LECTURE 5

CIRCULARLY SYMMETRIC FUNCTIONS
HANKEL TRANSFORM
) SECTION 2.7

CIRCULARLY SYMMETRIC FUNCTIONS

$$f(x,y) = f\left(\sqrt{x^2+y^2}\right) = f(r) \qquad r = \sqrt{x^2+y^2}$$

$$F(u,v) = F\left(\sqrt{u^2+v^2}\right) = F(q) \qquad q = \sqrt{u^2+v^2}$$

WHAT IS THE FOURIER TRANSFORM OF $f(r)$?

$$\mathcal{F}_{2D}\{f(r)\} = \mathcal{F}_{2D}\left\{f\left(\sqrt{x^2+y^2}\right)\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\sqrt{x^2+y^2}\right) e^{-j2\pi(ux+vy)}\, dx\,dy$$

$x = r\cos\theta$ $\qquad u = q\cos\phi$ $\qquad dx\,dy \Rightarrow r\,dr\,d\theta$

$y = r\sin\theta$ $\qquad v = q\sin\phi$

$r = \sqrt{x^2+y^2}$ $\qquad q = \sqrt{u^2+v^2}$

$$\mathcal{F}_{2D}\{f(r)\} = \int_0^{\infty}\int_0^{2\pi} f(r)\, e^{-j2\pi\overbrace{(r\cos\theta\, q\cos\phi + r\sin\theta\, q\sin\phi)}^{rq\cos(\theta-\phi)}}\, r\,dr\,d\theta$$

$$= \int_0^{\infty} f(r) \underbrace{\left[\int_0^{2\pi} e^{-j2\pi rq\cos(\theta-\phi)}\, d\theta\right]}_{2\pi J_0(2\pi rq)} r\,dr$$

$$\boxed{\begin{aligned} F(q) &= 2\pi\int_0^{\infty} f(r)\, J_0(2\pi rq)\, r\,dr \\ f(r) &= 2\pi\int_0^{\infty} F(q)\, J_0(2\pi rq)\, q\,dq \end{aligned}}$$

THIS IS CALLED THE <u>HANKEL TRANSFORM</u>

$$\mathcal{H}\{f(r)\} = F(q)$$
$$\mathcal{H}\{F(q)\} = f(r)$$

FUNCTION OF ONE VARIABLE (r or q), BUT FUNDAMENTALLY 2D FUNCTIONS ($r = \sqrt{x^2+y^2}$)

TRANSFORM PAIRS

2D GAUSSIAN

$$\boxed{e^{-\pi r^2} \Longleftrightarrow e^{-\pi q^2}}$$

$$e^{-\pi(x^2+y^2)} \Longleftrightarrow e^{-\pi(u^2+v^2)}$$

IMPULSE RING

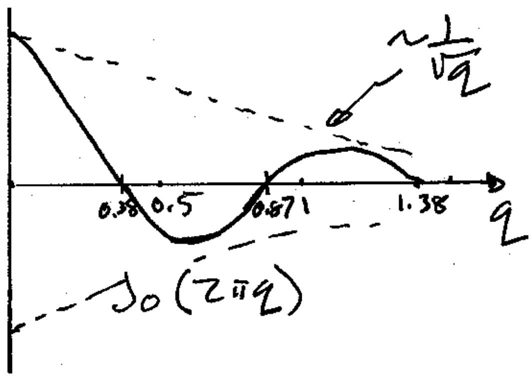
$$f(r) = \delta(r-r_0) \qquad r_0 > 0$$

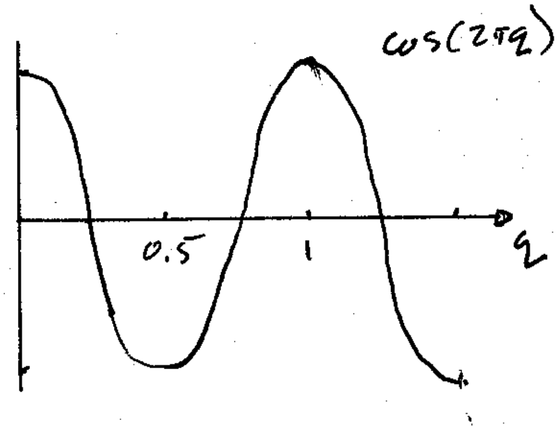
$$F(q) = 2\pi \int_0^\infty \delta(r-r_0)\, \underbrace{J_0(2\pi r q)\, r}\, dr$$

$$= 2\pi r_0\, J_0(2\pi q r_0)$$

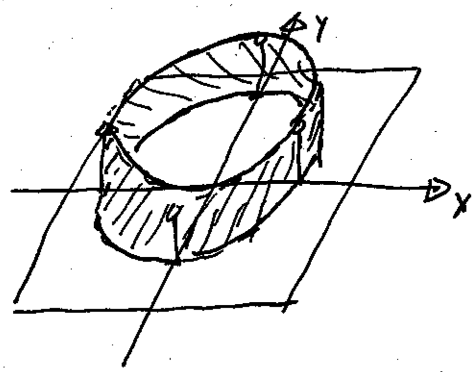
$$\boxed{\delta(r-r_0) \Longleftrightarrow 2\pi r_0\, J_0(2\pi r_0 q)}$$

