Scalable MCMC for Bayes Shrinkage Priors

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July 2, 2018

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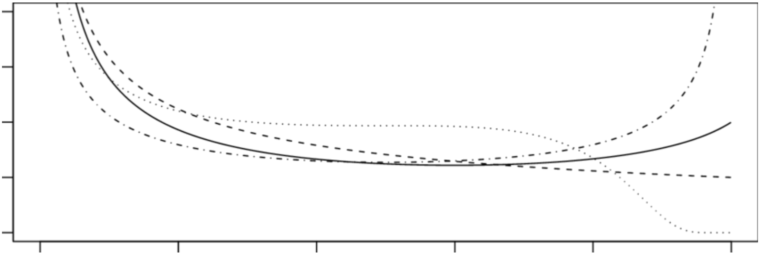

Joint work with James Johndrow and Anirban Bhattacharya

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  - Point mass mixture prior, *but*: computation is prohibitive

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- Global-local priors can achieve this (with some qualifications).
- But... they are still slow.
  - Lasso: $n \approx 1,000$, $p \approx 1,000,000$;
  - Global-local: $n \approx 1,000$, $p \approx 1,000$.

Model

- The Horseshoe model*:

$$y_i \mid \beta_j, \lambda_j, \tau, \sigma^2 \overset{ind}{\sim} N(x_i\beta, \sigma^2)$$

$$\beta_j \overset{ind}{\sim} N(0, \tau^2\lambda_j^2)$$

$$\lambda_j \overset{ind}{\sim} \text{Cauchy}_+(0,1)$$

$$\tau \sim \text{Cauchy}_+(0,1)$$

$$\sigma^2 \sim \text{InvGamma}(a_0/2, b_0/2)$$

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- It achieves the minimax-adaptive risk for squared error loss up to a constant.
- Suppose $X = I$, $\|\beta\|_0 = s_n$, then [van der Pas et al., 2014],

$$\sup_{\beta:\|\beta\|_0 \le s_n} \mathbb{E}_\beta \left[\|\hat{\beta}_{HS} - \beta\|_2^2 \right] \le 4\sigma^2 s_n \log \frac{n}{s_n} \cdot (1 + o(1)),$$

while, for any estimator $\hat{\beta}$, [Donoho et al., 1992] shows

$$\sup_{\beta:\|\beta\|_0 \le s_n} \mathbb{E}_\beta \left[\|\hat{\beta} - \beta\|_2^2 \right] \ge 2\sigma^2 s_n \log \frac{n}{s_n} \cdot (1 + o(1)).$$

Computation

- State-of-the-art: (i) $\tau \mid \beta, \sigma^2, \lambda$, (ii) $\left(\beta, \sigma^2\right) \mid \tau, \lambda$, (iii) slice sampling for λ.

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- First idea: **block** (β, σ^2, τ) to improve *mixing*;
  1. sample $(\beta, \sigma^2, \tau) \mid \lambda$ by block sampling: $\tau \mid \lambda$, then $\sigma^2 \mid \tau, \lambda$, and finally $\beta \mid \sigma^2, \tau, \lambda$;
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- Second idea: **truncate** some of the matrices involved to improve the *computational cost per step*.

Gibbs sampling

Let $M = X(\text{diag}(\xi\eta))^{-1}X^T + I$, $\xi = \tau^{-2}$, $\eta_j = \lambda_j^{-2}$, and **block update**:

- $p(\tau \mid \lambda, y) \propto \frac{1}{\sqrt{\xi}(1+\xi)} |M|^{-1/2} \left(y^T M^{-1} y + b_0\right)^{-\frac{n+a_0}{2}}$
- $p(\sigma^2 \mid \tau, \lambda, y) \sim \text{InvGamma}\left(\frac{n+a_0}{2}, \frac{1}{2}\left[y^T M^{-1} y + b_0\right]\right)$
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Then perform slice sampling:

- $p(\lambda \mid \beta, \sigma^2, \tau, y)$: (i) $U \mid \eta_j \sim \text{Unif}\left[0, \frac{1}{1+\eta_j}\right]$; (ii) $\eta_j \mid u \sim e^{-\frac{\xi\beta_j^2}{2\sigma^2}\eta_j} \mathbb{I}_{\left[\frac{1-u}{u} > \eta_j\right]}$.

