

# Pure-Strategy Equilibrium in the Generalized First-Price Auction\*

Michael Ostrovsky<sup>†</sup>      Andrzej Skrzypacz<sup>‡</sup>

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## Abstract

We revisit the classic result on the (non-)existence of pure-strategy Nash equilibria in the Generalized First-Price Auction for sponsored search advertising and show that the conclusion may be reversed when ads are ranked based on the product of stochastic quality scores and bid amounts, rather than solely on the bids or on the product of bids and deterministic quality scores. Moreover, the expected revenue in the pure strategy equilibrium of the Generalized First-Price Auction may substantially exceed that of the Generalized Second-Price Auction.

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<sup>†</sup>ostrovsky@stanford.edu. Stanford University and NBER.

<sup>‡</sup>skrz@stanford.edu. Stanford University.

# 1 Introduction

The initial design of sponsored search auctions, implemented in 1997 by the company GoTo (later renamed Overture and subsequently acquired by Yahoo), used a very simple payment rule. Namely, each advertiser, following a click from a user, paid to GoTo the bid that it submitted to the system: e.g., if an advertiser submitted a bid of \$1, then that was the amount charged every time some user clicked on that advertiser’s ad.

This format, dubbed the Generalized First-Price Auction (GFP), was intuitive and easy to explain to the advertisers, but it suffered from a serious shortcoming. The ads on the results page were sorted purely based on the submitted bids (in the descending order), and so each advertiser, given a particular position, had an incentive to outbid the next highest competitor only by the smallest possible amount – 1 cent. This incentive structure resulted in the non-existence of pure-strategy equilibria, and instead pronounced rapid “cycling” patterns, in which advertisers would continually outbid each other by 1 cent, then one advertiser would drop her bid by a large amount (when she decided that competing for the top position was no longer worth it, and settling for a lower one would be better) – and then immediately her competitor would drop his bid as well (because he did not want to outbid her by more than 1 cent). This behavior led to a stress on GoTo’s ad servers, as well as lowered ad auction revenues (Edelman and Ostrovsky, 2007).

When Google came out with their own sponsored search auction design in 2002, they introduced two major changes. First, instead of ranking ads purely based on the corresponding bids, they ranked them based on the product of each ad’s bid and its *quality score*. Initially, quality score was equal to the estimated “clickability” of the ad, i.e., the probability that the ad would be clicked by a user if it was shown in the top position, although in later iterations it started including other factors (such as, e.g., the quality of the landing page). The second change was the introduction of a different payment rule. Instead of paying his own bid, each advertiser would pay the smallest amount he would have to bid to outbid its nearest competitor (given their respective quality scores). With identical quality scores, the payment rule reduced to simply paying the bid of the next highest bidder, and the format was dubbed the Generalized Second-Price Auction (Edelman et al., 2007; Varian, 2007).

The Generalized Second-Price Auction format (GSP) has become enormously successful, becoming the de facto industry standard. Beyond Google, it has been deployed by most of the search engines worldwide. It is used by Amazon for its sponsored products offering (which is a major part of the company’s \$31 billion per year, and rapidly growing, advertising business<sup>1</sup>), and many other online retailers and marketplaces use it to power their “Retail Media” advertising offerings. DoorDash and UberEats use it to allow restaurants to advertise on their apps and websites<sup>2</sup>. Apple’s App Store and Google’s Play Store use it to let developers to promote their apps. Travel sites like TripAdvisor, Expedia, Booking, and others use it to allow hotels to promote themselves

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<sup>1</sup><https://www.cnn.com/2022/02/03/amazon-has-a-31-billion-a-year-advertising-business.html>

<sup>2</sup><https://digiday.com/marketing/the-holy-grail-of-e-commerce-advertising-why-doordash-is-bolstering-its-advertising-offerings/>

to travelers. Job search sites like Indeed allow firms to post Sponsored Jobs that receive premium placement when job-seekers search for relevant terms. Instacart is using GSP to power its booming ad auction business.<sup>3</sup>

Thus, until recently, it may have looked like the GSP has “won,” while GFP has fallen by the wayside, due to its poor equilibrium properties. In this paper, we show that this conclusion may be premature, and the situation is more nuanced. Specifically, we re-consider the question of equilibrium existence in the Generalized First-Price Auction when we include in the analysis a feature that is realistic (and common) in many real-world settings in which these types of auctions are used—stochastic quality scores. In the original Edelman and Ostrovsky (2007) paper, quality scores were not considered (or equivalently, were constant and equal to 1), but the paper’s conclusions on the non-existence of pure-strategy equilibria would have remained largely the same even if the scores were different from 1, but were constant. In practice, however, the scores are stochastic, and may vary widely from one impression to the next (e.g., one ad may perform particularly well with women in New York, while another ad for the same search term may perform particularly well with men in San Francisco). Other real-world features may also make the environment stochastic (e.g., bidders’ campaigns may be “paced” if their daily budgets are too low).<sup>4</sup> In such cases, the GFP strategy “bid 1 cent higher than my opponent” is in general no longer a best response, and the payoff functions become continuous, smooth, and better behaved. As a result, in such environments, we show that the Generalized First-Price Auction may in fact have a pure strategy equilibrium. Moreover, perhaps even more surprisingly, we show that the expected revenue under that equilibrium may substantially exceed that of the Generalized Second-Price Auction.<sup>5</sup> Together with other attractive features of the Generalized First-Price Auction, such as that it is easy to explain to the advertisers (and their managers) and that it does not suffer from potential auctioneer credibility problems (Akbarpour and Li, 2020), our results suggest that it may deserve another look in some settings.

We start out by setting up the basic model to illustrate the above features (Section 2). The model is deliberately “stripped down” to its bare essence, to illustrate the key driving forces behind our results and make the derivations as transparent as possible. The simplicity of the model makes the revenue non-equivalence result particularly striking. After proving the existence and uniqueness of the equilibrium of GFP in this basic setting (Section 3) and comparing the revenues of GFP and GSP (Section 4), we explore two extensions of the basic model.

In Section 5, we vary the degree of uncertainty in quality scores, from that of the basic model to the limit case of no uncertainty. Consistent with the intuition, we find that as the amount of uncertainty decreases, the range of other parameter values for which the pure-strategy equilibrium of GFP exists shrinks, completely disappearing in the limit as uncertainty goes away. Revenue

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<sup>3</sup><https://www.wsj.com/articles/instacart-goes-deeper-into-digital-advertising-as-grocery-delivery-slows-11630920600>

<sup>4</sup>See Athey and Nekipelov (2012) and Pin and Key (2011) for discussions and empirical evidence of randomness in quality scores and other auction parameters.

<sup>5</sup>In Section 4, where we present this comparison, we explain why this finding does not contradict the celebrated Revenue Equivalence Theorem of Myerson (1981), despite both formats resulting in identical allocations.

comparisons between GSP and GFP also become more subtle as the amount of uncertainty is reduced. These findings suggests that the generalized first-price auction might be particularly attractive in settings in which the amount of uncertainty in quality scores is high (such as, e.g., “display advertising” types of settings, in which user intent is not clear and the ad system is trying to predict user interests based on their highly variable characteristics), whereas the generalized second-price auction might be relatively more attractive in settings in which the amount of uncertainty in quality scores is lower (such as, e.g., in pure “sponsored search,” in which ads are shown directly in response to user queries which clearly contain a lot of information and thus make the additional information about user characteristics relatively less important—thus making overall quality scores less variable). This comparative static is consistent with the current state of advertising at such search engines as Google, whose sponsored search system on Google.com (AdWords) is using the generalized second-price auction, while its ad system for external publishers, closer to display advertising (AdSense) has recently shifted to the generalized first-price auction.<sup>6</sup>

In Section 6, we consider another extension to the basic model, where we allow the bidders’ per-click values to be heterogeneous. We find that even with this generalization, the equilibrium of the generalized first-price auction continues to exist. Moreover, equilibrium bids exhibit robust comparative statics with respect to key model parameters. The revenue comparison between GSP and GFP, however, is again more subtle.

In Section 7, we generalize our basic model in yet another direction, allowing for  $n \geq 2$  bidders. We show that a symmetric pure-strategy equilibrium of the Generalized First-Price Auction continues to exist in this setting, and can be characterized in closed form by a simple function of  $n$  (number of bidders) and  $\alpha$  (the visibility of the second slot on the screen).

In Section 8, we consider a general model with more than two bidders *and* slots, and general distributions of click-through rates. We show that in this setting, the unique candidate bid for the symmetric pure-strategy equilibrium of GFP can still be characterized by a linear equation in the (symmetric equilibrium) bid  $b^*$ , thus giving a straightforward algorithm for finding a pure-strategy symmetric equilibrium (or determining that it does not exist) for any specific GFP auction setting.

Section 9 concludes.

## 2 Basic Model

Two advertisers are competing for two advertising slots on an internet page. Each values a click from a user at \$1. The two slots vary in terms of visibility to the user. The better slot has visibility normalized to 1 and the lesser slot has visibility  $\alpha \in (0, 1)$ .

