \Omega_{\to i} \smallsetminus \Psi'_{\to i}$. Furthermore, $Y^*_{i\to} \subseteq Z^*_{i\to}$ and $p_\omega = p'_\omega$ for each $\omega \in \Omega_{i\to}$ imply $\Psi'_{i\to} \subseteq \Psi_{i\to}$.

The proof that Case 2 of (CFS) implies Case 2 of (DFS) is analogous.

(DFS) ⇒ (DEFS), (DFS) ⇒ (DCFS) We first show that Case 1 of (DFS) implies Case 1 of (DEFS). Take two price vectors p, p' such that $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and fix an arbitrary $\Psi \in D_i(p)$. We need to show that there exists a set $\Psi' \in D_i(p')$ that satisfies the conditions of Case 1 of (DEFS).

As the statement is trivial when $D_i(p') = \{\Xi : \Xi \subset \Omega_i\}$, we assume the contrary. In the following, let $\tilde{\Omega}_{\to i} = \{\omega \in \Omega_{\to i} : p'_\omega < p_\omega\}$. Let $\varepsilon_1 = V_i(p') - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(p')} U_i([\Xi; p'])$, and $\varepsilon_2 = \min_{\omega \in \tilde{\Omega}_{\to i}} (p_\omega - p'_\omega)$. Let $\varepsilon = \frac{\min\{\varepsilon_1, \varepsilon_2\}}{2|\Omega_i|}$. Note that by construction, $\varepsilon > 0$.

We now define a price vector q^1 by

$$q^1_\omega = \begin{cases} p_\omega - \varepsilon & \omega \in \Omega_{i\to} \smallsetminus \Psi \text{ or } \omega \in \Psi_{\to i} \\ p_\omega + \varepsilon & \omega \in \Omega_{\to i} \smallsetminus \Psi \text{ or } \omega \in \Psi_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^1) = \{\Psi\}$. Now define q^2 by $q^2_\omega = q^1_\omega$ for all $\omega \in \Omega \smallsetminus \tilde{\Omega}_{\to i}$ and $q^2_\omega = p'_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. We claim that $D_i(q^2) \subseteq D_i(p')$. To see this, fix an arbitrary $\Phi \in D_i(p')$ and an arbitrary $\Xi \notin D_i(p')$. Then we must have

$$U_i([\Phi; q^2]) \geq U_i([\Phi; p']) - |\Phi|\varepsilon > U_i([\Xi; p']) \geq U_i([\Xi; q^2]),$$

where the first and third inequalities follow directly from the definitions of q^2, and the second inequality follows from $|\Phi|\varepsilon \leq |\Omega_i|\varepsilon_1 < V_i(p') - U_i([\Xi; p']) = U_i([\Phi; p']) - U_i([\Xi; p'])$.

We will now show that the condition in Case 1 of Definition 2 is satisfied for any set of trades $\Psi' \in D_i(q^2)$. Take any such Ψ'. Similar to the above, we define $\delta_1 = V_i(q^1) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^1)} U_i([\Xi; q^1])$, $\delta_2 = V_i(q^2) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^2)} U_i([\Xi; q^2])$, and $\delta_3 = \min_{\omega \in \tilde{\Omega}_{\to i}}(q^1_\omega - p'_\omega)$. Let $\delta = \frac{\min\{\delta_1, \delta_2, \delta_3\}}{3|\Omega_i|}$, and define price vector q^3 as

$$q^3_\omega = \begin{cases} q^2_\omega - \delta & \omega \in \Omega_{i\to} \smallsetminus \Psi' \text{ or } \omega \in \Psi'_{\to i} \\ q^2_\omega + \delta & \omega \in \Omega_{\to i} \smallsetminus \Psi' \text{ or } \omega \in \Psi'_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^3) = \{\Psi'\}$. Now define q^4 by $q^4_\omega = q^3_\omega$ for all $\omega \in \Omega \smallsetminus \tilde{\Omega}_{\to i}$ and $q^4_\omega = q^1_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. Similar to the above, we can show that $D_i(q^4) \subseteq D_i(q^1)$, and therefore $D_i(q^4) = \{\Psi\}$. Since $q^3_\omega < q^4_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$ and $q^3_\omega = q^4_\omega$ for all $\omega \in \Omega \smallsetminus \tilde{\Omega}_{\to i}$, we can now apply Case 1 of (DFS) to conclude that Ψ' satisfies the condition in Case 1 of (DEFS).

The proofs that Case 2 of (DFS) implies Case 2 of (DEFS), and that (DFS) implies (DCFS) are completely analogous.

(DEFS) ⇒ (CEFS), (DCFS) ⇒ (CCFS) We first prove Case 1 of (CEFS). Take agent i and any sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$. Define *usable* and *unusable* trades in Y as follows. Take trade $\omega \in Y_{i\to}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' > r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Similarly, take trade $\omega \in Y_{\to i}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' < r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Note that an unusable trade cannot be a part of any contract involved in any optimal choice in $C_i(Y)$. The definitions of trades usable and unusable in Z are completely analogous.

We now construct preliminary price vectors q and q' as follows. First, for every trade $\omega \notin \Omega_i$, $q_\omega = q'_\omega = 0$. Second, for every trade ω unusable in Y, $q_\omega = 0$, and for every trade ω unusable in Z, $q'_\omega = 0$. Next, for any trade $\omega \in \Omega_{i\to}$ usable in Y, $q_\omega = \max\{r : (\omega, r) \in Y\}$, and similarly, for any trade $\omega \in \Omega_{i\to}$ usable in Z, $q'_\omega = \max\{r : (\omega, r) \in Z\}$. Finally, for any trade $\omega \in \Omega_{\to i}$ usable in Y, $q_\omega = \min\{r : (\omega, r) \in Y\}$ and for any trade $\omega \in \Omega_{\to i}$ usable in Z, $q'_\omega = \min\{r : (\omega, r) \in Z\}$.

We now construct price vectors p and p'. First, for any trade $\omega \notin \Omega_i$, $p_\omega = p'_\omega = 0$. Second, for any trade $\omega \in \Omega_i$ that is usable in both Y and Z, let $p_\omega = q_\omega$ and let $p'_\omega = q'_\omega$. Finally, we need to set prices for trades unusable in Y or Z. We already noted that for any trade ω unusable in set Y, it has to be the case that ω is not involved in any contract in any optimal choice in $C_i(Y)$; and likewise, if ω is unusable in Z, then ω is not involved in any contract in any optimal choice in $C_i(Z)$. Thus, in forming prices p and p', we will need to assign to these trades prices that are so large (or small, depending on which side the trade is on) that the corresponding trades are not demanded by agent i.