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Markov approximation

- We approximate $M = X\text{diag}((\xi\eta_j)^{-1})X^T + I$ with

$$M_\delta = XD_\delta X^T + I, \qquad D_\delta = \text{diag}((\xi\eta_j)^{-1}\mathbb{I}_{[(\xi_{\max}\eta_j)^{-1}>\delta]})$$

for $\delta \ll 1$, and $\xi_{\max}$ the maximum of the current and proposed ξ.

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Approximating Kernels

Let $\mathcal{P}_\delta(x, \cdot)$ and $\mathcal{P}(x, \cdot)$ denote the Markov operators for the approximate and exact algorithms, with $x = (\beta, \sigma^2, \tau, \lambda)$ the entire state vector. Then

$$\sup_x \|\mathcal{P}_\delta(x, \cdot) - \mathcal{P}(x, \cdot)\|_{\text{TV}} \leq \sqrt{\delta}\|X\|\sqrt{a + \frac{n + a_0}{b_0} + \frac{n}{2}\frac{\|y\|^2}{b_0}} + \mathcal{O}(\delta),$$

for sufficiently small $\delta > 0$.

Simulation

- We simulate data as follows:

$$x_i \overset{\text{iid}}{\sim} N_p(0, \Sigma)$$
$$y_i \sim N(x_i\beta, 4)$$
$$\beta_j = \begin{cases} 2^{-(j/4-9/4)} & \text{if } j < 24, \\ 0 & \text{if } j \geq 24. \end{cases}$$

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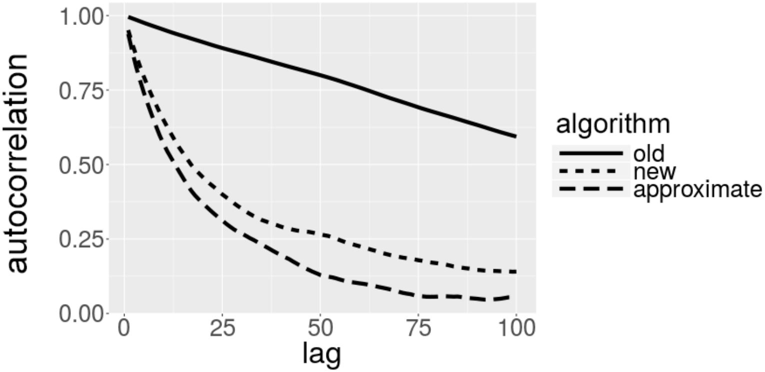
- We consider both $\boldsymbol{\Sigma} = I$ (independent design) and $\boldsymbol{\Sigma}_{ij} = 0.9^{|i-j|}$ (correlated design).