The bidders simultaneously submit bids  $\{b_1, b_2\}$ . Then a user arrives to the website. The website has information about user characteristics and the potential relevance of each ad to the user, and for each ad estimates the probability that the user will click on the ad conditional on noticing it. The probabilities for the two ads are (proportional to)  $\{q_1, q_2\}$ . Each  $q_i$  is drawn independently

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<sup>6</sup><https://blog.google/products/adsense/our-move-to-a-first-price-auction/>

from a uniform distribution  $[0, 1]$ .<sup>7</sup>

The auctioneer runs a generalized first-price auction (GFP) with scoring the bids by  $q_i$ 's. That is, the publisher after observing  $q_i$ 's and bids, ranks the bids based on the scores:

$$S_i = b_i q_i.$$

The bidder with the highest score wins the top slot and, conditional on a click, pays  $b_i$ . The bidder with the second-highest score wins the lesser slot and pays its bid if the user clicks on his ad.

The expected profit of the winner and loser, conditional on bid  $b$  and realized  $q$  are:

$$\begin{aligned} U_W(q, b) &= q(1 - b), \\ U_L(q, b) &= \alpha q(1 - b). \end{aligned}$$

We are interested in characterizing the pure-strategy Nash equilibrium of this game.

### 3 Existence of a Pure Strategy Equilibrium

Our first main result is:

**Theorem 1** *In the generalized first-price auction described above, there exists a unique symmetric pure-strategy equilibrium, with the equilibrium bids equal to*

$$b_1 = b_2 = \frac{2(1 - \alpha)}{4 - \alpha}.$$

Before we present the proof, we discuss the intuition. When  $q$ 's are known (or not used at all as in the original auctions), there is no pure strategy equilibrium because at a tie, one of the bidders wants to either deviate to an  $\epsilon$  higher bid or to zero. Unequal bids cannot be an equilibrium either since then the loser wants to deviate to bid zero, the winner to almost match him, but then the loser should “jump over.”

When bidders do not know  $q$ 's, the probability of winning changes smoothly in bid. Deviating to an  $\epsilon$ -higher bid changes the probability of getting the top position only slightly. When there is enough uncertainty about  $\frac{q_i}{q_j}$ , the best response becomes concave in own bid, and there exists a fixed point.<sup>8</sup>

The proof uses the fact that with the distribution of  $q_i$ 's having support starting at zero, the ratio of  $q$ 's ranges from 0 to  $\infty$ . As we show in Section 5, it is possible to generalize the existence

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<sup>7</sup>Since all payoffs in the model are proportional to the probability of a click,  $q_i = 1$  does not mean that there are users with an estimated probability of click equal to 1. Instead, whatever is the highest probability of click, we normalize it to be 1, without loss of generality.

<sup>8</sup>As we show in Section 5 when we consider a generalization of the basic model, concavity is important—continuity alone is not enough for existence.

result for less variation in that ratio, but the existence of a pure strategy equilibrium depends then on  $\alpha$ . The smaller is  $\alpha$ , the smaller variation in the ratio is needed for existence.

**Proof.** Since the optimization problem is symmetric for the two players, we consider only player 1. Conditional on the two bids being  $\{b_1, b_2\}$ , the expected profit of bidder 1 is

$$\begin{aligned} EU_1(b_1, b_2) &= (1 - b_1)E[q_1(\alpha + (1 - \alpha)\mathbf{1}_{b_1q_1 > b_2q_2})] \\ &= (1 - b_1) \int_0^1 q(\alpha + (1 - \alpha)h(b_1, b_2, q))dq, \end{aligned} \tag{1}$$

where  $h(b_1, b_2, q)$  is the probability the score of bidder 1 is higher than the score of bidder 2, given the bids and  $q_1 = q$ .

In this expression,  $(1 - b_1)$  is the expected profit conditional on a click,  $q\alpha$  is the probability of a click in position 2 and the additional probability of a click  $q(1 - \alpha)$  is only conditional on winning the auction.

Player 1 wins whenever  $b_1q_1 > b_2q_2$ , so

$$h(b_1, b_2, q) = \min \left\{ 1, \frac{b_1}{b_2}q \right\}.$$

Simplifying the expressions for expected payoffs we get: if  $b_1 \geq b_2$ :

$$EU_1(b_1, b_2) = (1 - b_1) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{6} \left( 3 - \left( \frac{b_2}{b_1} \right)^2 \right) \right], \tag{2}$$

and if  $b_1 \leq b_2$ :

$$EU_1(b_1, b_2) = (1 - b_1) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{3} \frac{b_1}{b_2} \right]. \tag{3}$$

A necessary condition for equilibrium is that the FOC holds at the symmetric bidding profile  $b_1 = b_2 = b$ . By inspection,  $EU_1(b_1, b_2)$  is differentiable in  $b_1$  at symmetric bidding profiles and the derivative at such vectors is

$$\frac{\partial EU_1(b_1, b_2)}{\partial b_1} = \frac{2(1 - \alpha) - b(4 - \alpha)}{6b}.$$

The unique solution of the FOC is hence a unique candidate for the pure symmetric equilibrium:

$$\frac{\partial EU_1(b_1, b_2)}{\partial b_1} = 0 \text{ and } b_1 = b_2 = b \iff b_1 = b_2 = \frac{2(1 - \alpha)}{4 - \alpha}.$$

To check that this is actually an equilibrium we need to check that these bids are global best responses. Direct inspection of (3) shows that  $EU_1(b_1, b_2)$  is concave in  $b_1$  (since it is quadratic

with a negative coefficient on  $b_1$ ). Similarly, for any  $b_1 \leq 1$  the expression in (2) is concave in  $b_1$  (note that  $\frac{\partial^2 EU_1(b_1, b_2)}{\partial b_1^2} = (1 - \alpha)(b_1 - 3)\frac{b_2^2}{3b_1^4} > 0$  and bidding above  $b_1$  is a dominated strategy). Hence, the FOC is both necessary and sufficient for the optimality of the best response of bidder 1.

■

Given the equilibrium bids  $b$ , the equilibrium payoff is

$$EU_1(b, b) = (1 - b) \left( \frac{1}{6}\alpha + \frac{1}{3} \right)$$

The intuition is that the bidder gets payoff  $(1 - b)$  conditional on a click. They are half of the time the bidder with the higher score and half of the time with the lower score. Conditional on having the higher  $q_i$ , the expected  $q_i$  is  $\frac{2}{3}$ , which is the expected probability of a click in the top position. Conditional on having the lower  $q_i$ , the expected  $q_i$  is  $\frac{1}{3}$ , so the expected probability of a click on the lower position is  $\alpha/3$ . This yields the expression for the expected equilibrium number of clicks.

## 4 Revenue Comparison: GFP vs. GSP

Our second result compares bidding and revenues between the equilibrium of the GFP we characterized above and the equilibrium in a generalized second-price auction, GSP.

In the equilibrium of the GFP the expected revenues are  $b(\frac{\alpha}{3} + \frac{2}{3})$  because the expected  $q$  of the winner is  $\frac{2}{3}$  and the expected  $q$  of the loser is  $\frac{1}{3}$ . Plugging in the equilibrium bidding strategy we get

$$REV_{GFP} = \frac{1}{3}(1 - \alpha)\frac{2(\alpha + 2)}{4 - \alpha}. \quad (4)$$

In a GSP, the loser pays 0 and the winner  $i$  pays

$$P = b_j \frac{q_j}{q_i}.$$

That is the lowest amount the winner could have bid and still at least tied with the loser (with a bid  $b_i = P$  the scores of the two bidders are equal).

With two ad slots and two bidders, the equilibrium bidding strategies in GSP are straightforward to find. Each bidder is guaranteed to win the second position and pay 0, so an equilibrium strategy is to bid per click the full value of the incremental expected number of clicks from the upgrade to the top position:

$$b_{GSP} = 1 - \alpha.$$

To see this, suppose bidder 1 happens to know  $q_1$ ,  $q_2$ , and  $b_2 = (1 - \alpha)$ . Then his options are:

1. Lose the auction, pay 0, and get the second position (and thus  $\alpha q_1$  clicks), for the expected payoff of  $\alpha q_1$ ; or

2. Win the auction, pay  $b_2 q_2 = (1 - \alpha)q_2$  in expectation, and get  $q_1$  clicks, for the expected payoff of  $q_1 - (1 - \alpha)q_2$ .

The second option is more attractive to bidder 1 whenever  $q_1 > q_2$ —and that is exactly the outcome that the bidder will get if he bids  $b_1 = (1 - \alpha)$ .

The expected revenue in the GSP can be computed as:

$$REV_{GSP} = 2 \int_0^1 q \left( \int_0^q \left( (1 - \alpha) \frac{q_2}{q} \right) dq_2 \right) dq = \frac{1}{3} (1 - \alpha)$$

This expression is intuitive: given the realized first and second order statistic of  $q$ ,  $q^{(1)}$  and  $q^{(2)}$ , respectively, only the winner pays and the expected payment of the winner per impression is

$$q^{(1)} * b \frac{q^{(2)}}{q^{(1)}} = bq^{(2)}.$$

The equilibrium bid is  $b = (1 - \alpha)$  and the expected second-highest  $q$  is  $\frac{1}{3}$ .

Finally, observe that

$$REV_{GFP} = \frac{1 - \alpha}{3} \frac{2(\alpha + 2)}{4 - \alpha} > \frac{1 - \alpha}{3} = REV_{GSP},$$

because  $\frac{2(\alpha+2)}{4-\alpha} > 1$ .

