EE 169 INTRODUCTION TO BIOIMAGING

LECTURE 3

2D SYSTEMS (SECTION 2.3)

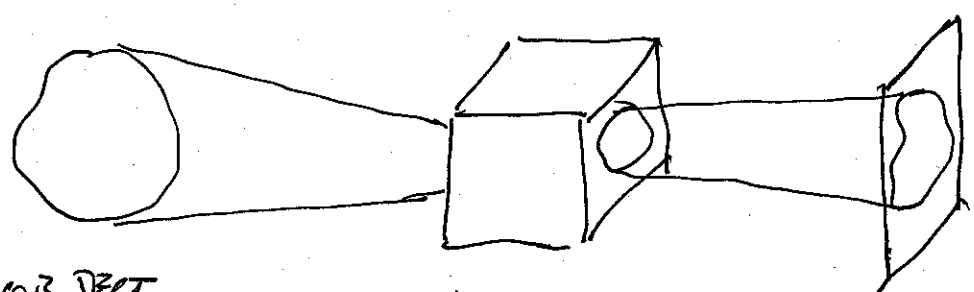
FOURIER TRANSFORM (SECTION 2.4, 2.5)

2D SYSTEMS

IN GENERAL A 2D SYSTEM DOES:

$$\underbrace{g(x,y)}_{\text{OUTPUT}} = S\left[\underbrace{f(x,y)}_{\text{INPUT}}\right]$$

$S[\cdot]$ OPERATES ON ALL OF $f(x,y)$ TO PRODUCE $g(x,y)$.



OBJECT $f(x,y)$ IMAGING SYSTEM $S[\cdot]$ IMAGE $g(x,y)$

LINEAR SYSTEM

$$\underbrace{S\underbrace{\left[\sum_{k=1}^{\infty} w_k f_k(x,y)\right]}_{\text{COMBINE INPUTS}}}_{\text{APPLY SYSTEM}} = \underbrace{\sum_{k=1}^{\infty} w_k \underbrace{S[f_k(x,y)]}_{\text{APPLY SYSTEM}}}_{\text{COMBINE OUTPUTS}}$$

SEEMS SIMPLE, IMPORTANT IMPLICATIONS

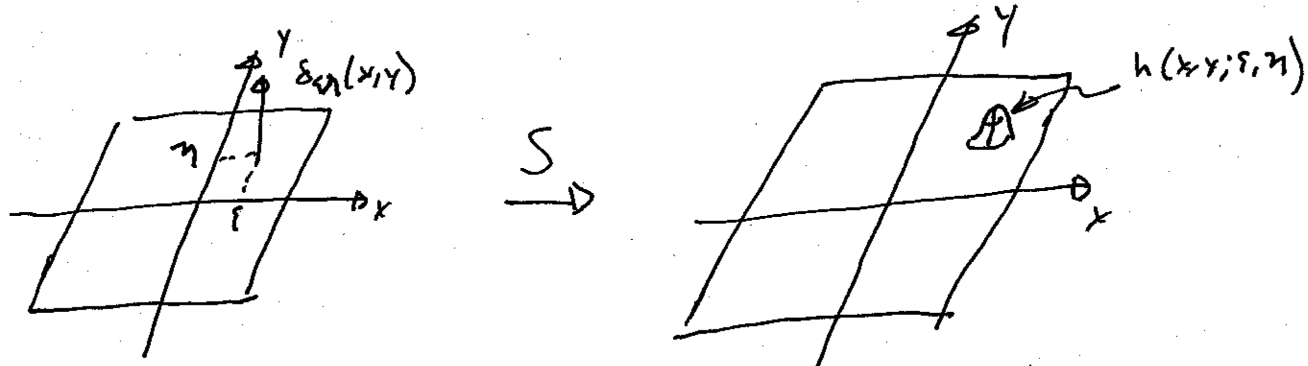
MANY SYSTEMS AREN'T LINEAR WHEN YOU LOOK CLOSELY

CAN BE APPROXIMATED AS LINEAR

IMPULSE RESPONSE

APPLY AN IMPULSE, SEE WHAT THE SYSTEM DOES

$$\underbrace{h(x,y;\underbrace{\xi,\eta}_{\text{IMPULSE LOCATION}})}_{\substack{\text{RESPONSE TO}\\ \text{IMPULSE AT } \xi,\eta}} = S\left[\underbrace{\delta_{\xi\eta}(x,y)}_{\delta(x-\xi,\,y-\eta)}\right]$$



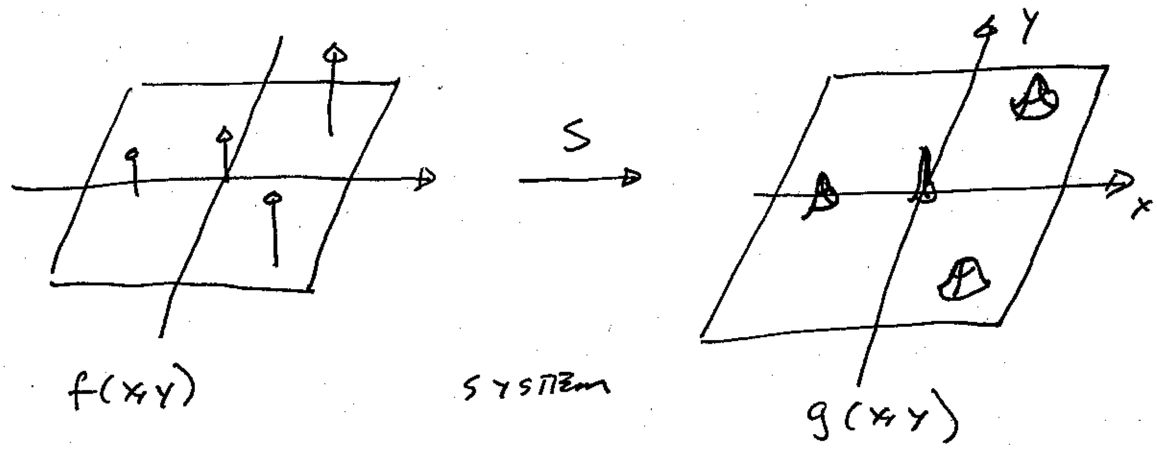
IMPULSE INPUT SYSTEM IMPULSE RESPONSE

ALSO CALLED POINT SPREAD FUNCTION (PSF)

IMPULSE RESPONSE COMPLETELY CHARACTERIZES A LINEAR SYSTEM

$$\begin{aligned} g(x,y) &= S\left[f(x,y)\right] \\ &= S\Big[\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\delta(x-\xi,y-\eta)\,d\xi\,d\eta}_{\text{SAMPLE OF } f(\xi,\eta) \text{ AT } (x,y)}\Big] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S\left[f(\xi,\eta)\,\delta(x-\xi,y-\eta)\right] d\xi\,d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,S\left[\delta(x-\xi,y-\eta)\right] d\xi\,d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,h(x,y;\xi,\eta)\,d\xi\,d\eta \end{aligned}$$

SUPERPOSITION INTEGRAL



IMPULSE RESPONSE CAN (AND USUALLY DOES!) VARY IN SPACE

EXAMPLE: OPTICAL LENSES
RESOLUTION DEGRADES OFF AXIS

SHIFT INVARIANCE

LET

$$f_{x_0,y_0}(x,y) = f(x-x_0, y-y_0)$$

AND

$$g(x,y) = S[f(x,y)]$$

IF $S[\cdot]$ IS SHIFT INVARIANT

$$\underbrace{g(x-x_0, y-y_0)}_{\text{SHIFTED OUTPUT}} = S\Big[\underbrace{f_{x_0,y_0}(x,y)}_{\text{SHIFTED INPUT}}\Big]$$

IF A LINEAR SYSTEM IS SHIFT INVARIANT, ITS IMPULSE RESPONSE IS

$$S\Big[\underbrace{\delta_{\xi\eta}(x,y)}_{\text{SHIFTED IMPULSE}}\Big] = \underbrace{h(x-\xi,\, y-\eta)}_{\text{SHIFTED IMPULSE RESPONSE}}$$

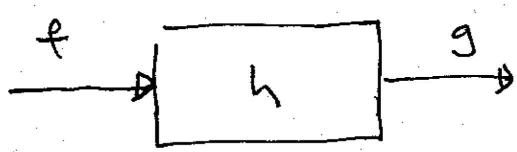
THE RESPONSE OF A LINEAR SHIFT INVARIANT (LSI) SYSTEM TO AN INPUT $f(x,y)$ IS

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(x,y;\xi,\eta)\, d\xi\, d\eta$$

$$\boxed{g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(x-\xi,\, y-\eta)\, d\xi\, d\eta}$$

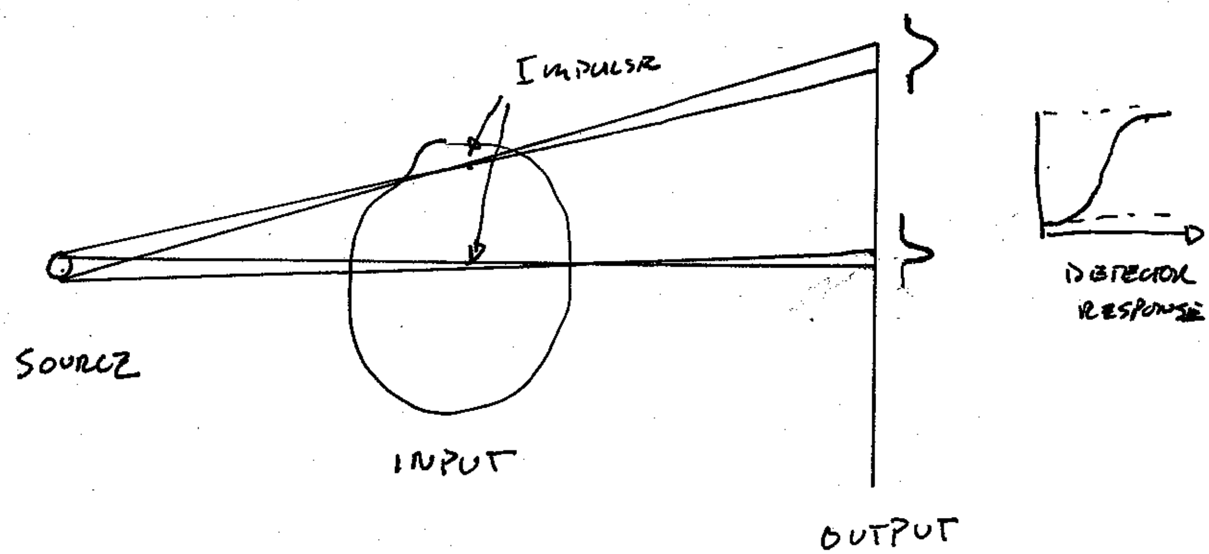
<u>2D CONVOLUTION</u>

$$g(x,y) = f(x,y) * h(x,y)$$
$$g = f * h$$



MANY SYSTEMS ARE NOT LSI, BUT CAN STILL BE STUDIED THIS WAY

X-RAY PROJECTION



SIMPLE SYSTEM, BUT

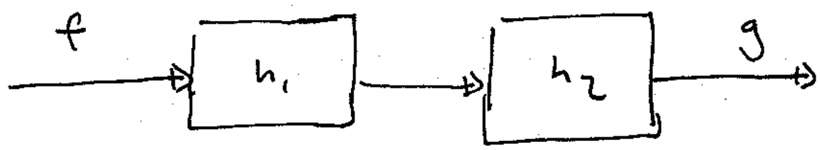
NON-LINEAR (DETECTOR RESPONSE, X-RAY ATTENUATION)

SPACE VARIANT (MAGNIFICATION, OBLIQUITY)

HOWEVER, APPROXIMATELY LINEAR, SPACE INVARIANT OVER AN INPUT RANGE, LOCAL REGION IN OUTPUT

COMPOSITION OF LSI SYSTEMS

SERIAL OR CASCADE CONNECTION

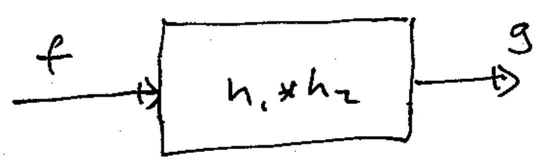


$$g = h_2 * (h_1 * f)$$

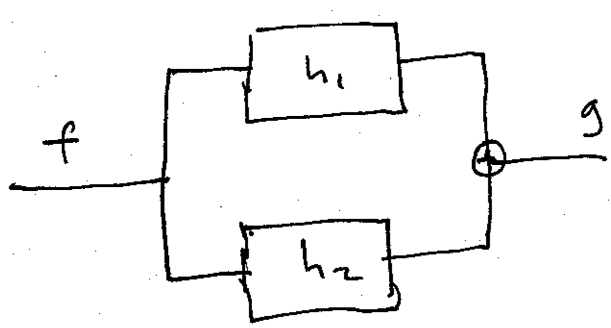
$$= (h_1 * h_2) * f$$

CONVOLUTION IS ASSOCIATIVE AND COMMUTATIVE

THIS IS THE SAME AS



PARALLEL CONNECTION

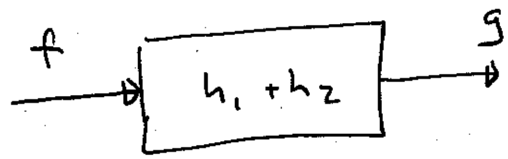


$$g = h_1 * f + h_2 * f$$

$$= (h_1 + h_2) * f$$

CONVOLUTION IS DISTRIBUTIVE

THIS IS THE SAME AS



SEPARABLE SYSTEMS

SEPARABLE IMPULSE RESPONSE

$$h(x,y) = h_1(x)\,h_2(y)$$

SIMPLIFIES CONVOLUTION

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, h_1(x-\xi)\, h_2(y-\eta)\, d\xi\, d\eta$$

$$= \underbrace{\int_{-\infty}^{\infty} h_2(y-\eta) \underbrace{\int_{-\infty}^{\infty} f(\xi,\eta)\, h_1(x-\xi)\, d\xi}_{\text{CONVOLUTION FOR EACH } x}\, d\eta}_{\text{CONVOLUTION IN } y \text{ FOR EACH } x}$$

OFTEN EASIER TO COMPUTE.

FOURIER TRANSFORMS

ω VS $2\pi f$

IN 102a WE USED ω FORM OF THE FOURIER TRANSFORM

$$\boxed{\begin{aligned} F_\omega(j\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\omega(j\omega)\, e^{+j\omega t}\, dt \end{aligned}}$$

HERE WE USE $2\pi f$

$$F_\omega(j2\pi f) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi f t}\, dt = F(f)$$

THE INVERSE TRANSFORM IS THEN

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_\omega(j\omega)\, e^{+j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} \underbrace{F_\omega(j2\pi f)}_{F(f)}\, e^{j2\pi f t}\, \underbrace{\frac{d\omega}{2\pi}}_{df}$$

$$= \int_{-\infty}^{\infty} F(f)\, e^{+j2\pi f t}\, df$$

THE FOURIER TRANSFORM PAIR IS THEN

$$\boxed{\begin{aligned} F(f) &= \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ft}\, dt \\ f(t) &= \int_{-\infty}^{\infty} F(f)\, e^{+j2\pi ft}\, df \end{aligned}}$$

EXACTLY SYMMETRIC, EXCEPT FOR +/- IN EXPONENT!

CONVERTING $F_\omega(j\omega)$ INTO $F(f)$ TRANSFORMS

$$\mathrm{rect}(t) \Longleftrightarrow \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

TAKE

$$F_\omega(j\omega) \xrightarrow[\omega \to 2\pi f]{} F(f)$$

AND SIMPLIFY

$$\mathrm{sinc}\left(\frac{\omega}{2\pi}\right) \xrightarrow[\omega \to 2\pi f]{} \mathrm{sinc}\left(\frac{2\pi f}{2\pi}\right) = \mathrm{sinc}(f)$$

RESULT

$$\boxed{\mathrm{rect}(t) \Longleftrightarrow \mathrm{sinc}(f)}$$

SIMILARLY

$$\mathrm{sinc}(t) \Longleftrightarrow \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

SO

$$\mathrm{sinc}(t) \Longleftrightarrow \mathrm{rect}\left(\frac{2\pi f}{2\pi}\right)$$

$$= \mathrm{rect}(f)$$

$$\boxed{\mathrm{sinc}(t) \Longleftrightarrow \mathrm{rect}(f)}$$

DUALITY

LET $g(\cdot)$ AND $h(\cdot)$ BE TWO FUNCTIONS (sinc, rect, etc)

ASSUME

$$g(t) = \int_{-\infty}^{\infty} h(f) e^{j2\pi ft} df \qquad \mathcal{F}^{-1}\{h(f)\} = g(t)$$

THEN

$$g(-t) = \int_{-\infty}^{\infty} h(f) e^{j2\pi f(-t)} df$$

$$= \underbrace{\int_{-\infty}^{\infty} h(f) e^{-j2\pi ft} df}_{\text{FORWARD TRANSFORM WITH } f \leftrightarrow t}$$

SWAP f AND t

$$g(-f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt \qquad \mathcal{F}\{h(t)\} = g(-f)$$

SO IF

$$g(t) \Longleftrightarrow h(f)$$

THEN

$$h(t) \Longleftrightarrow g(-f)$$

FROM ABOVE

$$\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}(f)$$

SO

$$\mathrm{sinc}(t) \Leftrightarrow \mathrm{rect}(-f) = \mathrm{rect}(f)$$

DUALITY IS VERY USEFUL, MUCH CLEARER WITH $2\pi f$!