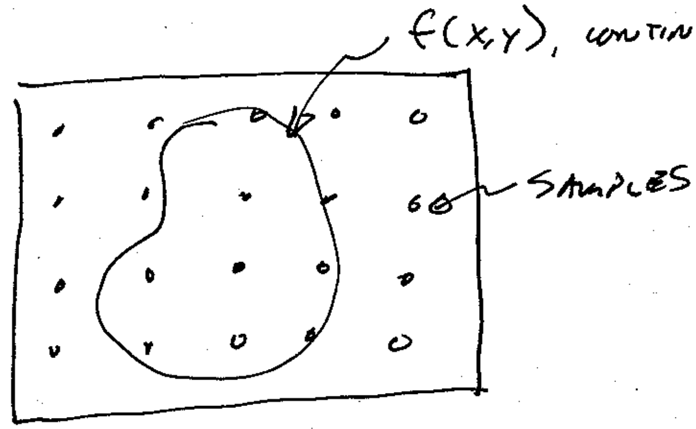
LECTURE 6

SAMPLING IN 1D AND 2D
RECONSTRUCTION IN 1D AND 2D } SECTION 2.8

EXAMPLE: AREA DETECTOR

SAMPLING

WE HAVE A 2D FUNCTION WE WANT TO REPRESENT BY SAMPLES



QUESTIONS:

HOW MANY SAMPLES DO I NEED?

HOW SHOULD SAMPLES BE SPACED?

HOW CAN I RECONSTRUCT f(x,y) FROM SAMPLES?

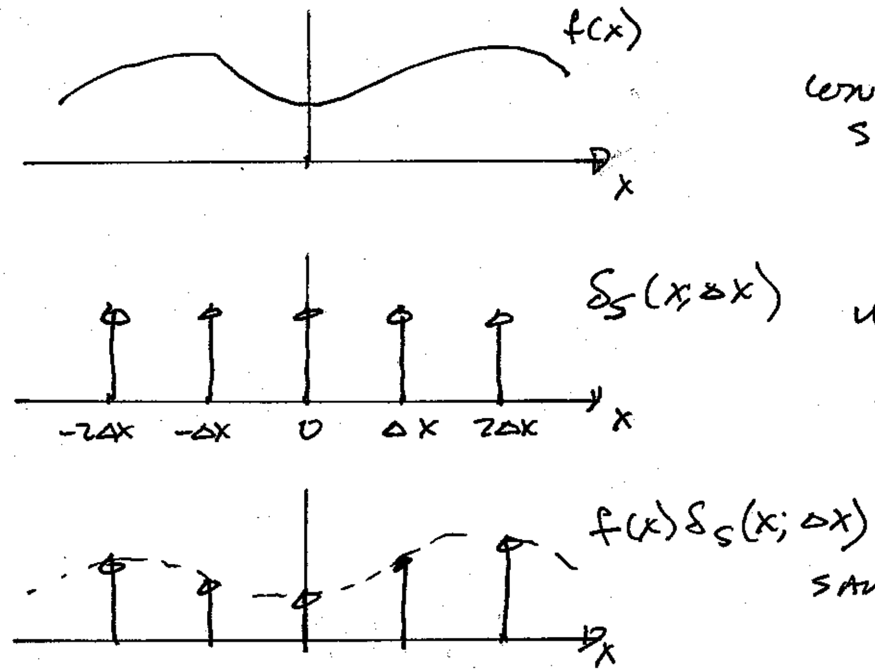
WHEN IS THIS POSSIBLE?

SIMILAR TO 1D, BUT SOME NEW TWISTS.