Summarizing the analysis above yields our second main result:

**Theorem 2** *The expected revenue in the Generalized First-Price Auction is equal to*

$$REV_{GFP} = \frac{2(1 - \alpha)(\alpha + 2)}{3(4 - \alpha)}.$$

*The expected revenue in the Generalized Second-Price Auction is equal to*

$$REV_{GSP} = \frac{1 - \alpha}{3}.$$

*For all  $\alpha \in (0, 1)$ ,*

$$REV_{GFP} > REV_{GSP}.$$

Note that as  $\alpha$  gets close to 0, the expected revenues from those two formats converge. But for all higher  $\alpha$  the expected revenues are not the same, despite both formats resulting in efficient (and thus identical) equilibrium allocations. The difference can be substantial. For example, suppose  $\alpha = 0.7$ , which is a fairly typical “dropoff” value in ad auctions, meaning that the second slot gets 70% as many clicks as the first one. Then the revenue in the generalized second-price auction is equal to 0.1, while the revenue for the generalized first-price auction is equal to  $0.1 \cdot \frac{5.4}{3.3} \approx 0.164$ —a difference of more than 60%.

## 4.1 Understanding Revenue Non-Equivalence

At first glance, this result on revenue differences seems to violate the Revenue Equivalence Theorem (RET) of Myerson (1981)—after all, in both formats, we end up with fully efficient (and thus identical) allocations: whichever bidder has the higher relevance  $q_i$  gets the top position (because in both GFP and GSP, equilibria are symmetric, and the bidders are submitting identical bids). Since the value per click of each bidder is the same (equal to 1), that is the efficient allocation. And yet the revenues are different!

Of course, there is no contradiction. The purely technical reason is that RET holds subject to the payoffs of the lowest types being equal. And since in our model each bidder has only one informational type (the value per click of each bidder is fixed at 1, and relevance  $q_i$  is not known to the bidder beforehand), which is thus automatically “the lowest,” RET has no bite.

More substantively, the machinery behind the proof of RET in fact helps us understand the difference in expected revenues between the two formats. Specifically, consider a slightly modified version of the GFP “auction.” Bidder 2 has value  $v_2 = 1$  and mechanically submits bid  $b_2 = \frac{2(1-\alpha)}{4-\alpha}$  (the same as the equilibrium bid in Theorem 1 of Section 3). Bidder 1’s value  $v_1$  can take values between 0 and 1. The exact distribution of bidder 1’s values is not important, but for concreteness, assume that it is uniform on  $[0, 1]$ . Computing the equilibrium in this auction reduces to a solving the single-agent bidding problem for every value  $v_1 \in [0, 1]$  of bidder 1. For  $v_1 = 1$ , we already know from Theorem 1 that the optimal best response  $b_1(v_1)$  is equal to the bid of bidder 2,  $\frac{2(1-\alpha)}{4-\alpha}$ . For lower values  $v_1 < 1$ , we know from the standard monotonicity arguments that the optimal bid  $b_1(v_1)$  must be weakly lower than  $b_1(1) = b_2$ . Thus, by the derivation parallel to that behind equation (3) above, the expected utility of bidder 1 with value  $v_1$  who submit bid  $b_1$  is equal to

$$\begin{aligned} EU_1 \left( v_1, b_1, b_2 = \frac{2(1-\alpha)}{4-\alpha} \right) &= (v_1 - b_1) \left[ \frac{\alpha}{2} + (1-\alpha) \frac{1}{3} \frac{b_1}{b_2} \right] \\ &= (v_1 - b_1) \left[ \frac{\alpha}{2} + (1-\alpha) \frac{1}{3} \frac{b_1}{\left( \frac{2(1-\alpha)}{4-\alpha} \right)} \right] \\ &= \frac{1}{6} (v_1 - b_1) [3\alpha + b_1 (4 - \alpha)]. \end{aligned}$$

Since  $b_1$  cannot be negative, and since  $\alpha \in [0, 1]$  (and thus  $4 - \alpha > 0$ ), this expression is maximized at the optimal bid

$$b_1(v_1) = \max \left\{ 0, \frac{v_1}{2} - \frac{3\alpha}{2(4-\alpha)} \right\}.$$

Thus, for small values of  $v_1$ , bidder 1 will optimally bid zero and settle for the second slot with no payment, while above the threshold value  $v_1 = 3\alpha/(4 - \alpha)$ , the bid is increasing linearly in  $v_1$ . And for  $v_1 = 1$ , as expected, we get  $b_1(v_1 = 1) = \max \left\{ 0, \frac{1}{2} - \frac{3\alpha}{2(4-\alpha)} \right\} = \frac{2(1-\alpha)}{4-\alpha}$ , which is equal to  $b_2$  and to the equilibrium bid of both bidders in the basic model of Sections 2 and 3.

Let  $a^{GFP}(v_1)$  be the expected number of clicks that bidder 1 receives in this auction when he

submits the optimal bid  $b_1(v_1)$ . That number is equal to

$$a^{GFP}(v_1) = \frac{\alpha}{2} + (1 - \alpha) \frac{1}{3} \frac{b_1(v_1)}{b_2}.$$

Consider now an analogous modified GSP auction, in which bidder 2's value is  $v_2 = 1$  and his bid is fixed at  $b_2^{GSP} = (1 - \alpha)$ . Bidder 1's value  $v_1$  is again distributed uniformly on  $[0, 1]$ . By the arguments analogous to those in the beginning of Section 4, the optimal bid for bidder 1 with value  $v_1$  is  $b_1^{GSP}(v_1) = (1 - \alpha)v_1$ . And the expected number of clicks that bidder 1 receives in this auction when behaving in this optimal way is equal to

$$a^{GSP}(v_1) = \frac{\alpha}{2} + (1 - \alpha) \frac{1}{3} \frac{b_1^{GSP}(v_1)}{b_2^{GSP}}.$$

Note that in both GSP and GFP, the expected payoff of bidder 1 whose value is zero is also equal to zero. Thus, by the standard envelope argument underlying the Revenue Equivalence Theorem, we know that the expected payoff of bidder 1 whose value is 1 is equal to  $\int_0^1 a^{GFP}(v_1) dv_1$  in the Generalized First-Price Auction and to  $\int_0^1 a^{GSP}(v_1) dv_1$  in the Generalized Second-Price Auction. (And of course, these numbers are precisely the expected payoffs obtained by each bidder under these formats in the basic model in which both bidders' values are equal to 1.)

Now, the key observation is that for all  $v_1$ , we have  $a^{GFP}(v_1) \leq a^{GSP}(v_1)$ . For the values of  $v_1$  for which  $b_1(v_1) = 0$ , this is immediate. For the remaining values of  $v_1$ , this inequality is equivalent to

$$\begin{aligned} \frac{b_1(v_1)}{b_2} &\leq \frac{b_1^{GSP}(v_1)}{b_2^{GSP}} \\ &\Leftrightarrow \\ \frac{\left(\frac{v_1}{2} - \frac{3\alpha}{2(4-\alpha)}\right)}{\left(\frac{1}{2} - \frac{3\alpha}{2(4-\alpha)}\right)} &\leq v_1 \\ &\Leftrightarrow \\ \frac{v_1}{2} - \frac{3\alpha}{2(4-\alpha)} &\leq \frac{v_1}{2} - v_1 \frac{3\alpha}{2(4-\alpha)}, \end{aligned}$$

which is immediate.

Thus, the expected payoff of bidder 1 whose value is 1 (and thus the expected payoff of each bidder in the basic model) is weakly lower in GFP than in GSP ( $\int_0^1 a^{GFP}(v_1) dv_1 \leq \int_0^1 a^{GSP}(v_1) dv_1$ ), and it is straightforward to further verify that the inequality is strict unless  $\alpha = 0$  or  $\alpha = 1$ . But as we observed above, in the basic model both formats result in efficient allocations, and thus have identical total surpluses—and the expected revenue of the auctioneer, which is equal to the total surplus less the expected payoffs of the bidders, is therefore higher in the Generalized First-Price Auction than in the Generalized Second-Price Auction.

To provide an illustration for the above arguments, consider a special case  $\alpha = \frac{1}{2}$ . Using

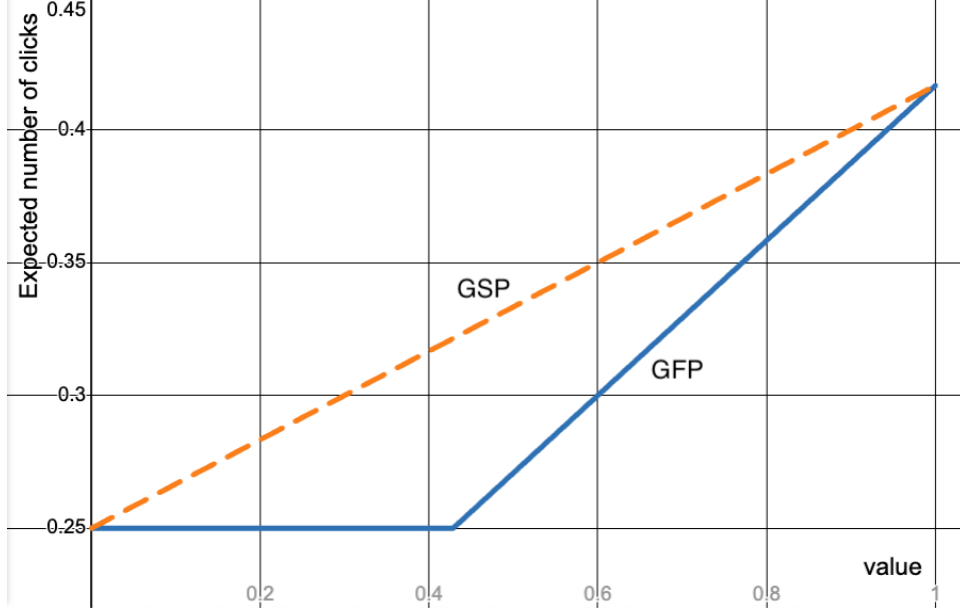


Figure 1: Expected number of clicks under GFP and GSP

the above derivations, for this case, the optimal bid in GFP is  $b_1^{GFP}(v_1) = 0$  for  $v_1 \leq \frac{3}{7}$  and  $b_1^{GFP}(v_1) = \frac{v_1}{2} - \frac{3}{14}$  for  $v_1 > \frac{3}{7}$ . The expected number of clicks for such a bidder is  $\frac{1}{4}$  when  $v_1 \leq \frac{3}{7}$  (the bidder gets the second slot for sure) and  $\frac{1}{4} + \frac{1}{6} \left( \frac{v_1 - 3/7}{1 - 3/7} \right)$  when  $v_1 > \frac{3}{7}$  (there is a positive chance of getting the first slot).