Let Π be a very large number. For instance, let

$$\Delta_1 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i([\Omega_1; q]) - U_i([\Omega_2; q])|,$$

$$\Delta_2 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i(\Omega_1; q') - U_i(\Omega_2; q')|,$$

and $\Pi = 1 + \Delta_1 + \Delta_2 + \max_{\omega \in \Omega_i} |q_\omega| + \max_{\omega \in \Omega_i} |q'_\omega|$. For all $\omega \in \Omega_{i\to}$ that are unusable in Y (and thus also in Z), let $p_\omega = p'_\omega = -\Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in both Y and Z, let $p_\omega = p'_\omega = \Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in Y but not in Z, let $p_\omega = \Pi$ and $p'_\omega = q'_\omega$. Finally, for all $\omega \in \Omega_{\to i}$ that are unusable in Z but not in Y, let $p_\omega = p'_\omega = q_\omega$. Note that for any such ω, since $Y \subset Z$, $(\omega, q_\omega) \in Z$; also, as ω is unusable in Z, there are no contracts involving ω in any optimal choice in $C_i(Z)$.

Now, $p'_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$. Take any $Y^* \in C_i(Y)$, and let $\Psi = \tau(Y^*)$. By construction, $\Psi \in D_i(p)$. By (DEFS), there exists $\Psi' \in D_i(p')$ such that $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Let $Z^* = \kappa([\Psi', p'])$. Again, by construction, $Z^* \in C_i(Z)$. We now show that this set of contracts satisfies the conditions in Case 1 of (CEFS).

First, take some $y \in Y_{\to i} \smallsetminus Y^*_{\to i}$ and suppose that contrary to what we want to show, $y \in Z^*_{\to i}$. The latter implies that $y = (\omega, p'_\omega)$ for some trade ω, which, in turn, implies that $p_\omega = p'_\omega$ (because $y = (\omega, p'_\omega) \in Y$ and, since $Y \subset Z$, $(\omega, r) \notin Y$ for any $r < p'_\omega$). But then, by construction, $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'_{\to i}$, contradicting $y \in Z^*_{\to i}$. Second, since $Y^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi_{i\to}\}$, $Z^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi'_{i\to}\}$, and $\Psi_{i\to} \subseteq \Psi'_{i\to}$, it is immediate that $Y^*_{i\to} \subseteq Z^*_{i\to}$. This completes the proof that Case 1 of (DEFS) implies Case 1 of (CEFS).

The proofs that Case 2 of (DEFS) implies Case 2 of (CEFS) and that (DCFS) implies (CCFS) are completely analogous.

(DEFS) + (DCFS) $\Rightarrow$ (IDFS) + (IIFS) We first show that (DEFS) and (DCFS) jointly imply (IDFS). Take two price vectors p, p' such that $p \leq p'$. Let $\Psi' \in D_i(p')$ be arbitrary. We have to show that there exists a set of trades $\Psi \in D_i(p)$ such that $e_{i,\omega}(\Psi') \geq e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

First, let p^1 be a price vector such that $p^1_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^1_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$. By (DCFS) there must exist a $\Psi^1 \in D_i(p^1)$ such that $\{\omega \in \Psi'_{i\to} : p^1_\omega = p_\omega\} \subseteq \Psi^1$ and $\Psi^1_{\to i} \subseteq \Psi'_{\to i}$. Now note that $p_\omega = p^1_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \le p^1_\omega$ for all $\omega \in \Omega_{\to i}$. By (DEFS), there must exist a $\Psi \in D_i(p)$ such that $\{\omega \in \Psi_{\to i} : p^1_\omega = p_\omega\} \subseteq \Psi^1$ and $\Psi^1_{i\to} \subseteq \Psi_{i\to}$. Combining this with what we know about Ψ^1, we obtain that $\{\omega \in \Psi_{\to i} : p_\omega = p'_\omega\} = \{\omega \in \Psi_{\to i} : p_\omega = p^1_\omega\} \subseteq \Psi^1_{\to i} \subseteq \Psi'_{\to i}$ and $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} = \{\omega \in \Psi'_{i\to} : p_\omega = p^1_\omega\} \subseteq \Psi^1_{i\to} \subseteq \Psi_{i\to}$. This implies $e_{i,\omega}(\Psi') \ge e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_i$ such that $p'_\omega = p^1_\omega$.

The proof that (DEFS) and (DCFS) jointly imply (IIFS) is completely analogous.

(IFS) ⇒ (DFS) This follows immediately, because the price change conditions in both Cases 1 and 2 of (DFS) are special cases of the price change condition of (IFS). □

C Proofs of the Results in Sections 4, 5, and 6

Proof of Theorem 2

We first show that if the preferences of an agent i are substitutable, then those preferences induce a submodular indirect utility function. It is enough to show that for any two trades $\varphi, \psi \in \Omega_i$ and any prices $p \in \mathbb{R}^\Omega$, $p^{\text{high}}_\varphi > p_\varphi$, and $p^{\text{high}}_\psi > p_\psi$ we have that[15]

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p^{\text{high}}_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p^{\text{high}}_\psi) \\ \ge V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p_\psi). \quad (4)$$

Suppose that $\varphi, \psi \in \Omega_{\to i}$.[16] There are three cases to consider:

1. Suppose that $\varphi \notin \Phi$ for any $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$. Then, by individual rationality, $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p_\psi)$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p^{\text{high}}_\varphi, p_\psi) = 0$$

[15] The definition of submodularity given in Definition 4 is equivalent to the pointwise definition given here; see, e.g., Schrijver (2002).

[16] The other three cases—

1. $\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i\to}$,
2. $\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i\to}$, and
3. $\varphi, \psi \in \Omega_{i\to}$—

are analogous.

and so equation (4) is satisfied, as the left side of (4) must be non-negative.

2. Suppose $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. Then, by individual rationality, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}})$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) = -(p_\varphi - p_\varphi^{\text{high}}) = p_\varphi^{\text{high}} - p_\varphi$$

and so equation (4) is satisfied, as the right side of (4) is (weakly) bounded from above by $p_\varphi^{\text{high}} - p_\varphi$ (with equality in the case that φ is demanded at both $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi)$).