$J_0(\cdot)$ VAGUELY SIMILAR TO COSINE

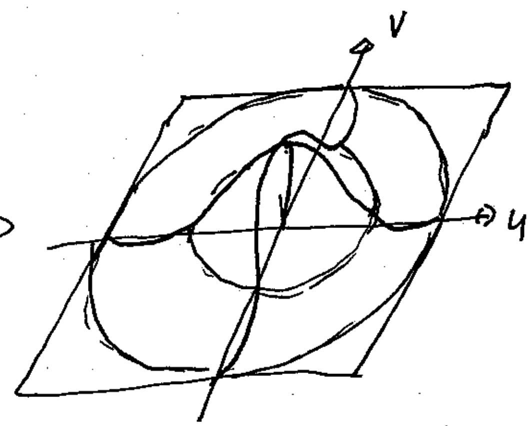




THE IMPULSE RING AND ITS TRANSFORM LOOK LIKE



$\Longleftrightarrow$



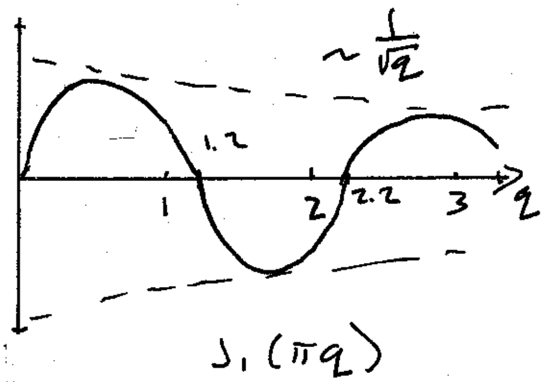
$\delta(r - r_0)$ $\Longleftrightarrow$ $2\pi r_0 J_0(2\pi r_0 q)$

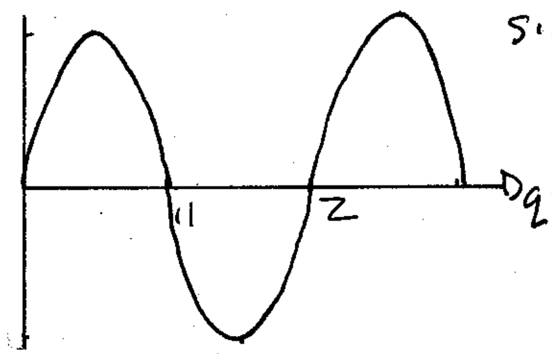
<u>CIRCULAR DISK</u>

$$\mathrm{rect}(r) \Longleftrightarrow \frac{J_1(\pi q)}{2q} = \mathrm{jinc}(q)$$

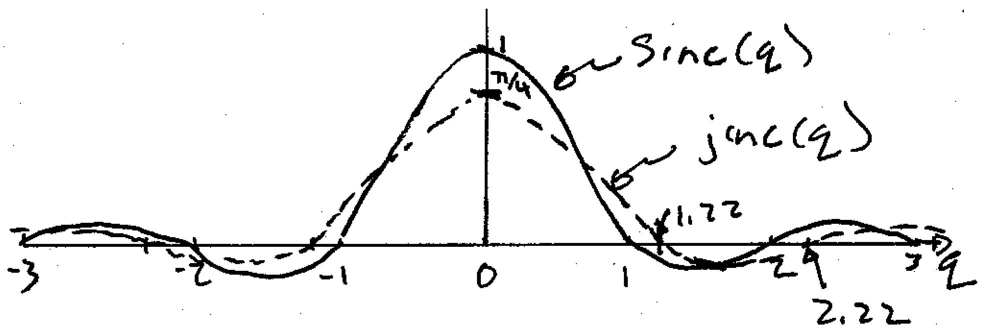
$$\boxed{\mathrm{rect}(r) \Longleftrightarrow \mathrm{jinc}(q)}$$

$J_1(\cdot)$ LOOKS SOMETHING LIKE $\sin(\cdot)$





jinc(·) LOOKS SOMETHING LIKE sinc(·)



jinc(·) IS

SLIGHTLY BROADER

SIDELOBES DECAY AS $\dfrac{J_1(\pi q)}{2q} \sim \dfrac{1/\sqrt{q}}{2q} \sim \dfrac{1}{(q)^{3/2}}$