In GSP, the optimal bid is  $b_1^{GSP}(v_1) = \frac{v_1}{2}$  for all  $v_1$ . The expected number of clicks is  $\frac{1}{4} + \frac{1}{6}v_1$ .

Figure 1 plots the expected numbers of clicks under GFP and under GSP for  $v_1 \in [0, 1]$ . For  $v_1 = 0$ , both values are the same—under both formats, the bidder gets the second slot for sure (and thus gets  $\frac{1}{4}$  clicks in expectation). For  $v_1 = 1$ , both values are also the same—under both formats, the bidder bids the same amount as his or her opponent, and thus gets the top slot whenever his or her click-through rate is higher than that of the opponent (which of course is exogenous and is independent of the format). In that case, under both formats, the bidder gets  $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$  clicks in expectation. The difference between the two formats is in the middle. Under GSP, the bidder bids truthfully for fraction  $\frac{1}{2} = 1 - \alpha$  of a click (which is the marginal value of moving from the second slot to the first one), and so even for small values of  $v_1$ , the bid is positive and there is a positive chance of the bidder getting the top slot (when his or her click-through rate happens to be high and the opponent's click-through rate happens to be low). The expected number of clicks is a linear function connecting the points  $(0, \frac{1}{4})$  and  $(1, \frac{5}{12})$ . Under GFP, for small values of  $v_1$ , the bidder does not bid a positive amount—doing so would have little upside (very low chance of getting the top slot) and substantial downside (higher pay-click payment even when getting the second slot). So for values below a threshold,  $v_1^* = \frac{3}{7}$  in this case, the optimal bid is zero and the expected number of clicks won is flat at  $\frac{1}{4}$ . For values higher than  $v_1^*$ , the optimal bid becomes positive, there is a positive chance of getting a top slot, and the expected number of clicks starts rising linearly (at a steeper rate than under GSP) until getting to the same  $(1, \frac{5}{12})$  point as in the graph of the expected

number of clicks under GSP. Of course, it is immediate that for all intermediate values of  $v_1$ , the expected number of clicks won under GFP is lower than that under GSP. Thus the integral of the former function is strictly lower than the integral of the latter one, which by the arguments above (adapting the canonical Myerson (1981) reasoning) translates into the corresponding differences in expected bidder payoffs and expected auction revenues under the two formats.

## 5 Pure-Strategy Equilibria with Less Heterogeneity in Click-Through Rates

In our baseline model we have assumed that  $q_i$ 's are drawn from a uniform distribution  $[0, 1]$ . That implies that the range of possible ratios of  $\frac{q_1}{q_2}$  is  $[0, \infty]$ . In this section we characterize necessary and sufficient conditions for the existence of a pure strategy equilibrium in the GFP when the range of possible ratios is less extreme.

To this end, assume that the probabilities of clicks are proportional to  $q_i$  which are drawn from a uniform distribution  $[u, u + 1]$ , for some parameter  $u \geq 0$ . When  $u = 0$  we have as a special case our previous model and as  $u$  gets larger, there is less and less ex-ante uncertainty about the ratio of the CTRs.<sup>9</sup>

The rest of the model remains unchanged.

**Expected Payoffs** Given bids  $(b_1, b_2)$ , the expected payoff of player 1 is still as in equation (1), but the expression for  $h(b_1, b_2, q)$  changes. Player 1 wins whenever  $b_1 q_1 > b_2 q_2$ , so

$$h(b_1, b_2, q) = \min \left\{ 1, \max \left\{ \frac{b_1}{b_2} q - u, 0 \right\} \right\}.$$

That is, when the probability is interior, it is  $\frac{b_1}{b_2} q - u$ .

Next we simplify the range of bids we consider. First, bidding so high that  $\frac{b_1}{b_2} (u + 1) - u > 1$  is never optimal. Second, conditional on bidding such that the probability of winning the top position is zero, the optimal bid is 0. Hence, without loss of generality in solving for the best response we need to only consider bids in the set

$$b_1 \in \{0\} \cup \left[ b_2 \frac{u}{u+1}, b_2 \frac{1+u}{u} \right].$$

Bidding 0 yields a payoff

$$EU_1(0, b_2) = \int_u^{u+1} q \alpha dq = \alpha \left( u + \frac{1}{2} \right).$$

Bidding in the range  $\left[ b_2 \frac{u}{u+1}, b_2 \frac{1+u}{u} \right]$  yields payoff:

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<sup>9</sup>Note that we can normalize the CTRs by dividing them by  $u + 1$ , so that the model is equivalent to assuming that the distribution of the CTRs is uniform  $\left[ \frac{u}{u+1}, 1 \right]$ . As  $u \rightarrow 1$ , the uncertainty about the CTRs disappears.

$$\begin{aligned}
& \text{If } b_1 \geq b_2 \\
EU_1(b_1, b_2) &= (1 - b_1) \left[ \alpha \left( u + \frac{1}{2} \right) + (1 - \alpha) \left( \int_u^{(u+1)\frac{b_2}{b_1}} q \left( \frac{b_1}{b_2} q - u \right) dq + \int_{(u+1)\frac{b_2}{b_1}}^{u+1} q dq \right) \right]. \\
& \text{If } b_1 \leq b_2 \\
EU_1(b_1, b_2) &= (1 - b_1) \left[ \alpha \left( u + \frac{1}{2} \right) + (1 - \alpha) \left( \int_u^{u+1} q \left( \frac{b_1}{b_2} q - u \right) dq \right) \right].
\end{aligned}$$

Simplifying the integrals we get:

$$\begin{aligned}
& \text{If } b_1 \geq b_2 \\
EU_1(b_1, b_2) &= (1 - b_1) \left[ \alpha \left( u + \frac{1}{2} \right) + (1 - \alpha) \frac{1}{6} \left( 3u^3 + 3(u+1)^2 - 2u^3 \frac{b_1}{b_2} - \left( \frac{b_2}{b_1} \right)^2 (u+1)^3 \right) \right]. \\
& \text{If } b_1 \leq b_2 \\
EU_1(b_1, b_2) &= (1 - b_1) \left[ \alpha \left( u + \frac{1}{2} \right) + (1 - \alpha) \frac{1}{6} \left( -3u(u+1)^2 + 2(u+1)^3 \frac{b_1}{b_2} + \left( \frac{b_2}{b_1} \right)^2 u^3 \right) \right].
\end{aligned}$$

When  $b_1 = b_2$ , the profits simplify to

$$EU_1(b_1, b_1) = (1 - b_1) \left[ \frac{1}{2} u (\alpha + 1) + \frac{1}{6} \alpha + \frac{1}{3} \right].$$

**FOC and Candidate Equilibrium** We start with a necessary condition for a pure strategy equilibrium in GFP is that at the candidate equilibrium the first-order conditions are satisfied. Differentiating the expected profit functions above with respect to  $b_1$  and solving for them at  $b_1 = b_2 = b$  we get that the unique candidate for the equilibrium is

$$b_{GFP} = (1 - \alpha) \frac{6u(u+1) + 2}{3u(1+2u)(1-\alpha) + 6u + 4 - \alpha}. \quad (5)$$

It can be shown that this candidate is between 0 and 1 for all  $u$  and  $\alpha$ .

**Necessary and Sufficient Conditions for Existence** We now provide necessary and sufficient conditions for the candidate  $b_{GFP}$  to be indeed the unique pure-strategy equilibrium of the GFP.

The first necessary condition (and as we show later a sufficient as well!) is that at the bidding profile no bidder wants to deviate to bidding zero. Bidding zero yields payoff  $\alpha(u + \frac{1}{2})$  since the average  $q_i$  is  $(u + \frac{1}{2})$ . Therefore the necessary condition is:

$$\alpha \left( u + \frac{1}{2} \right) \leq (1 - b_{GFP}) \left[ \frac{1}{2} u (\alpha + 1) + \frac{1}{6} \alpha + \frac{1}{3} \right]. \quad (6)$$

Substituting our candidate  $b_{GFP}$  we can solve this condition for the critical level of  $\alpha$  for any

$u$ : this necessary condition is satisfied when  $\alpha \leq \alpha^*$  where

$$\alpha^* = \frac{9u^2 + 12u + 4}{36u^3 + 45u^2 + 21u + 4} \in [0, 1] \quad (7)$$

When  $u = 0$  as in our baseline model,  $\alpha^* = 1$  and hence this condition is always satisfied. When the heterogeneity/uncertainty of CTRs is smaller,  $\alpha^*$  gets smaller and in the limit as  $u$  gets large,  $\alpha^*$  converges to zero (consistent with the non-existence result when  $q_i$ 's are known).