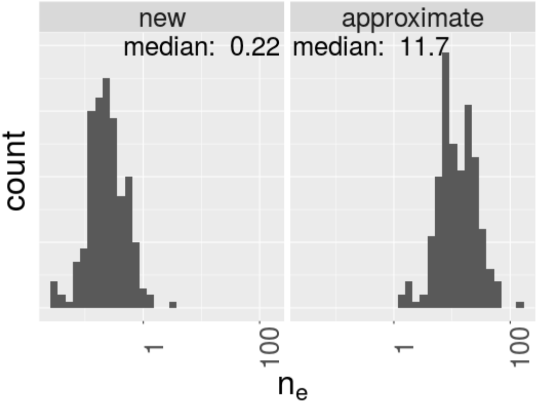
Autocorrelation



Autocorrelation for $\log(\xi) = -2 \log \tau$

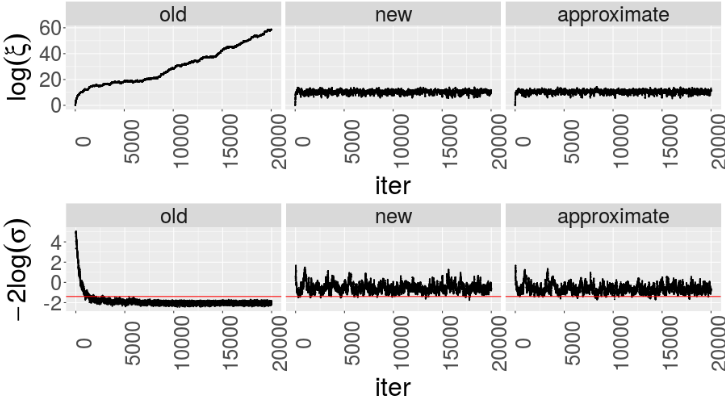
Effective samples per second

- Approximate algorithm is 50× more efficient with $n = 2,000$ and $p = 20,000$.



Accuracy

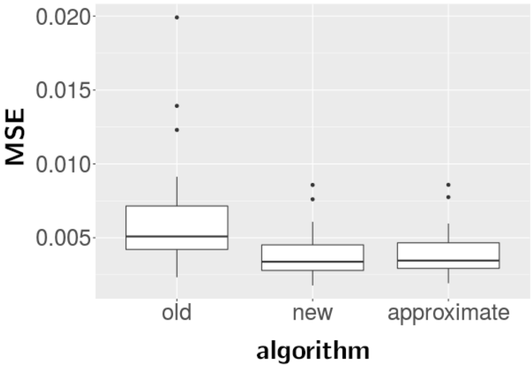
- Existing algorithms failed to converge, due to numerical underflow.



Trace plots for $-2\log(\sigma)$ and $\log(\xi) = -2\log(\tau)$; truth in red

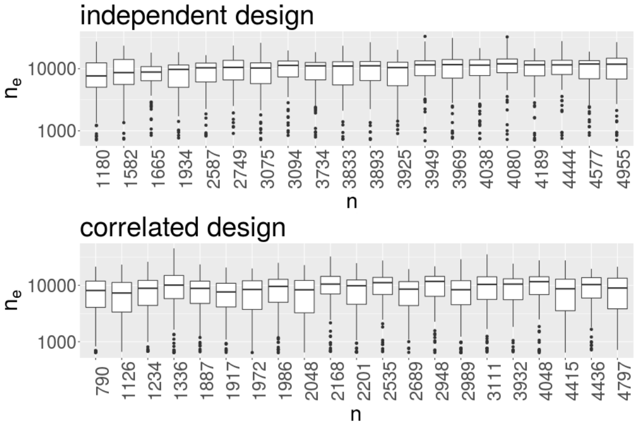
Accuracy

- In terms of MSE, the approximation costs us little.



Dependence on p and n

- Effective sample sizes seem independent of n and p.



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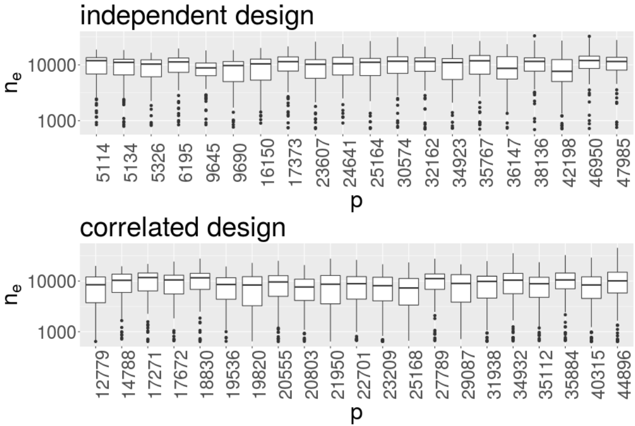

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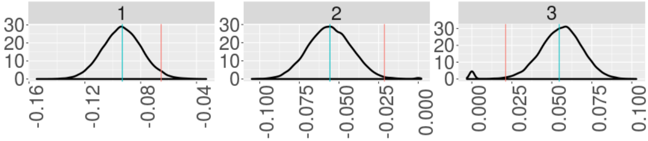