3. Suppose that $\varphi \in \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $\varphi \notin \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. In this case, as the preferences of i are fully substitutable, there exists a unique price $p_\varphi^{\uparrow}$ such that there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\uparrow}, p_\psi^{\text{high}})$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \le p_\varphi^{\uparrow} \le p_\varphi^{\text{high}}$. Similarly, let $p_\varphi^{\downarrow}$ be the unique price at which there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\downarrow}, p_\psi)$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \le p_\varphi^{\downarrow} \le p_\varphi^{\text{high}}$. By the definition of the utility function, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi < p_\varphi^{\uparrow}$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi > p_\varphi^{\uparrow}$; similarly, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi < p_\varphi^{\downarrow}$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi > p_\varphi^{\downarrow}$.

   Since the preferences of i are fully substitutable, $p_\varphi^{\downarrow} \le p_\varphi^{\uparrow}$. Hence,

$$\begin{aligned}
&V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\uparrow}, p_\psi^{\text{high}}) + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\uparrow}, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \\
&\quad = -p_\varphi + p_\varphi^{\uparrow} - 0 \\
&\quad \ge -p_\varphi + p_\varphi^{\downarrow} - 0 \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\downarrow}, p_\psi) + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\downarrow}, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi),
\end{aligned}$$

   which is exactly (4).

Now, suppose that the preferences of i are not substitutable. We suppose moreover that the preferences of i fail the first condition of Defintion 2.[17] Hence, for some price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p_\omega \ge p'_\omega$ for all $\omega \in \Omega_{\to i}$, we have that for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, either $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\to i}$ or

[17] The case where the preferences of i fail the second condition of Defintion 2 is analogous.

$\Psi_{i\to} \not\subseteq \Psi'_{i\to}$. We suppose that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\to i}$; the latter case is analogous. Let $\varphi \in \Psi_{\to i} \smallsetminus \{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\}$. Let p^{high}_φ be a price for trade φ high enough such that φ is not demanded at either $(p^{\text{high}}_\varphi, p_{\Omega\smallsetminus\{\varphi\}})$ or $(p^{\text{high}}_\varphi, p'_{\Omega\smallsetminus\{\varphi\}})$. Hence,

$$V_i(p_\varphi, p'_{\Omega\smallsetminus\{\varphi\}}) - V_i(p^{\text{high}}_\varphi, p'_{\Omega\smallsetminus\{\varphi\}}) = 0$$

while

$$V_i(p_\varphi, p_{\Omega\smallsetminus\{\varphi\}}) - V_i(p^{\text{high}}_\varphi, p_{\Omega\smallsetminus\{\varphi\}}) > 0.$$

Thus we see that V_i is not submodular.

Proof of Theorem 3

The proof is an adaptation of the proof of Theorem 1 in Sun and Yang (2009) to our setting. However, our model is more general, and as we do not impose either monotonicity or boundedness on the valuation functions, and do not require that the seller values each bundle at 0 and thus sells everything that he could sell, we have to carefully ensure that the approach of Sun and Yang (2009) remains valid.

"If" Direction We show first that (IDFS) and (IIFS) imply the single improvement property. Fix an arbitrary price vector $p \in \mathbb{R}^\Omega$ and a set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$. Fix a set of trades $\Xi \in D_i(p)$. We focus exclusively on the trades in Ψ and Ξ by rendering all other trades that agent i is involved in irrelevant. To this end, we first define a very high price Π,

$$\Pi \equiv \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i([\Omega_1; p]) - U_i([\Omega_2; p])| + \max_{\omega \in \Omega_i} |p_\omega| + 1,$$

and then, starting from p, we construct a preliminary price vector p' as follows:

$$p'_\omega = \begin{cases} p_\omega & \omega \in \Psi \cup \Xi,\ \omega \notin \Omega_i \\ p_\omega + \Pi & \omega \in \Omega_{\to i} \smallsetminus (\Psi \cup \Xi) \\ p_\omega - \Pi & \omega \in \Omega_{i\to} \smallsetminus (\Psi \cup \Xi). \end{cases}$$

Observe that $\Psi \notin D_i(p')$ and $\Xi \in D_i(p')$. As $\Psi \neq \Xi$, we have to consider two cases (each with several subcases), which taken together will show that there exists a set of trades $\Phi' \neq \Psi$ that satisfies conditions 2 and 3 of Definition 5 and, in addition, $U_i[(\Phi'; p)] \geq U_i([\Psi; p])$.

"Only If" Direction Case 1: $\Xi \setminus \Psi \neq \emptyset$.

Select a trade $\xi_1 \in \Xi \setminus \Psi$. Without loss of generality, assume that agent i is a buyer in ξ_1 (the case where i is a seller is completely analogous).
Starting from p', construct a modified price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \setminus ((\Xi_{\to i} \setminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i\to}),\ \omega \notin \Omega_i \\ p'_\omega + \Pi & \omega \in (\Xi_{\to i} \setminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i\to}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\xi_1 \in \Xi$, and $p'_{\xi_1} = p''_{\xi_1}$, full substitutability (definition A.5) implies that there exists $\Xi'' \in D_i(p'')$ such that $\xi_1 \in \Xi''$. Second, observe that following the price change from p' to p'', $(\Xi''_{\to i} \setminus \Psi_{\to i}) \subseteq \{\xi_1\}$ and $\Psi_{i\to} \subseteq \Xi''_{i\to}$. Thus, $\Xi''_{\to i} \setminus \Psi_{\to i} = \{\xi_1\}$ and $\Psi_{i\to} \subseteq \Xi''_{i\to}$. We consider three subcases.

Subcase (a): If $\Xi''_{i\to} \setminus \Psi_{i\to} \neq \emptyset$, let $\xi_2 \in \Xi''_{i\to} \setminus \Psi_{i\to}$.

Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \setminus ((\Xi_{i\to} \setminus (\Psi_{i\to} \cup \{\xi_2\})) \cup \Psi_{\to i}),\ \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i\to} \setminus (\Psi_{i\to} \cup \{\xi_2\})) \cup \Psi_{\to i}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\xi_2 \in \Xi''$, and $p''_{\xi_2} = p'''_{\xi_2}$, full substitutability (definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\xi_2 \in \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \subseteq \Xi'''$ and $\Xi''' \setminus \Psi \subseteq \{\xi_1, \xi_2\}$. Thus, $\Psi \setminus \Xi''' = \emptyset$ and $\Xi''' \setminus \Psi = \{\xi_1, \xi_2\}$ or $\{\xi_2\}$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are making one new sale ξ_2, i.e., $e_{i,\xi_2}(\Psi) > e_{i,\xi_2}(\Xi''')$ with $\xi_2 \in \Omega_{i\to} \setminus \Psi$, and (possibly) making one new purchase ξ_1, i.e. $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \setminus \Psi$.