The second necessary condition is that the bidder does not want to deviate to a bid that wins for sure,  $b = b_{GFP} \frac{1+u}{u}$ . Such a bid yields expected payoff  $(1 - b_{GFP} \frac{1+u}{u})(u + \frac{1}{2})$ . This necessary condition is hence

$$(1 - b_{GFP} \frac{1+u}{u})(u + \frac{1}{2}) \leq (1 - b_{GFP}) \left[ \frac{1}{2}u(\alpha + 1) + \frac{1}{6}\alpha + \frac{1}{3} \right]. \quad (8)$$

Straightforward (though tedious) algebra shows that this condition is satisfied for all  $u$  and  $\alpha$

The only remaining condition to check if both players bidding  $b_{GFP}$  is a mutual best response is to check if there is a bid  $b$  that results in an interior probability of winning that is profitable. To show that not, we compute the second derivative of the expected profit function for  $b \in \left[ b_2 \frac{u}{u+1}, b_2 \frac{1+u}{u} \right]$ .

$$\begin{aligned} \frac{\partial^2}{\partial b_1^2} EU_1(b_1, b_2) &= \frac{1-\alpha}{3b_1^4 b_2} \left( (2u^3) b_1^4 + (u+1)^3 (b_1 - 3) b_2^3 \right) \text{ for } b_1 \geq b_2 \\ \frac{\partial^2}{\partial b_1^2} EU_1(b_1, b_2) &= -\frac{1-\alpha}{3b_1^4 b_2} \left( 2(u+1)^3 b_1^4 + u^3 b_2^3 (b_1 - 3) \right) \text{ for } b_1 \leq b_2 \end{aligned}$$

The details of these expressions do not matter, just that the numerators change sign at most once. That leads to the following observations: In the range  $b_1 \leq b_{GFP}$  the expected profit function is either concave or convex or switches once from convex to concave. In the first case (concave), there is no profitable deviation. In the second and third case (convex or first convex and then concave), if there is a profitable deviation then deviating to the bottom of the range is also profitable. But then deviating to bidding zero would be even more profitable. So if deviating to bidding zero is not profitable, deviating to any other bid below  $b_{GFP}$  is not profitable either.

Similarly, in the range  $b_1 \geq b_{GFP}$  the expected profit function is also either everywhere concave, everywhere convex or first concave and then convex. That means that there is either no profitable deviation or deviating to a bid that wins with probability 1 is profitable. But we have already argued that deviating from the candidate  $b_{GFP}$  to a bid that always wins is never profitable.

Combining all these observations we get:

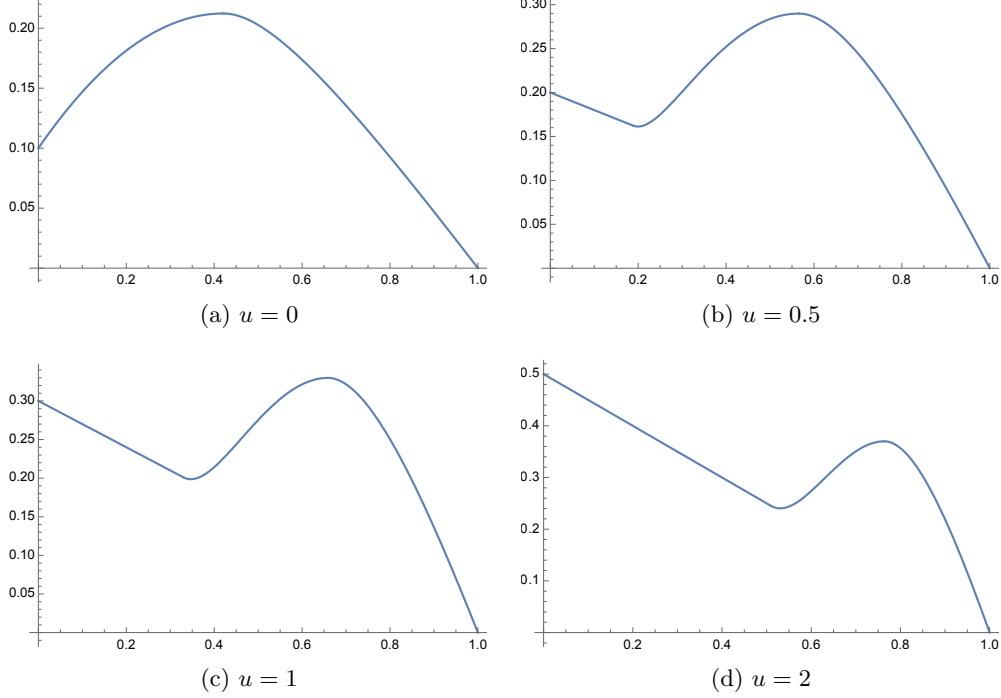


Figure 2: Expected payoffs from responding to a candidate equilibrium bid

**Theorem 3** *With click probabilities proportional to  $q_i$  distributed uniformly  $[u, u + 1]$ , there exists a unique symmetric pure-strategy equilibrium in the generalized first-price auction if and only if:*

$$\alpha \leq \alpha^* = \frac{9u^2 + 12u + 4}{36u^3 + 45u^2 + 21u + 4} \in [0, 1].$$

*When this condition is satisfied, the equilibrium bids are equal to:*

$$b_1 = b_2 = (1 - \alpha) \frac{6u(u + 1) + 2}{3u(1 + 2u)(1 - \alpha) + 6u + 4 - \alpha}.$$

To get the intuition for the reasons behind pure-strategy equilibrium existence (or non-existence) for different levels of uncertainty in click-through rates, in Figure 2 we present graphs of expected payoffs of bidder 1 as a function of her own bid (varying from 0 to 1). We set  $\alpha = 0.2$  and vary the level of uncertainty from the maximum possible in our model ( $u = 0$ ) to lower ones ( $u = 2$ , corresponding to the case in which click-through rates only vary from the lowest to the highest by a factor of  $\frac{2}{3}$ , and two values in between,  $u = 0.5$  and  $u = 1$ ). In each panel, we assume that bidder 1's opponent, bidder 2, bids according to the “candidate” equilibrium bid  $b_{GFP}$  that satisfies the first-order condition (see Equation (5)), and thus to check whether the “candidate” is indeed an equilibrium, we need to verify that the local maximum that corresponds to it in the expected payoff graph is in fact a global one. For the case of maximum uncertainty,  $u = 0$ , the graph is simple: it is concave, and the global optimality (and thus equilibrium existence) is immediate. Once the

amount of uncertainty is reduced, however, the picture becomes more subtle. Below a certain level (specifically, below  $b_{GFP} \frac{u}{u+1}$ ), it does not make sense for bidder 1 to bid any positive amount: she is guaranteed to get the second slot (because even if she gets the highest possible score and her opponent gets the lowest one, she would still not be able to outbid him for the top slot), and so any positive bid amount simply reduces her payoff from winning that second slot. So the real test of global optimality is whether for bidder 1, the expected payoff from bidding  $b_{GFP}$  (and thus trying to win the top slot with probability 50%) is higher than the payoff from bidding zero and settling for the second slot at no cost. For intermediate values of uncertainty in click-through rates ( $u = 0.5$  and  $u = 1$ ) that is indeed the case, and so both bidders bidding  $b_{GFP}$  is a pure-strategy equilibrium. However, once this uncertainty is reduced further ( $u = 2$ ), bidder 1 is better off bidding zero instead of bidding  $b_{GFP}$ , and a pure-strategy equilibrium no longer exists. Of course, in the limit, as  $u \rightarrow +\infty$ , we converge back to the deterministic case of no uncertainty in click-through rates (Edelman and Ostrovsky, 2007), in which a pure-strategy equilibrium likewise does not exist.

## 5.1 Revenue Comparisons

We can also extend our analysis of the revenue comparisons between GFP and GSP to this more general range of  $u$ , for the parameter values for which a pure-strategy equilibrium exists. In GSP, it is still optimal for bidders to bid  $b_{GSP} = (1 - \alpha)$ , because this is the value of improving from position 2 to position 1. As before, only the winner pays and since they pay the minimum amount they could have bid and still win, the expected revenue is

$$REV_{GSP} = (1 - \alpha)(u + \frac{1}{3}).$$

The expected revenue in the GFP is

$$REV_{GFP} = b_{GFP} \left( \alpha(u + \frac{1}{3}) + u + \frac{2}{3} \right)$$

because the expected number of clicks in position 2 is  $\alpha(u + \frac{1}{3})$  and the expected number of clicks in position 1 is  $u + \frac{2}{3}$ . Substituting the expression for  $b_{GFP}$  we get that

$$REV_{GSP} - REV_{GFP} = (1 - \alpha) \frac{u(u+1) - (3u+1)(3u+4u^2+1)\alpha}{(6u^2+9u+4) - (6u^2+3u+1)\alpha}$$

The denominator is always positive, so the sign depends only on the sign of the numerator.