Bimodal posterior distribution for $\beta \mid y$; Lasso (red) shrinks more than Horseshoe (blue)

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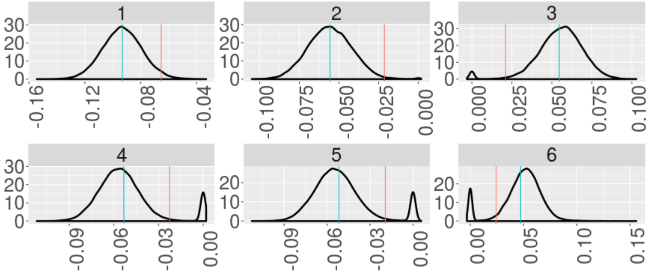

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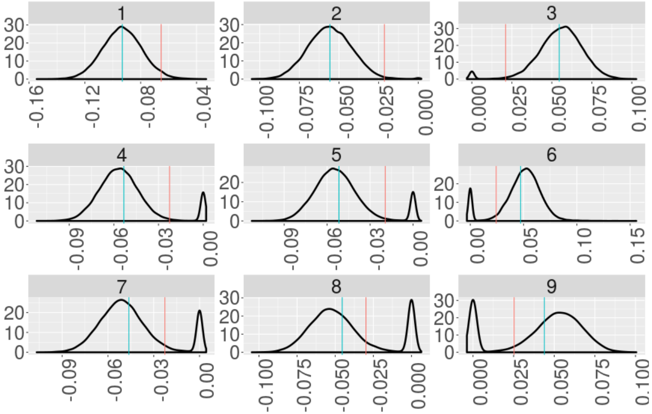
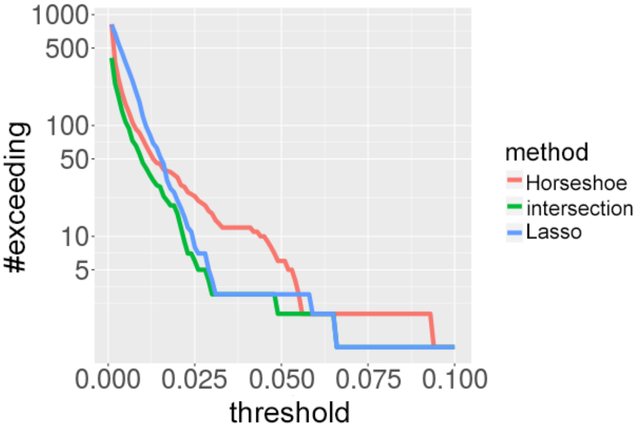


Bimodal posterior distribution for $\beta \mid y$; Lasso (red) shrinks more than Horseshoe (blue)

Variable selection with Horseshoe



Number of variables for which $\hat{\beta}_{\mathrm{HS},j} = \mathbb{E}[\beta_j \mid y] > t$ or $\hat{\beta}_{\mathrm{Lasso},j} > t$ vs threshold t; both methods largely agree on the identities of the signals

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- We manage to do that for the Horseshoe prior with two ideas: blocking and truncation.
- We observed interesting and novel statistical phenomena, e.g., bimodality of β.
- There is likely more room for improvement.

References

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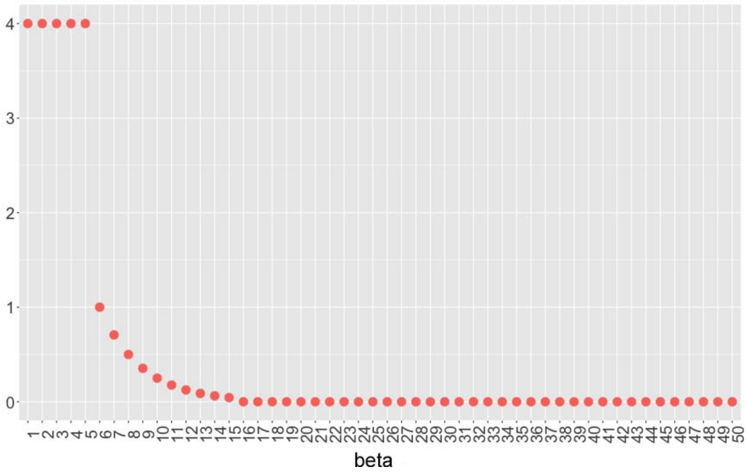
Extra slides

- More simulation results
- Why “Horseshoe”?