Subcase (b): If $\Xi''_{i\to} \setminus \Psi_{i\to} = \emptyset$ and $\Psi_{\to i} \setminus \Xi''_{\to i} \neq \emptyset$, let $\psi \in \Psi_{\to i} \setminus \Xi''_{\to i}$.

Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \setminus ((\Xi_{i\to} \setminus \Psi_{i\to}) \cup (\Psi_{\to i} \setminus \{\psi\})) \text{ or } \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i\to} \setminus \Psi_{i\to}) \cup (\Psi_{\to i} \setminus \{\psi\}). \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi \notin \Xi''$, and $p''_\psi = p'''_\psi$, by full substitutability (definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \setminus \Xi''' \subseteq \{\psi\}$ and $\Xi''' \setminus \Psi \subseteq \{\xi_1\}$. Thus, $\Psi \setminus \Xi''' = \{\psi\}$ and $\Xi''' \setminus \Psi = \{\xi_1\}$ or $\emptyset$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one purchase ψ, i.e., $e_{i,\psi}(\Psi) > e_{i,\psi}(\Xi''')$ with $\psi \in \Psi_{\to i}$, and (possibly) making one new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Subcase (c): $\Xi'' = \Psi \cup \{\xi_1\}$.
In this subcase, let $p''' = p''$ and $\Xi''' = \Xi''$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only difference from Ψ to Ξ''' is making a new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Case 2: $\Xi \smallsetminus \Psi = \emptyset$ and $\Psi \smallsetminus \Xi \neq \emptyset$.

Select a trade $\psi_1 \in \Psi \smallsetminus \Xi$. Without loss of generality, assume that agent i is a buyer in ψ_1 (the case where i is a seller is completely analogous).
Starting from p', construct price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{\to i} \smallsetminus \{\psi_1\}) \text{ or } \omega \notin \Omega_i \\ p'_\omega - \Pi & \omega \in \Psi_{\to i} \smallsetminus \{\psi_1\}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\psi_1 \notin \Xi$, and $p'_{\psi_1} = p''_{\psi_1}$, full substitutability (definition A.6) implies that there exists $\Xi'' \in D_i(p'')$ such that $\psi_1 \notin \Xi''$. Second, observe that following the price change from p' to p'', $\Xi'' \subseteq \Psi$ and $\Psi_{\to i} \smallsetminus \Xi''_{\to i} \subseteq \{\psi_1\}$. Thus, $\Psi_{\to i} \smallsetminus \Xi''_{\to i} = \{\psi_1\}$ and $\Xi'' \subseteq \Psi$. We consider two subcases.

Subcase (a): If $\Psi_{i\to} \smallsetminus \Xi''_{i\to} \neq \emptyset$, let $\psi_2 \in \Psi_{i\to} \smallsetminus \Xi''_{i\to}$.

Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{i\to} \smallsetminus \{\psi_2\}) \text{ or } \omega \notin \Omega_i \\ p''_\omega + \Pi & \omega \in \Psi_{i\to} \smallsetminus \{\psi_2\}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi_2 \notin \Xi''$, and $p''_{\psi_2} = p'''_{\psi_2}$, full substitutability (definition A.5) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi_2 \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Xi''' \subseteq \Psi$ and $\Psi \smallsetminus \Xi''' \subseteq \{\psi_1, \psi_2\}$. Thus, $\Xi''' \smallsetminus \Psi = \emptyset$ and $\Psi \smallsetminus \Xi''' = \{\psi_1, \psi_2\}$ or $\{\psi_2\}$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one sale ψ_2, i.e., $e_{i,\psi_2}(\Psi) < e_{i,\psi_2}(\Xi''')$ with $\psi_1 \in \Omega_{i\to} \smallsetminus \Psi$, and (possibly) canceling one purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) > e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Psi_{\to i}$.

Subcase (*b*): $\Xi'' = \Psi \smallsetminus \{\psi_1\}$.

In this subcase, let $p''' = p''$ and $\Xi''' = \Xi''$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only difference from Ψ to Ξ''' is canceling purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) < e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Taking together all the final statements from each subcase where for notational convenience in each subcase we let $\Phi' \equiv \Xi'''$, we obtain that we always have a price vector p''' and the sets Ψ and Φ' that satisfy conditions (2) and (3) of definition 5. Moreover, since we always have $\Phi \in D_i(p''')$, $U_i([\Phi', p''']) \geq U_i([\Psi, p'''])$.

Next, we show that $U_i([\Phi', p''']) - U_i([\Psi, p''']) \geq 0$ implies $U_i([\Phi', p]) \geq U_i([\Psi, p])$. First, observe that when taking the difference the prices of all trades $\omega \in \Phi' \cap \Psi$ cancel each other out. Thus, replacing the prices p'''_ω with p_ω for all trades $\omega \in \Phi' \cap \Psi$ leaves the difference unchanged. Second, observe that in all previous subcases, the construction of p''' implies that for all $\omega \in ((\Psi \smallsetminus \Phi') \cup (\Phi' \smallsetminus \Psi))$, $p_\omega = p'''_\omega$. Combining the two observations above, $U_i([\Phi', p''']) - U_i([\Psi, p''']) = U_i([\Phi', p]) - U_i([\Psi, p])$, and therefore $U_i([\Phi', p]) \geq U_i([\Psi, p])$.

We now show that there exists a set of trades Φ that satisfies all conditions of Definition 5. Since $\Psi \notin D_i(p)$, $V_i(p) > U_i([\Psi; p])$. Since i's utility is continuous in prices, there exists $\varepsilon > 0$ such that $V_i(q) > U_i([\Psi; q])$ where q is defined as follows:

$$q_\omega = \begin{cases} p_\omega + \varepsilon & \omega \in (\Omega_{\to i} \smallsetminus \Psi_{\to i}) \cup \Psi_{i \to} \\ p_\omega - \varepsilon & \omega \in (\Omega_{i \to} \smallsetminus \Psi_{i \to}) \cup \Psi_{\to i}. \end{cases}$$

Our arguments above imply that there exists a set of trades $\Phi \neq \Psi$ such that $U_i([\Phi; q]) \geq U_i([\Psi; q])$. Using the construction of q, we obtain $U_i([\Phi; p]) - U_i([\Psi; p]) = U_i([\Phi; q]) - U_i([\Psi; q]) + \varepsilon |(\Psi \smallsetminus \Phi) \cup (\Phi \smallsetminus \Psi)| > U_i([\Phi; q]) - U_i([\Psi; q]) \geq 0$. Thus, $U_i([\Phi; p]) > U_i([\Psi; p])$. This completes the proof that (IDFS) and (IIFS) imply the single improvement property.