That yields the following result:

**Theorem 4** *Suppose click probabilities are proportional to  $q_i$  distributed uniformly  $[u, u + 1]$ . Suppose  $\alpha \leq \alpha^*$  defined in (7). (The pure strategy equilibrium) expected revenues in the GFP are higher than in the GSP if and only if*

$$\alpha > \hat{\alpha} = \frac{u(u+1)}{(3u+1)(3u+4u^2+1)}.$$

Note that when  $u = 0$ ,  $\hat{\alpha} = 0$  and we get our previous result that GFP yields a higher expected revenue for all  $\alpha$ . However, for  $u > 0$ ,  $\hat{\alpha}$  is positive. Hence, for small  $\alpha$ , GSP yields higher revenues than GFP. Moreover we can verify that for all  $u > 0$ ,  $\hat{\alpha} < \alpha^*$  so that for every  $u > 0$  the range  $\alpha \in [\hat{\alpha}, \alpha^*]$  is non-empty. In that range, GFP yields higher expected revenues than GSP. In other words, GSP tends to dominate when the second position is much weaker than the first, but when they get closer in value (but not too close for the GFP equilibrium to disappear), GFP starts to dominate.<sup>10</sup>

## 6 Heterogeneous Bidder Values

Another natural question is to what extent our results remain valid if the per-click values of the two bidders are different. To answer this question, we go back to the basic model in which the quality scores are distributed uniformly on  $[0, 1]$ , but now assume that the value of the second bidder,  $v_2$ , is equal to some value  $v > 0$ , while the value of the first bidder,  $v_1$ , is equal to  $V \geq v$ .

Notably, the existence result continues to hold.

**Theorem 5** *In the generalized first-price auction described above, there exists a unique pure-strategy equilibrium.*

**Proof.** Conditional on the two bids being  $\{b_1, b_2\}$ , the expected profit of bidder 1 is:

$$EU_1(b_1, b_2) = (V - b_1) E[q_1(\alpha + (1 - \alpha)1_{b_1q_1 > b_2q_2})].$$

whereas the profit for bidder 2 is:

$$EU_2(b_1, b_2) = (v - b_2) E[q_2(\alpha + (1 - \alpha)1_{b_1q_1 < b_2q_2})].$$

As before, if we simplify the expressions for expected payoffs, we get that if  $b_1 \geq b_2$ :

$$EU_1(b_1, b_2) = (V - b_1) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{6} \left( 3 - \left( \frac{b_2}{b_1} \right)^2 \right) \right] \quad (9)$$

$$EU_2(b_1, b_2) = (v - b_2) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{3} \frac{b_2}{b_1} \right], \quad (10)$$

whereas if  $b_1 \leq b_2$ :

$$EU_1(b_1, b_2) = (V - b_1) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{3} \frac{b_1}{b_2} \right] \quad (11)$$

$$EU_2(b_1, b_2) = (v - b_2) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{6} \left( 3 - \left( \frac{b_1}{b_2} \right)^2 \right) \right]. \quad (12)$$

---

<sup>10</sup>The difference in expected revenues between GSP and GFP is monotone in  $\alpha$  when GSP dominates, but can be non-monotone when GFP dominates, but the difference crosses zero exactly once.

If in equilibrium  $b_1 \geq b_2$  (as will be the case), then equations (9) and (10) are the relevant ones.

Advertiser 1's problem is:

$$\max_{b_1} (V - b_1) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{6} \left( 3 - \left( \frac{b_2}{b_1} \right)^2 \right) \right]$$

with FOC:

$$\frac{-3b_1^3 + b_2^2(2V - b_1)(1 - \alpha)}{6b_1^3} = 0. \quad (13)$$

The sign of this derivative depends on the sign of the numerator and that changes once from positive to negative. Hence, for every  $b_2$  there is a unique best response  $b_1$  that is a solution to:

$$-3b_1^3 + b_2^2(2V - b_1)(1 - \alpha) = 0. \quad (14)$$

We can solve for the inverse of the best response:

$$b_2 = \sqrt{\frac{3b_1^3}{(1 - \alpha)(2V - b_1)}}.$$

Advertiser 2's problem is:

$$\max_{b_2} (v - b_2) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{3} \frac{b_2}{b_1} \right]$$

with FOC:

$$\frac{2(1 - \alpha)(v - 2b_2) - 3\alpha b_1}{6b_1} = 0. \quad (15)$$

This derivative is decreasing in  $b_2$ . So the best response of bidder 2 is

$$b_2 = \max \left\{ 0, \frac{1}{2}v - \frac{3\alpha b_1}{4(1 - \alpha)} \right\}. \quad (16)$$

The equilibrium bid  $b_1^*$  is the solution to:

$$\sqrt{\frac{3b_1^3}{(1 - \alpha)(2V - b_1)}} = \max \left\{ \frac{2v(1 - \alpha) - 3\alpha b_1}{4(1 - \alpha)}, 0 \right\}. \quad (17)$$

The function on the left is increasing and the function on the right is decreasing in  $b_1$ . At  $b_1 = 0$  the LHS is 0 and the RHS is positive. At  $b_1 = V$  the ranking is the opposite. So, there is a unique solution and at the solution  $b_1 > 0$ . Given the unique solution for  $b_1^*$ , there is a unique solution for  $b_2^*$  as well (that can be found using (16)). That pair  $(b_1^*, b_2^*)$  is thus the unique equilibrium for the

case  $b_1 \geq b_2$ .

To rule out the cases in which  $b_1 < b_2$  in equilibrium, we repeat the above computation for equations (11) and (12), and find that for all possible pairs of values such that  $v < V$  and all possible values of  $\alpha$ , all candidate solutions have  $b_1 > b_2$ , leading to a contradiction. This completes the proof. ■

In addition to showing equilibrium existence and uniqueness, we also can use the expressions in the proof to establish the comparative statics of equilibrium bids with respect to bidder values.

First, observe that equilibrium bids  $(b_1^*, b_2^*)$  are both increasing in  $v$  (as long as  $v$  remains smaller than  $V$ ). To see this, note that the RHS of equation (17) increases in  $v$ , so the solution  $b_1$  increases in  $v$ . Moreover, the LHS is the equilibrium bid  $b_2$ , and that is increasing in  $b_1$ .

Second, the comparative statics of equilibrium bids with respect to the higher value,  $V$ , are more subtle. Again from equation (17), we observe that if  $V$  increases, equilibrium bid  $b_1^*$  has to increase as well. To determine the change in the other bid,  $b_2^*$ , we now consider equation (16), and observe that as  $b_1^*$  increases,  $b_2^*$  has to *decrease*. In words, if the value of the weaker bidder increases, both bidders increase their equilibrium bids. By contrast, if the value of the stronger bidder increases, then that bidder's equilibrium bid also goes up, but the weaker bidder's equilibrium best response goes down.

Finally, we can also show that both equilibrium bids  $(b_1^*, b_2^*)$  are decreasing in  $\alpha$ : as the second position becomes more valuable, the competition for the first one becomes less intense. To see this, rewrite equation (17) as

$$\sqrt{\frac{3b_1^3}{(1-\alpha)(2V-b_1)}} + \frac{3\alpha b_1}{4(1-\alpha)} = \frac{1}{2}v. \quad (18)$$

(We can do this because in equilibrium, the “max” expression on the RHS of equation (17) is always equal to its first term, and can thus be simplified.) Next, observe that the expression on the LHS of equation (18) is increasing in  $b_1$  and also increasing in  $\alpha$ , and since this expression has to be constant (equal to the RHS), as  $\alpha$  increases,  $b_1$  has to decrease. To show that  $b_2$  also has to decrease in  $\alpha$ , note that by construction,  $b_2$  is equal to the first of the two terms in the sum on the LHS of equation (18). Since the sum of the two terms is constant, if we want to show that the first one of them is decreasing in  $\alpha$ , it is sufficient to show that *the ratio* of the first term to the second term is decreasing in  $\alpha$ . Dropping constant terms, this ratio is proportional to

$$\sqrt{\frac{(1-\alpha)b_1}{\alpha^2(2V-b_1)}}, \quad (19)$$

and since we've already established that  $b_1$  is decreasing in  $\alpha$ , it is immediate by inspecting the terms under the square root that the expression in equation (19) is also decreasing in  $\alpha$ .