More simulations

- We let $n = 1000$ and $p = 20,000$.



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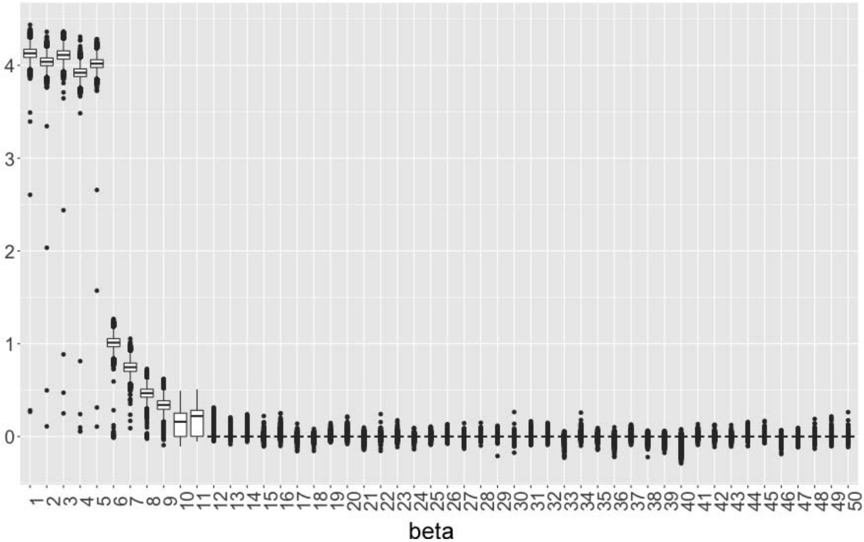

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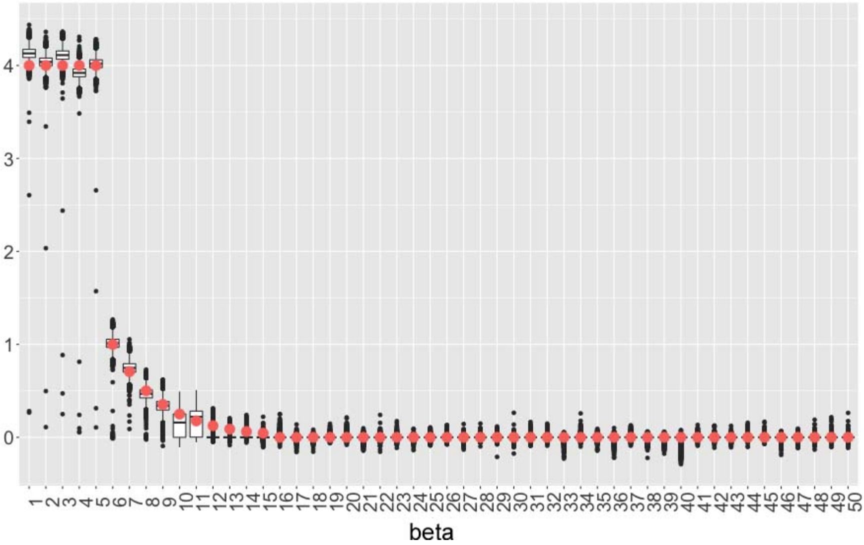