We now show that the single improvement property implies full substitutability (DCFS). More specifically, we will establish that single improvement implies the first condition of Definition A.4. The proof that the second condition of Definition A.4 is also satisfied uses a completely analogous argument.

Let $p \in \mathbb{R}^\Omega$ and $\Psi \in D_i(p)$ be arbitrary. It is sufficient to establish that for any $p' \in \mathbb{R}^\Omega$ such that $p'_\psi > p_\psi$ for some $\psi \in \Omega_{\to i}$ and $p'_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$, there exists a set of trades $\Psi' \in D_i(p')$ that satisfies the first condition of Definition A.4.

Fix one $p' \in \mathbb{R}^\Omega$ that satisfies the conditions mentioned in the previous paragraph and let $\psi \in \Omega_{\to i}$ be the one trade for which $p'_\psi > p_\psi$. Note that if either $\psi \notin \Psi$ or $\Psi \in D_i(p')$, there is nothing to show. From now on, assume that $\psi \in \Psi$ and $\Psi \notin D_i(p')$.

For any real number $\varepsilon > 0$ define a price vector $p^\varepsilon \in \mathbb{R}^\Omega$ by setting $p^\varepsilon_\psi = p_\psi + \varepsilon$ and $p^\varepsilon_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. Let $\Delta \equiv \max\{\varepsilon : \Psi \in D_i(p^\varepsilon)\}$. Note that Δ is well defined since i's utility function is continuous in prices. Furthermore, given that $\Psi \notin D_i(p')$, we must have $\Delta < p'_\psi - p_\psi$.

Next, for any integer n, define a price vector $p^n \in \mathbb{R}^\Omega$ by setting $p^n_\psi = p_\psi + \Delta + \frac{1}{n}$ and $p^n_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. By the definition of Δ we must have $\Psi \notin D_i(p^n)$ for all $n > 0$. By the single improvement property, this implies that for all $n > 0$, there exists a set of trades Φ^n such that the following conditions are satisfied:

1. $U_i([\Psi, p^n]) < U_i([\Phi^n, p^n])$,

2. there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi^n)$, and

3. there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi^n)$.

Note that we must have $\psi \notin \Phi^n$ for all $n \geq 1$. This follows since for for any $n \geq 1$ and any set of trades Φ such that $\psi \in \Phi$, $U_i([\Phi; p^n]) = U_i([\Phi; p]) - \Delta - \frac{1}{n} \leq U_i([\Psi; p]) - \Delta - \frac{1}{n} = U_i([\Psi; p^n])$ given that $\Psi \in D_i(p)$.

Conditions 2 and 3 imply that for all $n > 0$, we must have $\{\omega \in \Psi_{\to i} : p'_\omega = p_\omega\} = \{\omega \in \Psi_{\to i} : p^n_\omega = p_\omega\} \subseteq \Phi^n_{\to i}$ and $\Phi^n_{i \to} \subseteq \Psi_{i \to}$.

Since the set of trades is finite, it is without loss of generality to assume that there is a set of trades $\Phi^* \in \Omega_i$ and an integer $\bar{n}$ such that $\Phi^n = \Phi^*$ for all $n \geq \bar{n}$. Since i's utility function is continous with respect to prices and $p^n \to p^\Delta$, we must have $U_i([\Phi^*; p^\Delta]) \geq U_i([\Psi; p^\Delta])$. Since $\Psi \in D_i(p^\Delta)$, this implies $\Phi^* \in D_i(p^\Delta)$. Since $\Delta < p'_\psi - p_\psi$ and V_i is decreasing in the prices of trades for which i is a buyer, we must have $V_i(p^\Delta) \geq V_i(p')$. Since $\psi \notin \Phi^*$, we have that $U_i([\Phi^*; p']) = U_i([\Phi^*; p^\Delta]) = V_i(p^\Delta)$. Hence, $\Phi^* \in D_i(p')$ and setting $\Psi' \equiv \Phi^*$ yields a set that satisfies the first condition of Definition A.4.

Proof of Theorem 4

The proof is an adaptation of the proof of Theorem 1 in Gul and Stacchetti (1999). Since we do not impose either monotonicity or boundedness on valuation functions, we need to check that their proof strategy continues to work in our setting.

Throughout the proof, for any price vector $p \in \mathbb{R}^\Omega$, we denote by $\tilde{D}_i(p)$ the sets of objects that correspond to the optimal sets of trades in $D_i(p)$.

We show first that the single improvement property in object-language implies the no complementarities condition. Let p be an arbitrary price vector and $\mathbf{\Phi}, \mathbf{\Psi} \in \tilde{D}_i(p)$ be arbitrary. Let $\bar{\mathbf{\Psi}} \subseteq \mathbf{\Psi} \smallsetminus \mathbf{\Phi}$ be arbitrary. Let $\mathbf{\Xi} \in \tilde{D}_i(p)$ be a set of objects such that $\mathbf{\Xi} \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$ and $\mathbf{\Psi} \smallsetminus \bar{\mathbf{\Psi}} \subseteq \mathbf{\Xi}$, and such that there is no $\mathbf{\Xi}' \in \tilde{D}_i(p)$ for which $\mathbf{\Xi}' \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$, $\mathbf{\Psi} \setminus \bar{\mathbf{\Psi}} \subseteq \mathbf{\Xi}'$, and $|\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}| < |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|$. If $\mathbf{\Xi} \cap \bar{\mathbf{\Psi}} = \emptyset$, we are done. If not, let Π be a very large number[18] and define $p(\varepsilon)$ by setting $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = \Pi$ if $\boldsymbol{\omega} \in \mathbf{\Omega}_{\to i} \smallsetminus (\mathbf{\Phi} \cup \mathbf{\Psi})$, $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = -\Pi$ if $\boldsymbol{\omega} \in \mathbf{\Omega}_{i\to} \smallsetminus (\mathbf{\Phi} \cup \mathbf{\Psi})$, $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})}$ if $\boldsymbol{\omega} \in (\mathbf{\Phi} \cup \mathbf{\Psi}) \smallsetminus \bar{\mathbf{\Psi}}$, and $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})} + \varepsilon$ if $\boldsymbol{\omega} \in \bar{\mathbf{\Psi}}$. Note that for all $\varepsilon > 0$ we must have $\mathbf{\Phi} \in \tilde{D}_i(p(\varepsilon))$ (since $\bar{\mathbf{\Psi}} \subseteq \mathbf{\Psi} \setminus \mathbf{\Phi}$) and $U_i([\mathbf{\Phi}; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$. Since $\mathbf{\Xi} \in \tilde{D}_i(p)$, we must have $u_i(\mathbf{\Xi}) \neq -\infty$. Hence, we can apply the single improvement property (in object-language) to infer that there must exist a set of objects $\mathbf{\Xi}'$ such that $|\mathbf{\Xi}' \setminus \mathbf{\Xi}| \leq 1$, $|\mathbf{\Xi} \setminus \mathbf{\Xi}'| \leq 1$, and $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$. Given the definition of $p(\varepsilon)$ and Π, we must have $\mathbf{\Xi}' \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$. Since $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$ holds for arbitrarily small values of ε, we must have $\mathbf{\Xi}' \in \tilde{D}_i(p)$. But $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$ is equivalent to $U_i([\mathbf{\Xi}'; p]) - |\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}|\varepsilon > U_i([\mathbf{\Xi}; p]) - |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|\varepsilon$. Given that $\mathbf{\Xi}, \mathbf{\Xi}' \in \tilde{D}_i(p)$, the last inequality is equivalent to $|\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}| < |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|$ and we thus obtain a contradiction to the definition of $\mathbf{\Xi}$. Hence, it has to be the case that $\mathbf{\Xi} \cap \bar{\mathbf{\Psi}} = \emptyset$ and this completes the proof that single improvement implies no complementarities.

Next, we show that the generalized no complementarities condition implies object-language full substitutability. Let p, p' be two price vectors such that $p \leq p'$. Let $\mathbf{\Psi} \in \tilde{D}_i(p)$ be arbitrary.[19] Let $\tilde{\mathbf{\Omega}}_i = \{\boldsymbol{\omega} \in \mathbf{\Omega}_i : p_{\mathsf{t}(\boldsymbol{\omega})} < p'_{\mathsf{t}(\boldsymbol{\omega})}\}$. The proof will proceed by induction on $|\tilde{\mathbf{\Omega}}_i|$. Consider first the case of $|\tilde{\mathbf{\Omega}}_i| = 1$ and let $\tilde{\mathbf{\Omega}}_i = \{\boldsymbol{\omega}\}$. Clearly, if $\boldsymbol{\omega} \notin \mathbf{\Psi}$ or $\mathbf{\Psi} \in \tilde{D}_i(p')$, there is nothing to show. So suppose that $\boldsymbol{\omega} \notin \mathbf{\Psi}$ and that $\mathbf{\Psi} \notin \tilde{D}_i(p')$. For any $\varepsilon \geq 0$, define a price vector $p(\varepsilon)$ by setting $p_{\mathsf{t}(\boldsymbol{\varphi})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\varphi})}$ for all $\boldsymbol{\varphi} \neq \boldsymbol{\omega}$, and $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})} + \varepsilon$. Let $\bar{\varepsilon} = \max\{\varepsilon : \mathbf{\Psi} \in \tilde{D}_i(p(\varepsilon))\}$ and note that $\bar{\varepsilon} < p'_{\mathsf{t}(\boldsymbol{\omega})} - p_{\mathsf{t}(\boldsymbol{\omega})}$ given that $\mathbf{\Psi} \notin \tilde{D}_i(p')$. Consider some $\varepsilon > \bar{\varepsilon}$ and fix a set of objects $\mathbf{\Phi} \in \tilde{D}_i(p(\varepsilon))$. It is easy to see that $\boldsymbol{\omega} \notin \mathbf{\Phi}$ and that $\mathbf{\Phi} \in \tilde{D}_i(p(\bar{\varepsilon}))$. By the generalized no complementarities condition, there must exist a set of objects $\mathbf{\Xi} \subseteq \mathbf{\Phi}$ such that $\mathbf{\Psi}' := \mathbf{\Psi} \setminus \{\boldsymbol{\omega}\} \cup \Xi \in \tilde{D}_i(p(\bar{\varepsilon}))$. Clearly, we must also have $\mathbf{\Psi}' \in \tilde{D}_i(p')$ and this completes the proof in case of $|\tilde{\mathbf{\Omega}}_i| = 1$. Now suppose that the statement has already been established for all pairs of price vectors p, p' such that $|\tilde{\mathbf{\Omega}}_i| \leq K$ for some $K \geq 1$. Consider two price vectors p, p' such that $|\tilde{\mathbf{\Omega}}_i| = K + 1$. Fix a set of objects $\mathbf{\Psi} \in \tilde{D}_i(p)$. Let $\boldsymbol{\omega} \in \tilde{\mathbf{\Omega}}_i$ be arbitrary and consider a

[18]For instance, let

$$\Delta = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i(\Omega_1; p) - U_i(\Omega_2; p)|,$$

and $\Pi = 1 + \Delta + \max_{\omega \in \Omega_i} |p_\omega|$.

[19]There is no need to rule out the possibility of several optimal bundles of objects in this proof.

price vector p'' such that $p''_{\mathrm{t}(\boldsymbol{\omega})} = p_{\mathrm{t}(\boldsymbol{\omega})}$ and $p''_{\mathrm{t}(\boldsymbol{\varphi})} = p'_{\mathrm{t}(\boldsymbol{\varphi})}$ for all $\boldsymbol{\varphi} \neq \boldsymbol{\omega}$. By the inductive assumption, there is a set $\boldsymbol{\Psi}'' \in \tilde{D}_i(p'')$ such that $\{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p''_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} \subseteq \boldsymbol{\Psi}''$. Note that $\{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p'_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} = \{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p''_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} \setminus \{\boldsymbol{\omega}\}$. Applying the inductive assumption one more time, there has to be a set $\boldsymbol{\Psi}' \in \tilde{D}_i(p')$ such that $\boldsymbol{\Psi}'' \setminus \{\boldsymbol{\omega}\} \subseteq \boldsymbol{\Psi}'$. Combining this with the previous arguments, we obtain $\{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p'_{\mathrm{t}(\boldsymbol{\varphi})} = p_{\mathrm{t}(\boldsymbol{\varphi})}\} \subseteq \boldsymbol{\Psi}'$. This completes the proof.

Proof of Theorem 5

As Ω is finite and non-empty, for each agent i the domain of u_i is bounded and non-empty. Hence, by Part (b) of Theorem 7 of Murota and Tamura (2003), we see that u_i is $M^\natural$-concave over objects if and only if the preferences of i have the single-improvement property.[20] The result then follows from Theorem 3.

Proof of Theorem 6

The indirect utility function for $\hat{u}_i^{(\Phi,p_\Phi)}$ is given by

$$\begin{aligned}
\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi}) &\equiv \max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi) + \sum_{\xi\in\Xi_{\to i}} p_\xi - \sum_{\xi\in\Xi_{\to i}} p_\xi\right\} + \sum_{\psi\in\Psi_{\to i}} p_\xi - \sum_{\psi\in\Psi_{\to i}} p_\xi\right\} \\
&= \max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi) + \sum_{\lambda\in\Xi_{\to i}\cup\Psi_{\to i}} p_\lambda - \sum_{\lambda\in\Xi_{i\to}\cup\Psi_{i\to}} p_\lambda\right\}\right\} \\
&= \max_{\Lambda\subseteq\Omega}\left\{u_i(\Lambda) + \sum_{\lambda\in\Lambda_{\to i}} p_\lambda - \sum_{\lambda\in\Lambda_{i\to}} p_\lambda\right\}.
\end{aligned}$$

Hence, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi}) = V_i(p_{\Omega\smallsetminus\Phi}, p_\Phi)$. Now, $V_i(p)$ is submodular over $\mathbb{R}^\Omega$ by Theorem 2. As a submodular function restricted to a subspace of its domain is still submodular, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi})$ is submodular over $\mathbb{R}^{\Omega\smallsetminus\Phi}$. Hence, by Theorem 2, we see that $\hat{u}_i^{(\Phi,p_\Phi)}$ is fully substitutable.

Proof of Theorem 7

We suppose, by way of contradiction, that u_J does not induce fully substitutable preferences over trades in $\Omega \smallsetminus \Omega^J$. By Corollary 1 of Hatfield et al. (2013), there exist fully substitutable

[20]Strictly speaking, Theorem 7(b) shows the equivalence of $M^\natural$-convexity and the (M$^\natural$-SI) property of a function f. It is, however, immediate that this result implies the equivalence of $M^\natural$-concavity and the single-improvement property for a function $g = -f$.

preferences $\tilde{u}_i$ for the agents $i \in I \setminus J$ such that no competitive equilibrium exists for the *modified economy* with

1. set of agents $(I \setminus J) \cup \{J\}$,
2. set of trades $\Omega \setminus \Omega^J$,
3. and valuation function for agent J given by u_J.[21]

Now, we consider the *original economy* with

1. set of agents I,
2. set of trades Ω,
3. valuations for $i \in I \setminus J$ given by $\tilde{u}_i$, and
4. valuations for $j \in J$ given by u_j.

Let $[\Psi; p]$ be a competitive equilibrium of this economy; such an equilibrium must exist by Theorem 1 of Hatfield et al. (2013).

Claim. $[\Psi \setminus \Omega^J; p_{\Omega \setminus \Omega^J}]$ *is a competitive equilibrium of the modified economy.*

Proof. It is immediate that $[\Psi \setminus \Omega^J]_i \in D_i(p_{\Omega \setminus \Omega^J})$ for all $i \in I \setminus J$. Moreover, since Ψ is individually-optimal for each $j \in J$ in the original economy (at prices p),

$$u_j(\Psi) + \sum_{\psi \in \Psi_{j\to}} p_\psi - \sum_{\psi \in \Psi_{\to j}} p_\psi \geq u_j(\Phi) + \sum_{\varphi \in \Phi_{j\to}} p_\varphi - \sum_{\varphi \in \Phi_{\to j}} p_\varphi \tag{5}$$

for every $\Phi \subseteq \Omega$. Summing (5) over all $j \in J$ and simplifying, we obtain

$$\sum_{j \in J} \left(u_j(\Psi) + \sum_{\psi \in \Psi_{j\to}} p_\psi - \sum_{\psi \in \Psi_{\to j}} p_\psi \right) \geq \sum_{j \in J} \left(u_j(\Phi) + \sum_{\varphi \in \Phi_{j\to}} p_\varphi - \sum_{\varphi \in \Phi_{\to j}} p_\psi \right)$$

$$\sum_{j \in J} \left(u_j(\Psi) + \sum_{\psi \in [\Psi \setminus \Omega^J]_{j\to}} p_\psi - \sum_{\psi \in [\Psi \setminus \Omega^J]_{\to j}} p_\psi \right) \geq \sum_{j \in J} \left(u_j(\Phi) + \sum_{\varphi \in [\Phi \setminus \Omega^J]_{j\to}} p_\varphi - \sum_{\varphi \in [\Phi \setminus \Omega^J]_{\to j}} p_\psi \right)$$

$$\sum_{j \in J} u_j(\Psi) + \sum_{\psi \in [\Psi \setminus \Omega^J]_{J\to}} p_\psi - \sum_{\psi \in [\Psi \setminus \Omega^J]_{\to J}} p_\psi \geq \sum_{j \in J} u_j(\Phi) + \sum_{\varphi \in [\Phi \setminus \Omega^J]_{J\to}} p_\varphi - \sum_{\varphi \in [\Phi \setminus \Omega^J]_{\to J}} p_\psi. \quad \square$$

[21] Technically, in order to apply Corollary 1 of Hatfield et al. (2013), we must have that for every pair (i, j) of distinct agents in I, there exists a trade ω such that $b(\omega) = i$ and $s(\omega) = j$. For any pair (i, j) of distinct agents in I such that no such trade ω exists, we can augment the economy by adding the requisite trade ω and, if $i \in J$, letting $\bar{u}_i(\Psi \cup \{\omega\}) = u^i(\Psi)$ (and similarly for j). It is immediate that $\bar{u}_i$ is substitutable if and only if u_i is substitutable.

The preceding claim shows that $[\Psi \smallsetminus \Omega^J; p_{\Omega \smallsetminus \Omega^J}]$ is a competitive equilibrium of the modified economy, contradicting the earlier conclusion that no competitive equilibrium exists in the modified economy. Hence, we see that u_J must be fully substitutable.

Proof of Theorem 8

The proof of this result is very close to Step 1 in the proof of Theorem 1 of Hatfield et al. (2013). The only differences are that in that paper, all trades could be bought out, and the price for buying them out was set to a very large number—the same for all trades. By contrast, in Theorem 8 of the current paper we allow for the possibility that only a subset of trades can be bought out, and that the prices at which these trades can be bought out can be different, and are not necessarily large. Adapting Step 1 of the proof of Theorem 1 of Hatfield et al. (2013) to the current more general setting is straightforward, but we include the proof for completeness.

Consider the fully substitutable valuation function u_i, and take any trade $\varphi \in \Omega_{i\to} \cap \Phi$. Consider a modified valuation function u_i^φ:

$$u_i^\varphi(\Psi) = \max\left\{u_i(\Psi), u_i(\Psi \smallsetminus \{\varphi\}) - \Pi_\varphi\right\}.$$

That is, this valuation function allows (but does not require) agent i to pay Π_φ instead of forming one particular trade, φ. Let us show that the valuation function u_i^φ is fully substitutable.

To see this, consider utility U_i^φ and demand D_i^φ corresponding to valuation u_i^φ. We show that D_i^φ satisfies the (IFS) condition (Definition 3). Fix two price vectors p and p' such that $p \le p'$ and $|D_i^\varphi(p)| = |D_i^\varphi(p')| = 1$. Take the unique $\Psi \in D_i^\varphi(p)$ and $\Psi' \in D_i^\varphi(p')$. We need to show that for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$, $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$.

Let price vector q coincide with p on all trades other than φ, and set $q_\varphi = \min\{p_\varphi, \Pi_\varphi\}$. Note that if $p_\varphi < \Pi_\varphi$, then $p = q$ and $D_i^\varphi(p) = D_i(p)$. If $p_\varphi > \Pi_\varphi$, then under utility U_i^φ, agent i always wants to form trade φ at price p_φ, and the only decision is whether to "buy it out" or not at the cost Π_φ; i.e., the agent's effective demand is the same as under price vector q. Thus, $D_i^\varphi(p) = \{\Xi \cup \{\varphi\} : \Xi \in D_i(q)\}$. Finally, if $p_\varphi = \Pi_\varphi$, then $p = q$ and $D_i^\varphi(p) = D_i(p) \cup \{\Xi \cup \{\varphi\} : \Xi \in D_i(p)\}$. Construct price vector q' corresponding to p' analogously.

Now, if $p_\varphi \le p'_\varphi < \Pi_\varphi$, then $D_i^\varphi(p) = D_i(p)$, $D_i^\varphi(p') = D_i(p')$, and thus $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ follows directly from (IFS) for demand D_i.

If $\Pi_\varphi \le p_\varphi \le p'_\varphi$, then (since we assumed that D_i^φ was single-valued at p and p') it has to be the case that D_i is single-valued at the corresponding price vectors q and q'. Let $\Xi \in D_i(q)$ and $\Xi' \in D_i(q')$. Then $\Psi = \Xi \cup \{\varphi\}$, $\Psi' = \Xi' \cup \{\varphi\}$, and the statement follows from the (IFS) condition for demand D_i, because $q \leq q'$.

Finally, if $p_\varphi < \Pi_\varphi \leq p'_\varphi$, then $p = q$, Ψ is the unique element in $D_i(p)$, and Ψ' is equal to $\Xi' \cup \{\varphi\}$, where Ξ' is the unique element in $D_i(q')$. Then for $\omega \neq \varphi$, the statement follows from (IFS) for demand D_i, because $p \leq q'$. For $\omega = \varphi$, the statement does not need to be checked, because $p_\varphi < p'_\varphi$.

Thus, in this case, valuation function u_i^φ is fully substitutable. The proof for the case when $\varphi \in \Omega_{\to i}$ is completely analogous.

To complete the proof of Theorem 8, it is now enough to note that valuation function $\hat{u}_i(\Psi) = \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}$ can be obtained from the original valuation u_i by allowing agent i to "buy out" all of the trades in set Φ, one by one, and since the preceding argument shows that each such transformation preserves substitutability (and Ω_i is finite), the valuation function $\hat{u}_i$ is substitutable as well.

Proof of Theorem 9

We prove the Law of Aggregate Demand; the proof of the Law of Aggregate Supply is analogous.

Fix a fully substitutable valuation function u_i for agent i. Take two finite sets of contracts Y and Y' such that $|C_i(Y)| = |C_i(Y')| = 1$, $Y_{i\to} = Y'_{i\to}$, and $Y_{\to i} \subseteq Y'_{\to i}$. Assume that for any $\omega \in \Omega_{i\to}$, $(\omega, r) \in Y_{i\to}$ and $(\omega, r') \in Y_{i\to}$ implies $r = r'$ (this is without loss of generality, because for a given trade in $\Omega_{i\to}$, agent i, as a seller, can only choose a contract with the highest price available for that trade, and thus we can disregard all other contracts involving that trade). Let $W \in C_i(Y)$ and $W' \in C_i(Y')$. Define a modified valuation $\tilde{u}_i$ on $\tau(Y'_i)$ for agent i by setting, for each $\Psi \subseteq \tau(Y'_i)$,

$$\tilde{u}_i(\Psi) = u_i(\Psi_{\to i} \cup (\tau(Y') \smallsetminus \Psi)_{i\to}).$$

Let $\tilde{C}_i$ denote the associated choice correspondence. By construction,

$$\tilde{u}_i(\Psi) = u_i(\mathfrak{o}_i(\Psi)), \tag{6}$$

where here the object operator is defined with respect to underlying set of trades $\tau(Y')$:

$$\mathfrak{o}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \tau(Y') \smallsetminus \Psi_{i\to}\}.$$

As the preferencs of i are fully substitutable, the restriction of those preferences to $\tau(Y')$ is fully substitutable, as well. Object-language full substitutability of those preferences, as well as (6), together imply that $\tilde{u}_i$ satisfies the gross substitutability condition of Kelso and Crawford (1982).

Now, we must have $\tilde{C}_i(Y) = \{W_{\to i} \cup (Y' \smallsetminus W)_{i\to}\}$ and $\tilde{C}_i(Y') = \{W'_{\to i} \cup (Y' \smallsetminus W')_{i\to}\}$. As we assume quasilinearity, the Law of Aggregate Demand for two-sided markets applies to $\tilde{C}_i$ (by Theorem 7 of Hatfield and Milgrom (2005)). As $Y \subseteq Y'$, this implies that $|W'_{\to i} \cup (Y' \smallsetminus W')_{i\to}| \geq |W_{\to i} \cup (Y' \smallsetminus W)_{i\to}|$. The last inequality is equivalent to $|W'_{\to i}| - |W_{\to i}| \geq |W'_{i\to}| - |W_{i\to}|$, which is precisely the Law of Aggregate Demand.

The proof that the Law of Aggregate Demand for the case in which choice correspondences are single-valued implies the more general case in which they can be multi-valued is analogous to the proof of the implication (DFS)⇒(DEFS) of Theorem B.1.

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