Next, given equilibrium values  $(b_1^*, b_2^*)$ , we can compute equilibrium payoffs:

$$\begin{aligned} EU_1(b_1^*, b_2^*) &= (V - b_1^*) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{6} \left( 3 - \left( \frac{b_2^*}{b_1^*} \right)^2 \right) \right] \\ EU_2(b_1^*, b_2^*) &= (v - b_2^*) \left[ \frac{\alpha}{2} + (1 - \alpha) \frac{1}{3} \frac{b_2^*}{b_1^*} \right] \end{aligned}$$

and the expected revenue for GFP:

$$\begin{aligned} REV_{GFP} &= \int_0^1 \int_{\frac{b_2^*}{b_1^*}}^1 (q_1 b_1^* + \alpha q_2 b_2^*) dq_1 dq_2 + \int_0^1 \int_0^{\frac{b_2^*}{b_1^*}} (q_2 b_2^* + \alpha q_1 b_1^*) dq_1 dq_2 \\ &= \frac{1}{2} b_1^* + \frac{1}{2} \alpha b_2^* + \frac{1 - \alpha}{6} \frac{(b_2^*)^2}{b_1^*}. \end{aligned}$$

As before, for GSP, the bidding strategies are straightforward to find. Specifically, it is easy to see that advertiser 1 should bid  $b_1 = V(1 - \alpha)$ , while advertiser 2 should bid  $b_2 = v(1 - \alpha)$ . The argument is identical to the one in Section 4. To compute the expected revenue in GSP, it is convenient to normalize  $V = 1$  (and  $v \leq V = 1$ ), and the revenue is then:

$$\begin{aligned} REV_{GSP} &= \int_0^1 \int_{q_2 v}^1 (1 - \alpha) v \frac{q_2}{q_1} q_1 dq_1 dq_2 + \int_0^1 \int_0^{q_2 v} (1 - \alpha) \frac{q_1}{q_2} q_2 dq_1 dq_2 \\ &= \frac{(1 - \alpha)v}{2} - \frac{(1 - \alpha)v^2}{6}. \end{aligned}$$

The relationship between the revenues in GFP and GSP is now more complex. Figure 3 plots the revenues in the two auction formats for  $V = 1$  and the values of  $v$  between 0 and 1. We show three cases illustrating a wide range of possible outcomes:  $\alpha \in \{0, 0.4, 0.8\}$ . As we saw earlier, the revenue in GFP is greater than in GSP for  $v = 1$  and any  $\alpha > 0$  (they are equal for  $\alpha = 0$  and  $v = 1$ ). However, for lower values of  $v$ , that is not always the case. For instance, when  $\alpha$  is low (see the graphs for  $\alpha = 0$ ), for relatively high (but lower than 1) values of  $v$ , the revenue in GSP is higher, and the ranking reverses for small  $v$ . For intermediate values of  $\alpha$  (e.g.,  $\alpha = 0.4$ ), GFP dominates for all  $v$ . Finally, for large  $\alpha$  (see the graph for  $\alpha = 0.8$ ), GFP dominates for large  $v$ , and the ranking flips for small  $v$  (but the differences are minimal for small  $v$  and large  $\alpha$ , so in the bottom-right panel we zoom to a range of small  $v$  to make the difference in revenues visible).

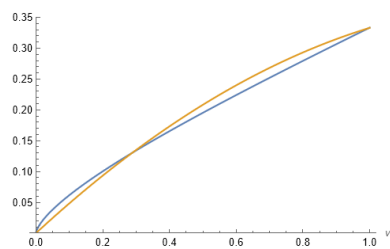
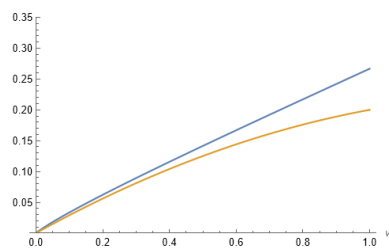
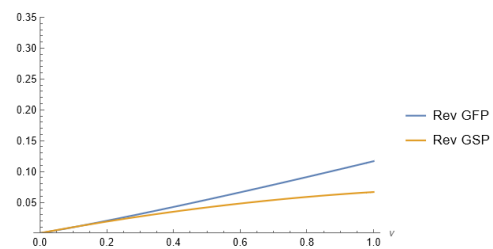
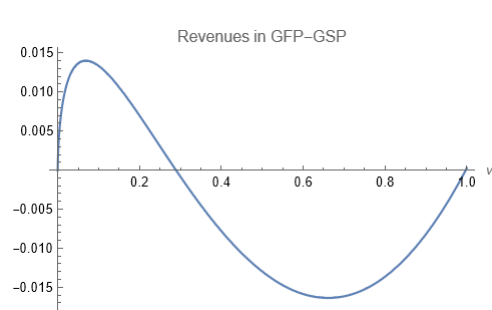
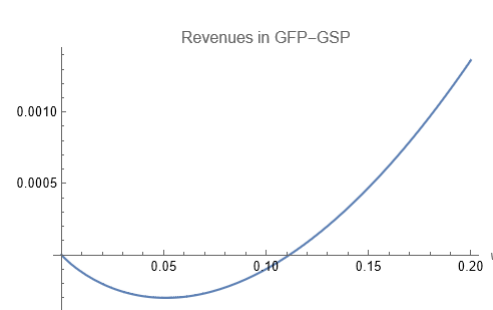
(a)  $\alpha = 0$ , Auction Revenues(b)  $\alpha = 0.4$ , Auction Revenues(c)  $\alpha = 0.8$ , Auction Revenues(d)  $\alpha = 0$ , Revenue Difference(e)  $\alpha = 0.4$ , Revenue Difference(f)  $\alpha = 0.8$ , Revenue Difference

Figure 3: Revenues in GFP and GSP Auctions

## 7 More than two bidders

In the basic model of Section 2, we restrict attention to the case of two bidders. We now show that our existence result continues to hold for any  $n \geq 2$ .

Suppose there are  $n \geq 2$  bidders, each with value  $v = 1$  per click. The ad of bidder  $i$  has probability  $q_i$  of being clicked (conditional on being seen by the user). As before, there are two slots on the screen, with the top slot having visibility 1 and the bottom slot having visibility  $\alpha \in (0, 1)$ . Bidders are ranked based on the product of their bid  $b_i$  and click-through rate  $q_i$ . If bidder  $i$  has the highest-ranked bid, her net payoff is thus  $q_i(1 - b_i)$ , while the payoff of the second-highest bidder  $j$  is  $\alpha q_j(1 - b_j)$ .

**Theorem 6** *In the auction described above, there exists a unique symmetric pure-strategy equilibrium, with bids given by*

$$b_i = \frac{n - 2\alpha}{\frac{n^2}{n-1} - \alpha}.$$

The proof of Theorem 6 is in the Appendix. As a “sanity check,” note that for  $n = 2$ , the expression is equal to the one we derived in Theorem 3.1 for the case of two bidders.

## 8 More than two slots, general distributions of click-through rates

Our final generalization is to the case in which the numbers of both bidders and slots are potentially greater than two, and the distributions of click probabilities  $q_i$  are general. For this general case, we show that there exists at most one symmetric pure-strategy equilibrium of GFP, and moreover, we show that that a candidate for this equilibrium (or more precisely, for the equilibrium bid of each advertiser) is characterized by a linear equation. Thus, while we cannot guarantee existence for all combinations of numbers of bidders and slots and for all distributions of values (and as we know from Section 5, a symmetric equilibrium may indeed fail to exist in some cases), for any specific setting, one can check whether such an equilibrium exist. First, the solution of the linear equation is the unique candidate for the symmetric equilibrium bid  $b^*$ . Next, it is straightforward to check whether for a particular advertiser, if everyone else bids  $b^*$ , it is a best response to also bid  $b^*$ . (Figure 2 in Section 5 illustrates this “candidate checking” approach for a particular setting of that section.) If it is, then  $b^*$  is the unique pure-strategy equilibrium; if it is not, then such an equilibrium does not exist.

Formally, suppose there are  $n \geq 2$  bidders, each with value  $v = 1$  per click, and  $n$  slots, with visibilities  $1 = \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$  (several of the slots can have visibility zero, allowing for the possibility that there are fewer positive-visibility slots than bidders). The click-through rate of each bidder  $i$ ,  $q_i$ , is drawn from distribution  $F [q, \bar{q}]$  that has a probability density function  $f$  that is continuous everywhere except perhaps at a finite number of points.

**Theorem 7** *The Generalized First-Price auction in the above setting has at most one symmetric pure-strategy equilibrium  $b^*$ .*

**Proof.** Consider bidder  $i$ . Suppose all the other bidders bid  $b^*$ . Suppose bidder  $i$  submits some bid  $b_i$ . Note that the expected number of clicks that  $i$  receives depends only on the ratio  $\frac{b_i}{b^*}$ , because if both  $b_i$  and  $b^*$  are multiplied by the same positive number, the expected number of clicks that  $i$  receives remains unchanged. Denote by  $C\left(\frac{b_i}{b^*}\right)$  the expected number of clicks that  $i$  receives.<sup>11</sup>

The expected payoff of bidder  $i$  is equal to

$$(1 - b_i)C\left(\frac{b_i}{b^*}\right).$$

Thus, the first-order condition for the optimal best response is given by

$$-C\left(\frac{b_i}{b^*}\right) + \frac{1 - b_i}{b^*}C'\left(\frac{b_i}{b^*}\right) = 0.$$

In a symmetric equilibrium, when  $b_i = b^*$ , this expression simplifies to

$$-C(1) + \frac{1 - b^*}{b^*}C'(1) = 0,$$

which is equivalent to the linear equation  $-b^*C(1) + (1 - b^*)C'(1) = 0$ , giving us a formula for the unique candidate bid profile  $b^*$ :

$$b^* = \frac{C'(1)}{C'(1) + C(1)}.$$

■

## 9 Conclusion

In this paper, we revisit the classic result on the (non-)existence of pure-strategy Nash equilibria in the Generalized First-Price Auction and show that the conclusion may be reversed when ads are ranked based on a product of stochastic quality scores and bid amounts, as is commonly done in practice—provided there is enough uncertainty in these quality scores. Moreover, the expected revenue in the pure strategy equilibrium of the Generalized First-Price Auction may substantially exceed that of the Generalized Second-Price Auction.