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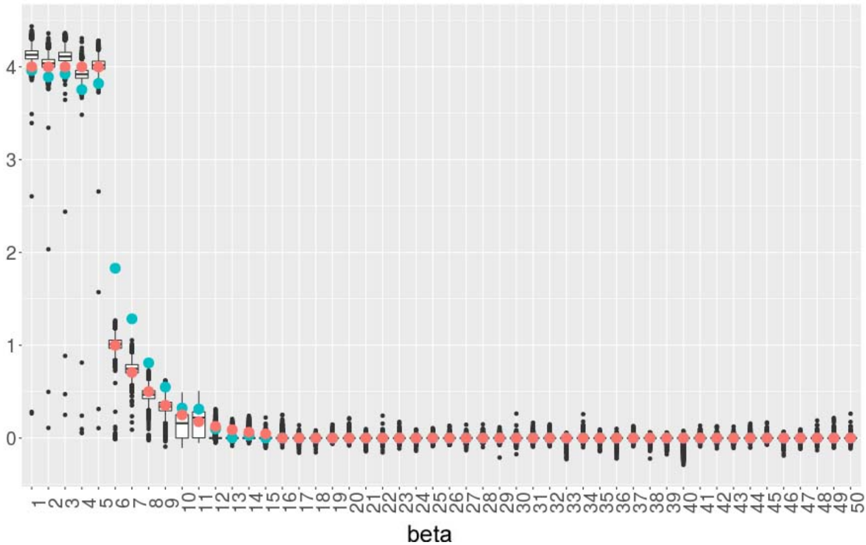
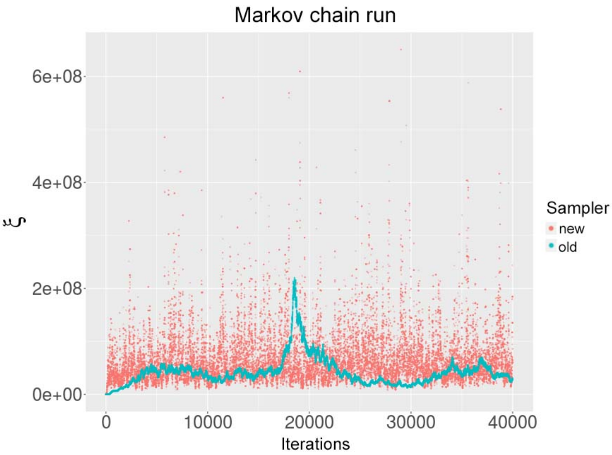


More simulations

- The new algorithm lead to significant improvement in the autocorrelation:



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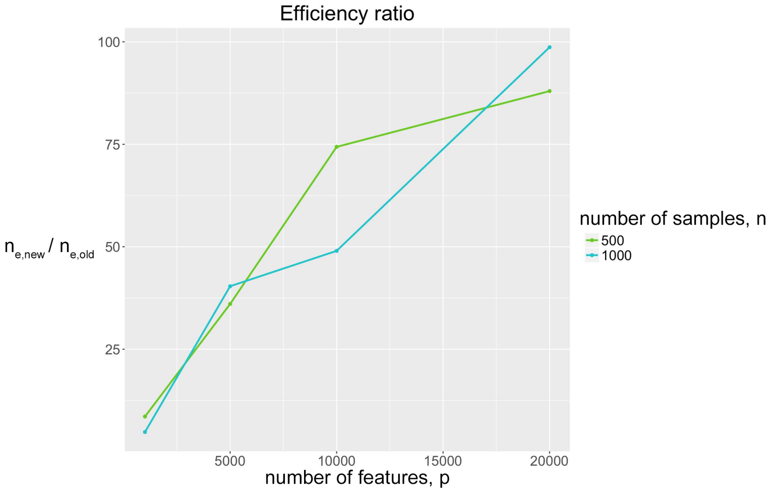

Why "Horseshoe"?

- In the orthogonal case with $n \geq p$ and $\sigma^2 = \tau = 1$, and defining a shrinkage profile $\kappa_j = 1/(1 + n\lambda_j^2)$, we can write $\mathbb{E}[\beta_j|y] = (1 - \mathbb{E}[\kappa_j|y])\hat{\beta}_j$.

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- Prior density for κ_j: