LECTURE 4

2D FOURIER TRANSFORMS (SECTION 2.5)

TRANSFER FUNCTIONS, LSI SYSTEMS (SECTION 2.6)

2D FOURIER TRANSFORMS

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$f(x,y) = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{F(u,v)}_{\text{WEIGHTS}}\, \underbrace{e^{+j2\pi(ux+vy)}}_{\text{COMPLEX EXPONENTIALS}}}_{\text{LINEAR COMBINATION}}\, du\, dv$$

THE SIGNAL $f(x,y)$ IS LINEAR COMBINATION OF BASIS

$$e^{j2\pi(ux+vy)}$$

WITH WEIGHTS $F(u,v)$.

NOTATION

BOOK USES

$$F(u,v) = \mathfrak{F}_{2D}(f)(u,v)$$
$$f(x,y) = \mathfrak{F}_{2D}^{-1}(F)(x,y)$$

AWKWARD! WE WILL USE

$$F(u,v) = \mathfrak{F}_{2D}\{f(x,y)\}$$
$$f(x,y) = \mathfrak{F}_{2D}^{-1}\{F(u,v)\}$$

MUCH MORE COMMON.

SPECTRUM

$F(u,v)$ IS THE SPECTRUM OF $f(x,y)$

REAL AND IMAGINARY

$$F(u,v) = \mathrm{Re}\{F(u,v)\} + j\,\mathrm{Im}\{F(u,v)\}$$

MAGNITUDE AND PHASE

$$F(u,v) = |F(u,v)|\, e^{j\angle F(u,v)}$$

POWER SPECTRUM

$$|F(u,v)|^2$$

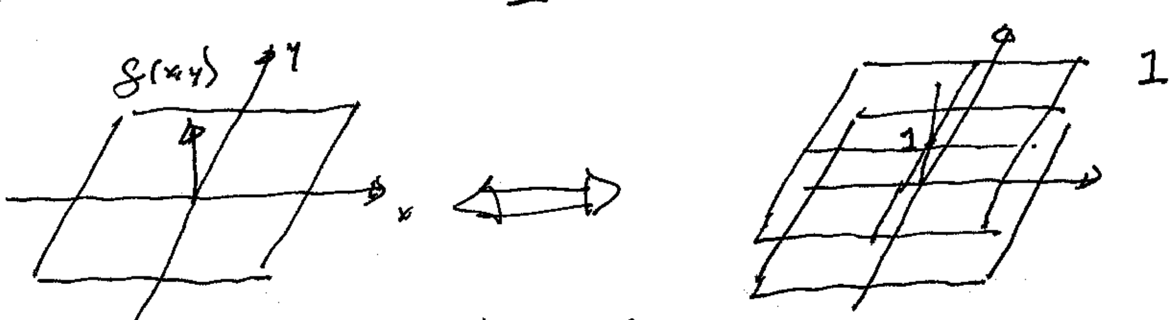
EXAMPLES

(1) $\delta(x,y)$

$$\mathcal{F}_{2D}\{\delta(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)\, e^{-j2\pi(ux+vy)}\, dx\,dy$$

$$= e^{-j2\pi(ux+vy)}\Big|_{x=0,\,y=0}$$

$$= 1$$



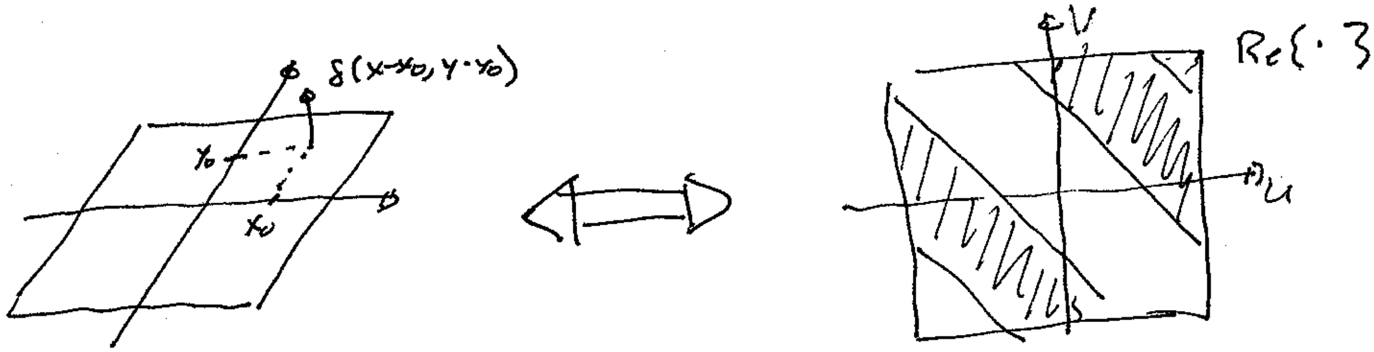
$$\delta(x,y) \Leftrightarrow 1$$

② $\delta(x-x_0, y-y_0)$

$$\mathcal{F}_{2D}\{\delta(x-x_0, y-y_0)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-x_0, y-y_0)\, e^{-j2\pi(ux+vy)}\, dx\,dy$$

$$= e^{-j2\pi(ux+vy)}\Big|_{\substack{x=x_0\\ y=y_0}}$$

$$= e^{-j2\pi(ux_0+vy_0)}$$



COMPLEX SPATIAL FREQUENCY

$$\boxed{\delta(x-x_0, y-y_0) \Longleftrightarrow e^{-j2\pi(ux_0+vy_0)}}$$

③ $\delta(u-u_0, v-v_0)$

$$\mathcal{F}^{-1}_{2D}\{\delta(u-u_0, v-v_0)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u-u_0, v-v_0)\, e^{+j2\pi(ux+vy)}\, dx\,dy$$

$$= e^{+j2\pi(ux+vy)}\Big|_{\substack{u=u_0\\ v=v_0}}$$

$$= e^{+j2\pi(u_0x+v_0y)}$$

$$\boxed{e^{+j2\pi(u_0x+v_0y)} \Longleftrightarrow \delta(u-u_0, v-v_0)}$$

FOURIER TRANSFORM THEOREMS

MANY SAME AS 1D, SEVERAL NEW ONES

SIMILAR THEOREMS

LINEARITY

$$\mathcal{F}_{2D}\{a f(x,y) + b g(x,y)\} = a F(u,v) + b G(u,v)$$

TRANSLATION

$$\mathcal{F}_{2D}\{f(x-x_0, y-y_0)\} = F(u,v)\, e^{-i2\pi(ux+vy)}$$

SCALING

$$\mathcal{F}_{2D}\{f(ax, by)\} = \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

CONVOLUTION

$$\mathcal{F}_{2D}\{f(x,y) * g(x,y)\} = F(u,v)\, G(u,v)$$

PRODUCT

$$\mathcal{F}_{2D}\{f(x,y)\, g(x,y)\} = F(u,v) * G(u,v)$$

NO 2π

DUALITY

IF

$$f(x,y) \Longleftrightarrow F(u,v)$$

THEN

$$F(x,y) \Longleftrightarrow f(-u,-v)$$

NEW THEOREMS

CONJUGATION

$$\mathcal{F}_{2D}\{f^*(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x,y)\, e^{-j2\pi(ux+vy)}$$

$$= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi((-u)x+(-v)y)}\, dx\,dy\right]^*$$

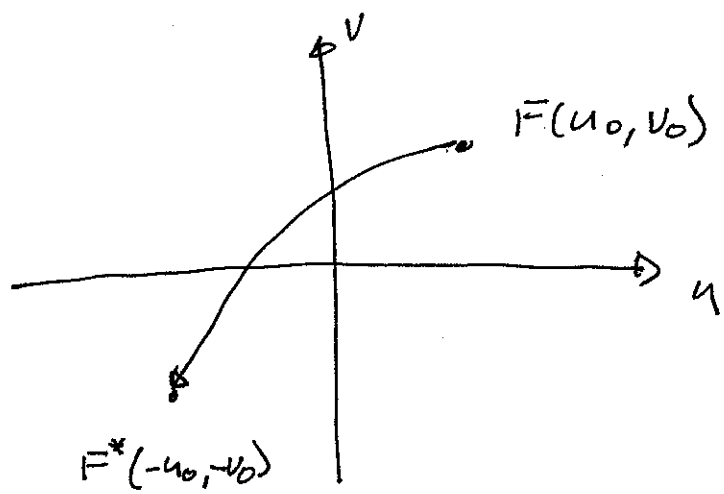
$$= F^*(-u,-v)$$

IMPORTANT SPECIAL CASE: IF $f(x,y)$ IS REAL

$f(x,y) = f^*(x,y)$, AND

$$F(u,v) = F^*(-u,-v)$$

HERMITIAN SYMMETRY



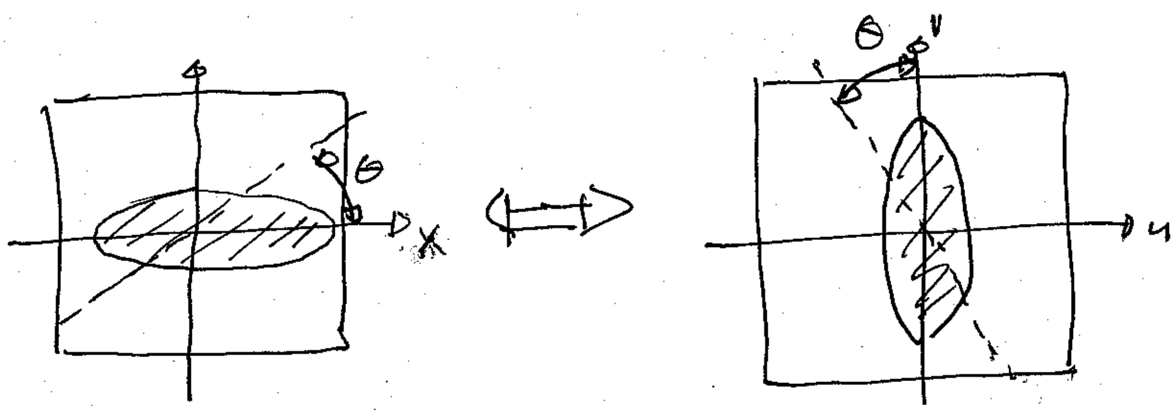
REFLECTED AROUND ORIGIN!

ROTATION

IF WE ROTATE $f(x,y)$ BY AN ANGLE θ

$$\mathcal{F}_{2D}\{f(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)\}$$
$$= F(u\cos\theta - v\sin\theta,\ u\sin\theta + v\cos\theta)$$

THE FOURIER TRANSFORM $F(u,v)$ ROTATES BY SAME ANGLE



SEPARABLE FUNCTIONS

IF

$$f(x,y) = f_1(x)\, f_2(y)$$

THEN

$$\mathcal{F}_{2D}\{f(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x)\, f_2(y)\, \underbrace{e^{-j2\pi(ux+vy)}}_{e^{-j2\pi ux}\, e^{-j2\pi vy}}\, dx\, dy$$

$$= \left[\int_{-\infty}^{\infty} f_1(x)\, e^{-j2\pi ux}\, dx\right]\left[\int_{-\infty}^{\infty} f_2(y)\, e^{-j2\pi vy}\, dy\right]$$

$$= F_1(u)\, F_2(v)$$

SEPARABLE SIGNALS HAVE SEPARABLE TRANSFORMS

EXAMPLE

$$\mathrm{rect}(x,y) = \mathrm{rect}(x)\,\mathrm{rect}(y)$$

$$\mathcal{F}_{2D}\{\mathrm{rect}(x,y)\} = \mathcal{F}_{1D}\{\mathrm{rect}(x)\}\,\mathcal{F}_{1D}\{\mathrm{rect}(y)\}$$

$$= \mathrm{sinc}(u)\,\mathrm{sinc}(v)$$

$$\boxed{\mathrm{rect}(x,y) \Longleftrightarrow \mathrm{sinc}(u)\,\mathrm{sinc}(v)}$$

THEN, USING DUALITY

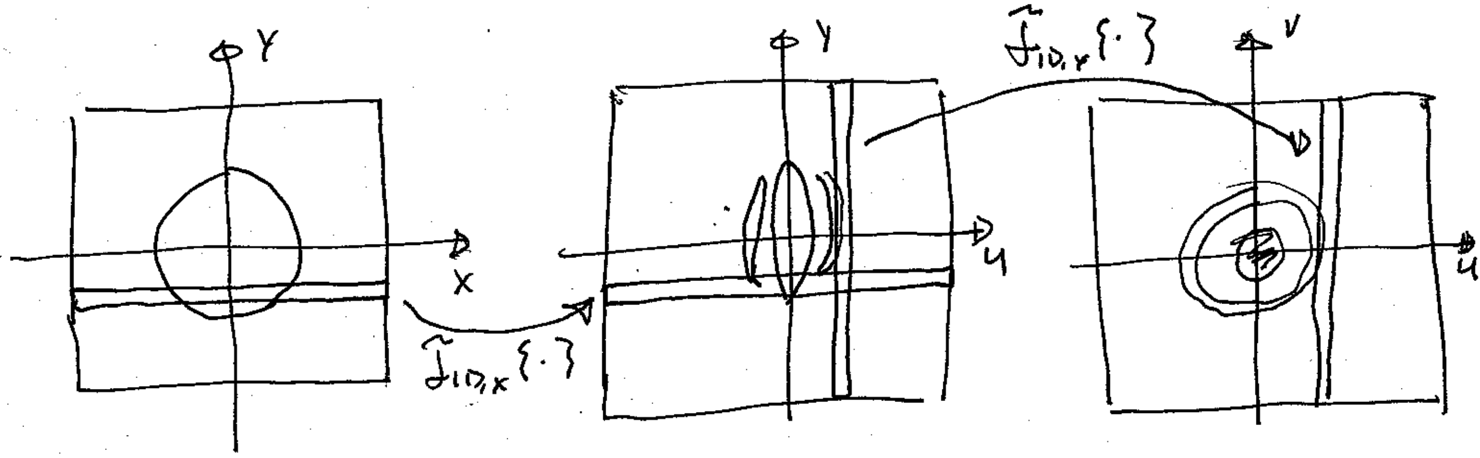
$$\mathrm{sinc}(x)\,\mathrm{sinc}(y) \Longleftrightarrow \mathrm{rect}(-u,-v) = \mathrm{rect}(u,v)$$

$$\boxed{\mathrm{sinc}(x)\,\mathrm{sinc}(y) \Longleftrightarrow \mathrm{rect}(u,v)}$$

THE 2DFT AS 1D FT'S

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$= \underbrace{\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi ux}\, dx\right]}_{\substack{\text{1D FT'S IN } x \text{ FOR EACH } y \\ \text{"HYBRID" SPACE } f_H(u,y)}} e^{-j2\pi vy}\, dy}_{\text{1D FT'S IN } y \text{ FOR EACH } u}$$



SPACE, $f(x,y)$

HYBRID SPACE $f_H(u,y)$

SPATIAL FREQUENCY $F(u,v)$

PARSEVAL'S THEOREM

ENERGY IS CONSERVED IN TWO DOMAINS

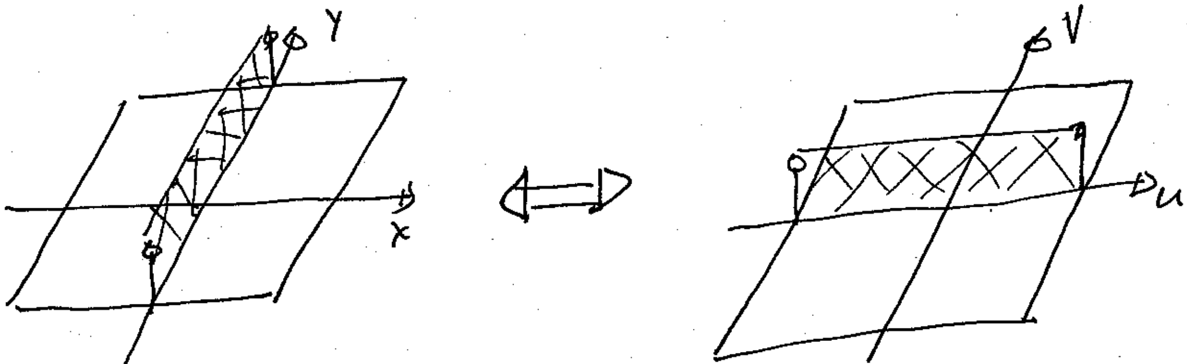
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)|^2\,du\,dv \qquad \underline{\text{NO } \tfrac{1}{2\pi}}$$

EXAMPLES

(1) $f(x,y) = \delta(x) = \delta(x)1(y)$ SEPARABLE

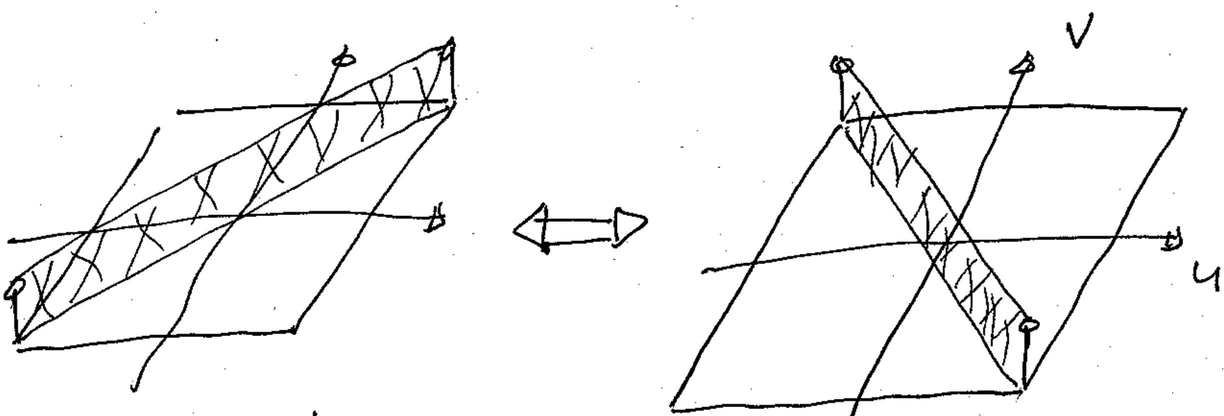
$$\delta(x)1(y) \Longleftrightarrow 1(u)\delta(v)$$

$$\delta(x) \Longleftrightarrow \delta(v)$$



$\delta(x)1(y)$ $1(u)\delta(v)$

(2)

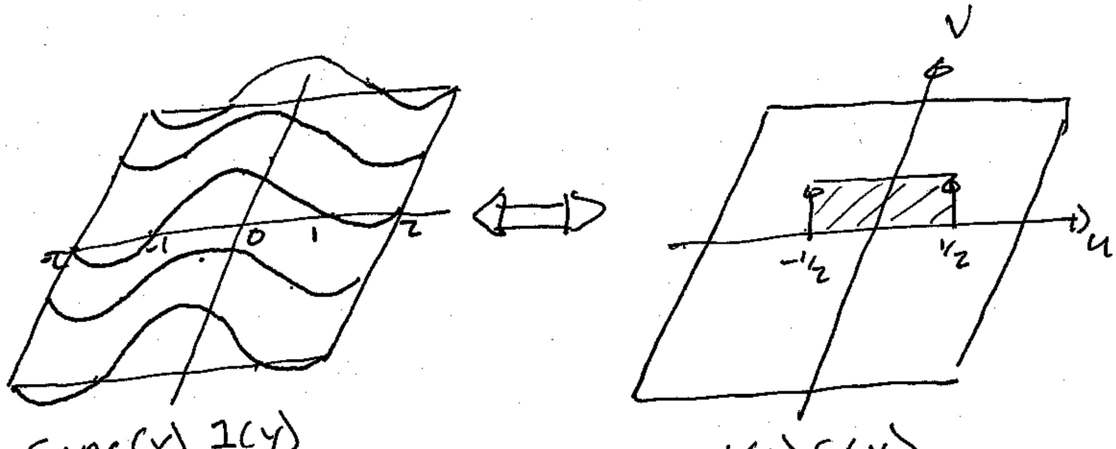


$\delta(x\frac{\sqrt{2}}{2} - y\frac{\sqrt{2}}{2})$ $\delta(x\frac{\sqrt{2}}{2} + y\frac{\sqrt{2}}{2})$

BOTH ROTATED BY $-45°$

(3) $f(x,y) = \text{sinc}(x)$

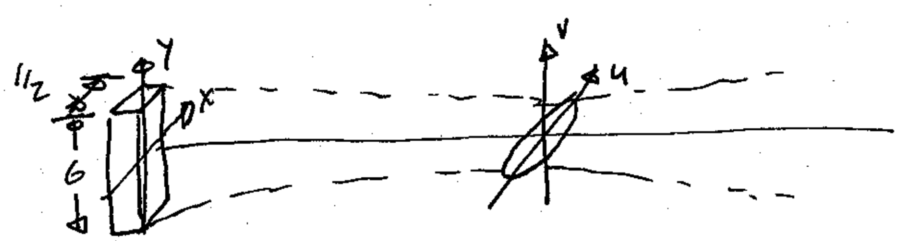
$$\text{sinc}(x)\, 1(y) \Longleftrightarrow \text{rect}(u)\, \delta(v)$$



$\text{sinc}(x)\, 1(y)$ $\text{rect}(u)\,\delta(v)$

(4) ULTRASOUND TRANSDUCER ELEMENT

BEAM PATTERN IN FAR FIELD IS FOURIER TRANSFORM OF APERTURE



$$f(x,y) = \text{rect}\left(\frac{x}{1/2}, \frac{y}{6}\right)$$
$$= \text{rect}\left(2x, \tfrac{1}{6}y\right)$$

$$F(u,v) = \mathcal{F}_{2D}\left\{\text{rect}\left(2x, \tfrac{1}{6}y\right)\right\}$$
$$= \frac{1}{2(1/6)}\, \text{sinc}\left(\frac{u}{2}\right) \text{sinc}(6v)$$
$$= 3\, \text{sinc}\left(\frac{u}{2}\right) \text{sinc}(6v)$$

LSI SYSTEMS

INPUT A COMPLEX EXPONENTIAL INTO AN IMAGING SYSTEM WITH AN IMPULSE RESPONSE $h(x,y)$

$$f(x,y) = e^{+j2\pi(ux+vy)}$$

$$g(x,y) = f(x,y) * h(x,y) = h(x,y) * f(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta)\, e^{+j2\pi(u(x-\xi)+v(y-\eta))}\, d\xi\, d\eta$$

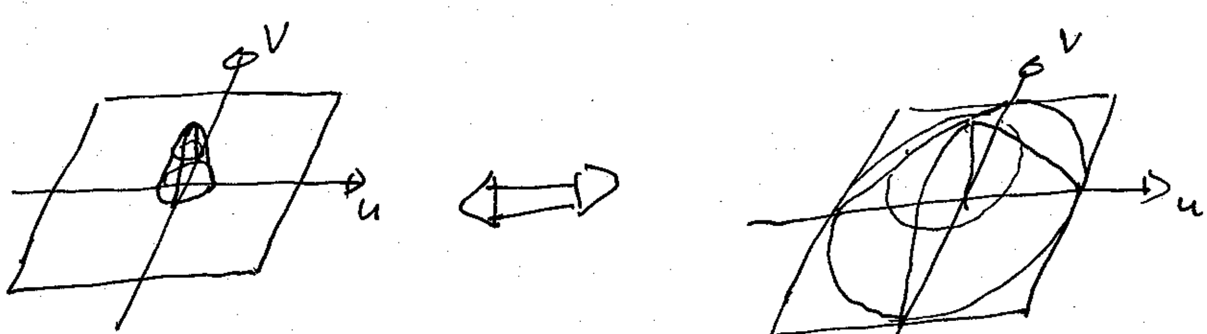
$$= \underbrace{e^{+j2\pi(ux+vy)}}_{\text{SAME AS INPUT}} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta)\, e^{-j2\pi(u\xi+v\eta)}\, d\xi\, d\eta}_{H(u,v)}$$

$H(u,v)$ IS CALLED

TRANSFER FUNCTION

FREQUENCY RESPONSE

MOST IMAGING SYSTEMS ARE LOWPASS



$h(x,y)$ IMPULSE RESPONSE

$H(u,v)$ FREQUENCY RESPONSE

OUTPUT IS BLURRED VERSION OF INPUT

FOR SOME ARBITRARY INPUT $f(x,y)$

$$g(x,y) = f(x,y) * h(x,y)$$

$$\underbrace{G(u,v)}_{\text{OUTPUT SPECTRUM}} = \underbrace{F(u,v)}_{\text{INPUT SPECTRUM}} \underbrace{H(u,v)}_{\text{FREQUENCY RESPONSE}}$$

NORMALIZED FREQUENCY RESPONSE IS CALLED THE MODULATION TRANSFER FUNCTION (MTF)

$$MTF(u,v) = \frac{H(u,v)}{H(0,0)}$$