SLIGHTLY FASTER THAN sinc(·)

THEOREMS

SCALING

$$f(ar) = f\left(a\sqrt{x^2+y^2}\right)$$
$$= f\left(\sqrt{(ax)^2+(ay)^2}\right)$$

THEN

$$\mathcal{F}_{2D}\left\{f\left(\sqrt{(ax)^2+(ay)^2}\right)\right\} = \frac{1}{|a|^2} F\left(\sqrt{\left(\frac{u}{a}\right)^2+\left(\frac{v}{a}\right)^2}\right)$$
$$= \frac{1}{|a|^2} F\left(\frac{q}{a}\right)$$

$$\boxed{f(ar) \Longleftrightarrow \frac{1}{|a|^2} F\left(\frac{q}{a}\right)}$$

DUALITY

SINCE FORWARD AND INVERSE HANKEL TRANSFORMS ARE THE SAME.

$$\boxed{\begin{array}{c} f(r) \Longleftrightarrow F(q) \\ F(r) \Longleftrightarrow f(q) \end{array}}$$

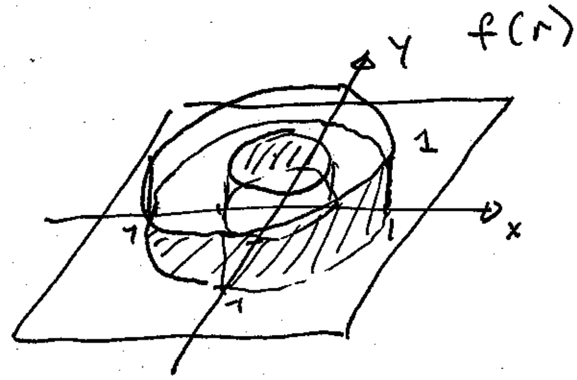
EXAMPLE

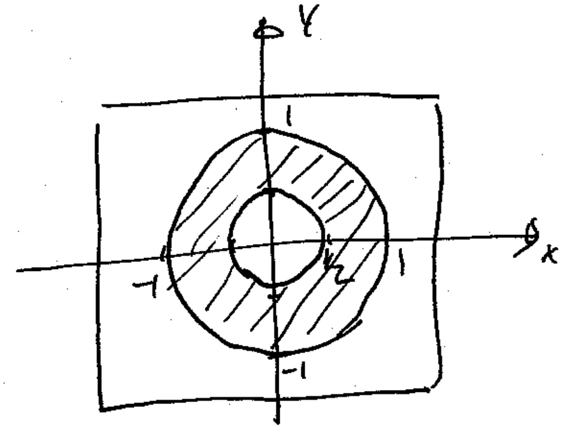
$$\mathrm{rect}(r) \Longleftrightarrow \mathrm{jinc}(q)$$

$$\boxed{\mathrm{jinc}(r) \Longleftrightarrow \mathrm{rect}(q)}$$

2D HANKEL TRANSFORM EXAMPLES

1





TOP VIEW

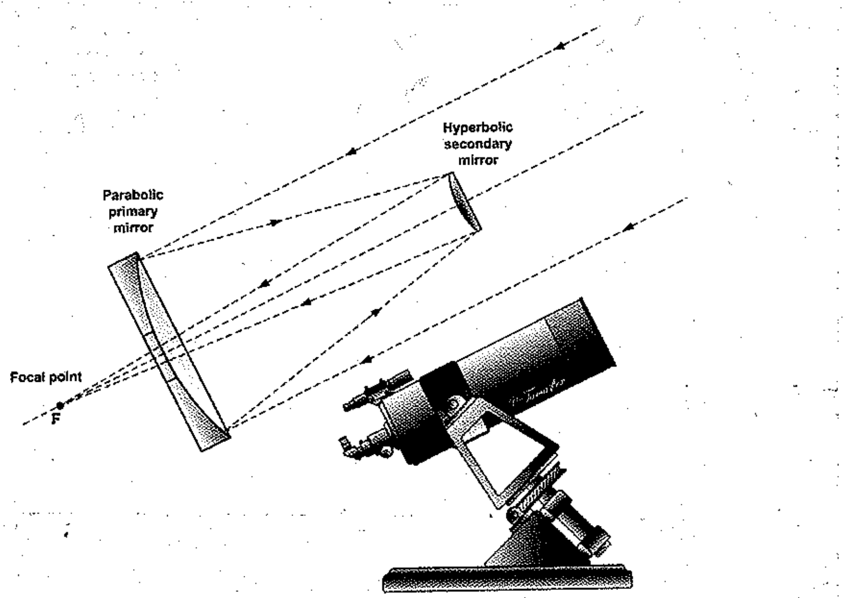
A DISK WITH DIAMETER 2 WITH A HOLE OF DIAMETER 1

$$f(r) = \text{rect}\left(\frac{r}{2}\right) - \text{rect}(r)$$

$$F(q) = \frac{1}{|1/2|^2}\,\text{jinc}\left(\frac{q}{1/2}\right) - \text{jinc}(q)$$

$$= \underline{4\,\text{jinc}(2q) - \text{jinc}(q)}$$

IMPULSE RESPONSE OF CASSEGRAIN TELESCOPE



WIKIPEDIA "CASSEGRAIN REFLECTOR"

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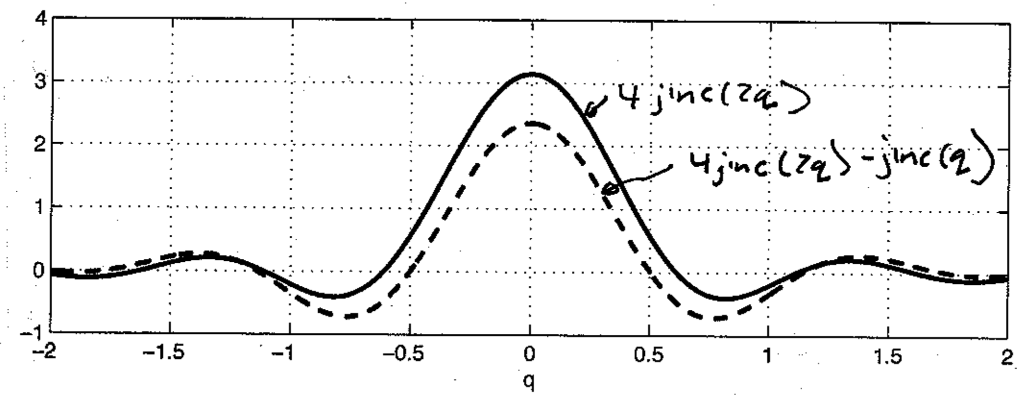
COMPARISON WITH jinc(·)



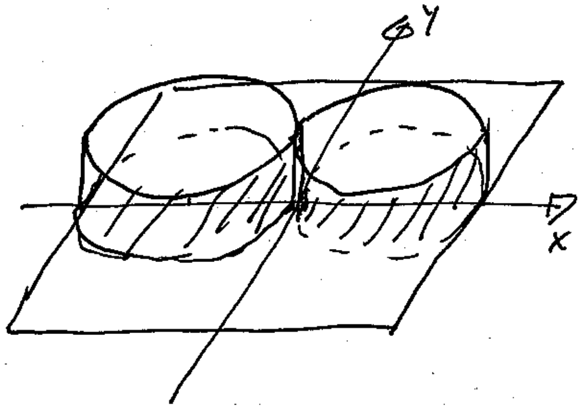
SLIGHTLY LESS AMPLITUDE

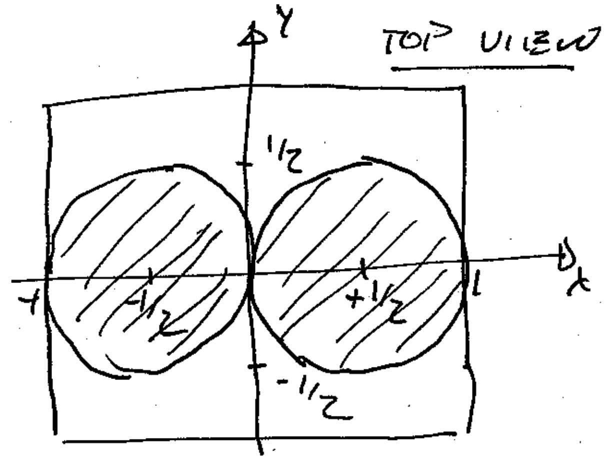
SLIGHTLY BETTER RESOLUTION

SLIGHTLY HIGHER SIDELOBES

L5

EXAMPLE 2





TWO DISKS CENTERED AT $\pm 1/2$

$$f(x,y) = \mathrm{rect}\left(\sqrt{(x-1)^2+y^2}\right) + \mathrm{rect}\left(\sqrt{(x+1)^2+y^2}\right)$$

$$F(u,v) = \mathrm{jinc}\left(\sqrt{u^2+v^2}\right) e^{-j2\pi u} + \mathrm{jinc}\left(\sqrt{u^2+v^2}\right) e^{+j2\pi u}$$

$$= \underline{2\,\mathrm{jinc}\left(\sqrt{u^2+v^2}\right) \cos(2\pi u)}$$