Our baseline model is deliberately simple and streamlined, to illustrate the essential driving forces behind our results. We also show that some of the important features of our results continue to hold in various generalizations. Of course, in practice, sponsored search auction settings can be even more complex. Nevertheless, the overall intuition of the stochastic click-through rates “smoothing out” payoff functions in the Generalized First-Price Auction—and thus making the

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<sup>11</sup>The expected number of clicks when bidder  $i$  bids  $b_i$  and everyone else bids  $b^*$  is equal to  $\int_{\underline{q}}^{\bar{q}} q \sum_{j=1}^n \alpha_j \binom{n-1}{j-1} F\left(\frac{b_i}{b^*}q\right)^{n-j} \left(1 - F\left(\frac{b_i}{b^*}q\right)\right)^{j-1} f(q) dq$ , with the notational convention that  $0^0 = 1$  for the terms in which  $j = n$  and  $F(\cdot) = 0$  or  $j = 1$  and  $1 - F(\cdot) = 0$ .

existence of a pure-strategy equilibrium more likely—should continue to hold in these more general settings.

## Appendix: Proof of Theorem 6

Let  $s_{(1)}, s_{(2)}, s_{(3)}, \dots$  denote the first, second, third, etc. highest scores, excluding bidder  $i$ . Then, the expected utility of a bidder  $i$  is:

$$\begin{aligned} \mathbb{E}U(b_i, b_{-i}) &= (1 - b_i) \mathbb{E} \left[ q_1 \left( \alpha \mathbb{1}_{b_i q_i > s_{(2)}} + (1 - \alpha) \mathbb{1}_{b_i q_i > s_{(1)}} \right) \right] \\ &= (1 - b_i) \left[ \int_0^1 q \alpha h_{(2)}(b_i, b_j, q) dq + \int_0^1 q (1 - \alpha) h_{(1)}(b_i, b_j, q) dq \right], \end{aligned}$$

where  $h_{(2)}(b_i, b_j, q)$  and  $h_{(1)}(b_i, b_j, q)$  are the probabilities that the score of bidder  $i$  is higher than the second highest or highest score of the other bidders (respectively), given that  $q_i = q$ .

We find the symmetric equilibrium for this general game by finding the best response of player  $i$  given that all other players play a symmetric strategy  $b$ , which is a necessary condition for the existence of such an equilibrium. Notice that this has the advantage that in this case player  $i$  wins if:

$$b_i q_i > s_{(1)} = b q_1 \Rightarrow \frac{b_i}{b} q_i > q_1,$$

and they end up second if:

$$b_i q_i > s_{(2)} = b q_2 \Rightarrow \frac{b_i}{b} q_i > q_2.$$

But this means that when setting up the problem looking for a symmetric equilibrium, we will be looking at the distributions of the first and second order statistics for the click-through rates. For the uniform case, this means we have two different cases to consider:

$$\begin{aligned} q < \frac{b}{b_i} &\Rightarrow h_{(2)}(b_i, b, q) = F_{(2)} \left( \frac{b_i}{b} q \right) \text{ and } h_{(1)}(b_i, b, q) = F_{(1)} \left( \frac{b_i}{b} q \right) \\ q > \frac{b}{b_i} &\Rightarrow h_{(2)}(b_i, b, q) = h_{(1)}(b_i, b, q) = 1, \end{aligned}$$

where  $F_{(1)}$  and  $F_{(2)}$  are the first and second order statistics for the click-through rates  $q$  of all other players. For the uniform case, this reduces to:

$$\begin{aligned} F_{(1)} \left( \frac{b_i}{b} q \right) &= \left( \frac{b_i}{b} q \right)^{n-1} \\ F_{(2)} \left( \frac{b_i}{b} q \right) &= (n-1) \left( \frac{b_i}{b} q \right)^{n-2} \left( 1 - \frac{b_i}{b} q \right) + \left( \frac{b_i}{b} q \right)^{n-1} \end{aligned}$$

Thus, the expected utility of bidder  $i$  is:

$$\mathbb{E}U(b_i, b_{-i}) = (1 - b_i) \left[ \int_0^{b/b_i} q\alpha \left( (n-1) \left( \frac{b_i}{b} q \right)^{n-2} \left( 1 - \frac{b_i}{b} q \right) + \left( \frac{b_i}{b} q \right)^{n-1} \right) dq + \int_{b/b_i}^1 q\alpha dq + \int_0^{b/b_i} q(1-\alpha) \left( \frac{b_i}{b} q \right)^{n-1} dq + \int_{b/b_i}^1 q(1-\alpha) dq \right].$$

Now, notice that we can simplify some parts:

$$\mathbb{E}U(b_i, b_{-i}) = (1 - b_i) \left[ \int_0^{b/b_i} q\alpha \left( (n-1) \left( \frac{b_i}{b} q \right)^{n-2} \left( 1 - \frac{b_i}{b} q \right) + \left( \frac{b_i}{b} q \right)^{n-1} \right) dq + \int_{b/b_i}^1 q\alpha dq + \int_0^{b/b_i} q(1-\alpha) \left( \frac{b_i}{b} q \right)^{n-1} dq + \int_{b/b_i}^1 q(1-\alpha) dq \right].$$

Here, the two parts in blue and the two parts in red simplify to get rid of the  $\alpha$  in those expressions:

$$\begin{aligned} \mathbb{E}U(b_i, b_{-i}) &= (1 - b_i) \left[ \int_0^{b/b_i} q \left( \alpha(n-1) \left( \frac{b_i}{b} q \right)^{n-2} \left( 1 - \frac{b_i}{b} q \right) + \left( \frac{b_i}{b} q \right)^{n-1} \right) dq + \int_{b/b_i}^1 q dq \right] = \\ (1 - b_i) &\left[ \alpha(n-1) \left( \int_0^{b/b_i} \left( \frac{b_i}{b} \right)^{n-2} q^{n-1} dq - \int_0^{b/b_i} \left( \frac{b_i}{b} \right)^{n-1} q^n dq \right) + \int_0^{b/b_i} \left( \frac{b_i}{b} \right)^{n-1} q^n dq + \int_{b/b_i}^1 q dq \right]. \end{aligned}$$

Evaluating these integrals yields:

$$\begin{aligned} \mathbb{E}U(b_i, b_{-i}) &= (1 - b_i) \left[ \alpha(n-1) \left( \frac{1}{n} \left( \frac{b}{b_i} \right)^2 - \frac{1}{n+1} \left( \frac{b}{b_i} \right)^2 \right) + \frac{1}{n+1} \left( \frac{b}{b_i} \right)^2 + \frac{1}{2} - \left( \frac{b}{b_i} \right)^2 \right] \\ &= (1 - b_i) \left[ \alpha(n-1) \left( \frac{1}{n} \left( \frac{b}{b_i} \right)^2 - \frac{1}{n+1} \left( \frac{b}{b_i} \right)^2 \right) + \frac{1}{n+1} \left( \frac{b}{b_i} \right)^2 + \frac{1}{2} - \frac{1}{2} \left( \frac{b}{b_i} \right)^2 \right]. \end{aligned}$$

We can simplify this further to get to:

$$\mathbb{E}U(b_i, b_{-i}) = (1 - b_i) \left[ \frac{1}{2} - \left( \frac{b}{b_i} \right)^2 \overbrace{\left( \frac{1}{2} - \frac{1}{n+1} - \frac{(n-1)\alpha}{(n+1)n} \right)}^{(*)} \right].$$

Notice that for all  $n \geq 2$  and all  $\alpha \in [0, 1]$ , we have that  $(*) \geq 0$ . This means that for all values  $b_i \in [0, 1]$ , the bidder's expected utility is strictly concave, which means that the FOC will produce the unique best response. Bidder  $i$ 's problem is:

$$\max_{b_i} \mathbb{E}U(b_i, b_{-i}) = (1 - b_i) \left[ \frac{1}{2} - \left( \frac{b}{b_i} \right)^2 \left( \frac{1}{2} - \frac{1}{n+1} - \frac{(n-1)\alpha}{(n+1)n} \right) \right].$$

The FOC is given by:

$$-\frac{1}{2} + \left(\frac{b}{b_i}\right)^2 \left(\frac{1}{2} - \frac{1}{n+1} - \frac{(n-1)\alpha}{(n+1)n}\right) + (1-b_i)(-2)\frac{b^2}{b_i^3} \left(\frac{1}{2} - \frac{1}{n+1} - \frac{(n-1)\alpha}{(n+1)n}\right) = 0.$$

Imposing symmetry we get:

$$-\frac{1}{2} + \frac{1}{2} - \frac{1}{n+1} - \frac{(n-1)\alpha}{(n+1)n} - 2(1-b_i)\frac{1}{b_i} \left(\frac{1}{2} - \frac{1}{n+1} - \frac{(n-1)\alpha}{(n+1)n}\right) = 0.$$

The resulting equation is linear in  $b_i$ , and thus the solution is:

$$b_i = \frac{1 - \frac{2}{n+1} - \frac{2(n-1)\alpha}{(n+1)n}}{1 - \frac{1}{n+1} - \frac{(n-1)\alpha}{(n+1)n}} = b_i = \frac{n - 2\alpha}{\frac{n^2}{n-1} - \alpha}.$$

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