MODELLING SAMPLING

MODEL SAMPLING AS MULTIPLYING BY IMPULSES

IN 1D



CONTINUOUS SIGNAL

UNIT IMPULSES AT $n\Delta x$

SAMPLED SIGNAL

THIS CAN BE SIMPLIFIED AS

$$
\begin{aligned}
f(x)\delta_s(x;\Delta x) &= f(x)\sum_{n=-\infty}^{\infty}\delta(x-n\Delta x) \\
&= \sum_{n=-\infty}^{\infty} f(x)\delta(x-n\Delta x) \\
&= \sum_{n=-\infty}^{\infty} f(n\Delta x)\,\delta(x-n\Delta x) \\
&= \sum_{n=-\infty}^{\infty} \underbrace{f_d(n)}_{\text{SIGNAL SAMPLES}}\,\delta(x-n\Delta x)
\end{aligned}
$$

IN 2D

$$f(x,y)\,\delta_s(x,y;\Delta x,\Delta y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \underbrace{f(n\Delta x, m\Delta y)}_{\text{IMAGE SAMPLES}}\,\delta(x-n\Delta x, y-m\Delta y)$$

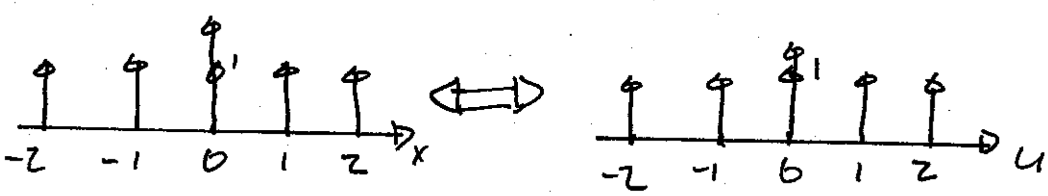
SPECTRUM OF SAMPLED SIGNAL

IN 1D

$$\tilde{\mathcal{F}}_{1D}\{f(x)\,\delta_s(x;\Delta x)\} = F(u) * \underbrace{\tilde{\mathcal{F}}_{1D}\{\delta_s(x;\Delta x)\}}_{\text{WHAT IS THIS?}}$$

RECALL

$$\mathrm{comb}(x) \Longleftrightarrow \mathrm{comb}(u)$$



NO 2π AS IN 102A

THE SAMPLING FUNCTION IS

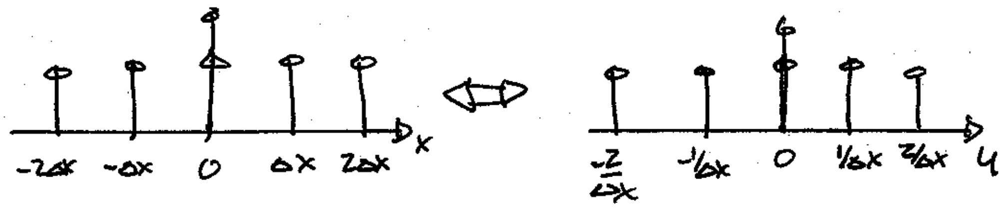
$$\delta_s(x;\Delta x) = \frac{1}{\Delta x}\,\mathrm{comb}\left(\frac{x}{\Delta x}\right)$$

BY THE SCALING THEOREM

$$\frac{1}{\Delta x}\,\mathrm{comb}\left(\frac{x}{\Delta x}\right) \Longleftrightarrow \mathrm{comb}(u\Delta x)$$

SO

$$\boxed{\delta_s(x;\Delta x) \Longleftrightarrow \mathrm{comb}(u\Delta x)}$$



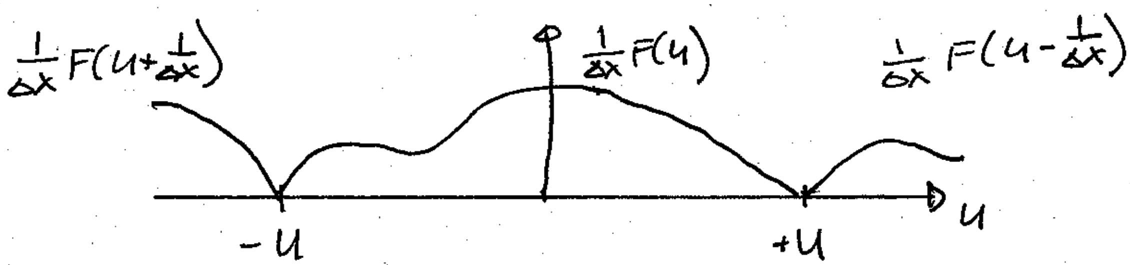
$\delta_s(x;\Delta x)$ $\qquad$ $\text{comb}(u\Delta x)$

THE SPECTRUM OF THE SAMPLED SIGNAL IS THEN

$$\mathcal{F}_{1D}\{f(x)\delta_s(x;\Delta x)\} = F(u) * \mathcal{F}_{1D}\{\delta_s(x;\Delta x)\}$$
$$= F(u) * \text{comb}(u\Delta x)$$
$$= F(u) * \left(\sum_{n=-\infty}^{\infty} \delta(u\Delta x - n)\right)$$
$$= F(u) * \left(\sum_{n=-\infty}^{\infty} \delta\left(\Delta x\left(u - \frac{n}{\Delta x}\right)\right)\right)$$
$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty} F\left(u - \frac{n}{\Delta x}\right)$$

THEN

$$\boxed{\mathcal{F}_{1D}\{f(x)\delta_s(x;\Delta x)\} = \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty} F\left(u - \frac{u}{\Delta x}\right)}$$



$\underbrace{\hspace{6cm}}_{\text{SIGNAL BANDWIDTH } 2U}$

WE CAN RECOVER $F(u)$ (AND $f(x)$) IF

$$F(u) = 0 \; ; \quad |u| > U = \frac{1}{2\Delta x}$$

$F(u)$ IS BANDLIMITED, AND

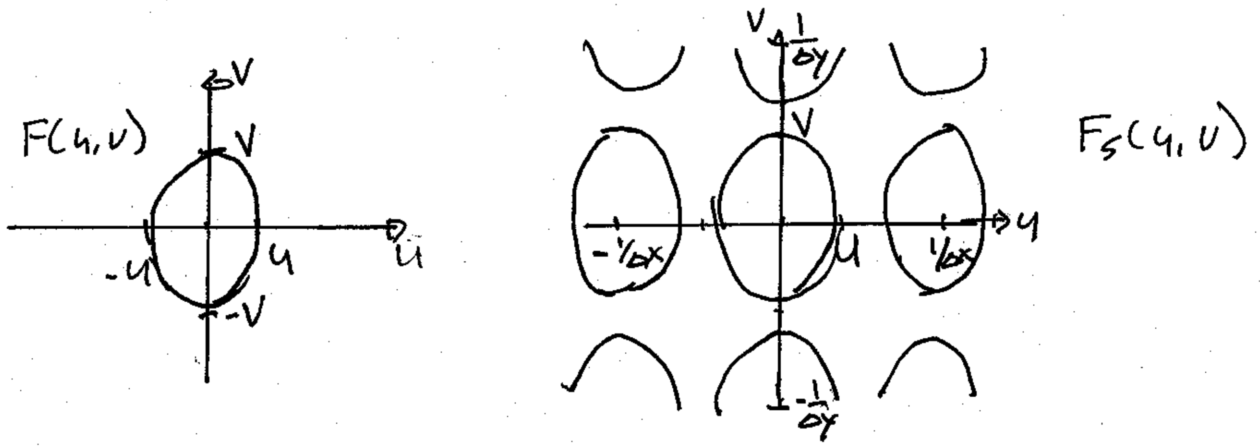
$$\boxed{\Delta x \le \frac{1}{2U}}$$

IN 2D

$$\mathcal{F}_{2D}\{ f(x,y)\,\delta_s(x,y;\Delta x,\Delta y)\}$$

$$= F(u,v) * \mathrm{comb}(u\Delta x, v\Delta y)$$

$$F_s(u,v) = \frac{1}{\Delta x \Delta y} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} F\left(u - \frac{n}{\Delta x},\, v - \frac{m}{\Delta y}\right)$$



CONTINUOUS SPECTRUM SAMPLED SPECTRUM

IMAGE $f(x,y)$ CAN BE RECOVERED IF

$$F(u,v) = 0 \qquad |u| > U \, , \; |v| > V$$

BANDLIMITED TO $\pm U, \pm V$

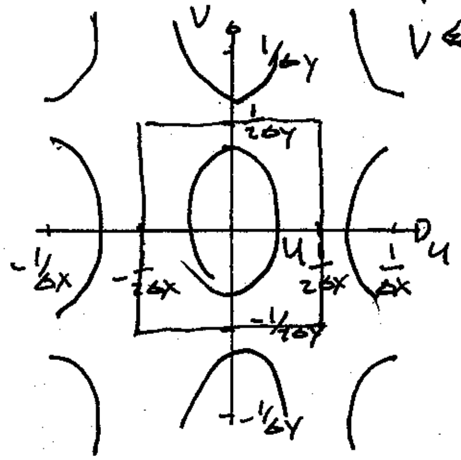
AND

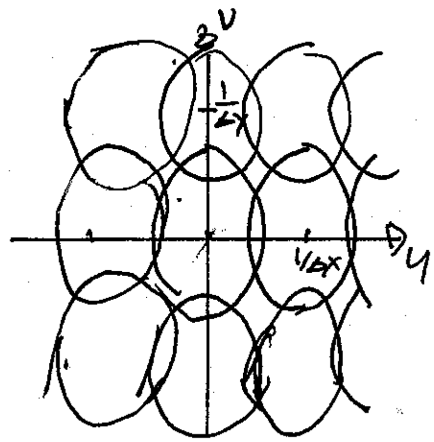
$$\Delta x \le \frac{1}{2U}$$
$$\Delta y \le \frac{1}{2V}$$

SAMPLED FASTER THAN $\frac{1}{2U}, \frac{1}{2V}$

NYQUIST RATE

ALIASING IN 2D





ABOVE CRITICAL SAMPLING

NO ALIASING

SIGNAL CAN BE RECOVERED

BELOW CRITICAL SAMPLING

ALIASING

SIGNAL CAN NOT BE RECOVERED (WITHOUT ADDITIONAL INFORMATION)

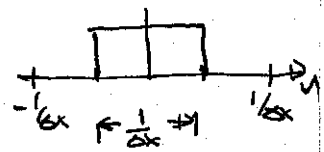
RECONSTRUCTION

LOWPASS FILTER SAMPLED SIGNAL

$$F(u,v) = F_s(u,v)\, H(u,v)$$

WHERE

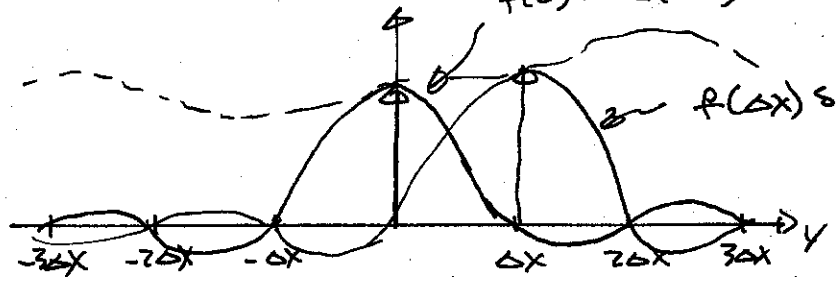
$$H(u,v) = (\Delta x \Delta y)\, \mathrm{rect}(u\Delta x, v\Delta y)$$



IMPULSE RESPONSE IS

$$\boxed{h(x,y) = \mathrm{sinc}\left(\frac{x}{\Delta x}\right)\mathrm{sinc}\left(\frac{y}{\Delta y}\right)}$$

IN 1D



EACH SINC IS 1 AT SAMPLE, ZERO AT OTHERS

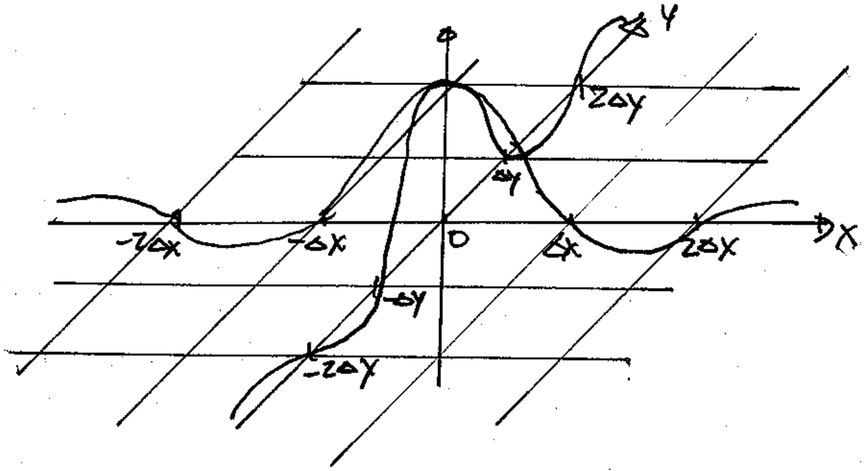
GUARANTEED TO

GO THROUGH SAMPLES

BE BANDLIMITED TO $\pm \frac{1}{2\Delta x}$

OPTIMAL SINC INTERPOLATOR

IN 2D



$\mathrm{sinc}\left(\frac{x}{\Delta x}\right)\mathrm{sinc}\left(\frac{y}{\Delta y}\right)$

2D SINC IS ZERO AT ALL OTHER GRID POINTS

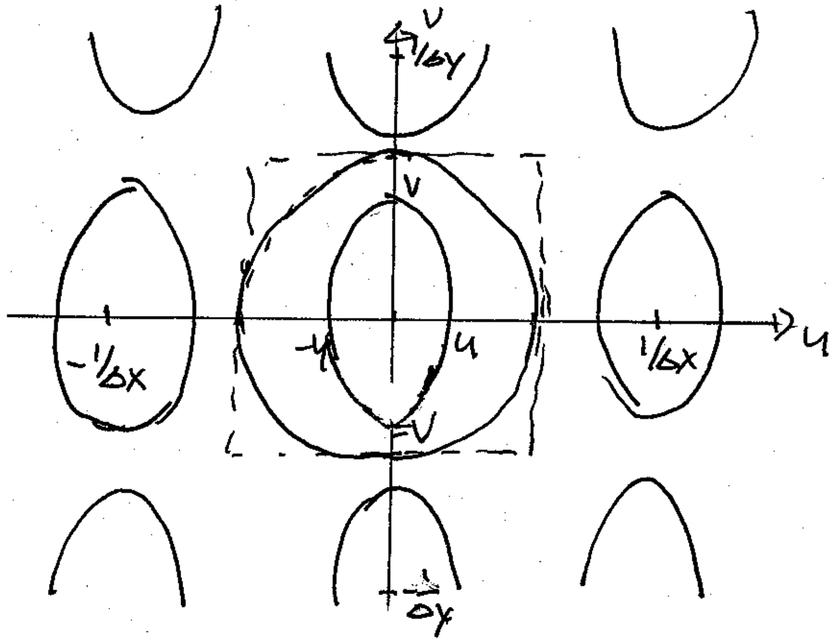
SINCE 2D SINC IS SEPARABLE WE CAN

INTERPOLATE FIRST IN X

THEN INTERPOLATE IN Y

IS THIS THE ONLY INTERPOLATION FILTER?

$F_s(u,v)$



$\Delta x > \frac{1}{2U}$

$\Delta y > \frac{1}{2V}$

MANY ALTERNATIVES

EXAMPLE

$$H(u,v) = \Delta x \Delta y \ \mathrm{rect}\left(\sqrt{(u\Delta x)^2 + (v\Delta y)^2}\right)$$

$$h(x,y) = \mathrm{jinc}\left(\sqrt{\left(\frac{x}{\Delta x}\right)^2 + \left(\frac{y}{\Delta y}\right)^2}\right)$$

SOME ALTERNATIVES HAVE

BETTER COMPUTATION CHARACTERISTICS

BETTER IMPULSE RESPONSE

FIELD OF VIEW, NUMBER OF SAMPLES

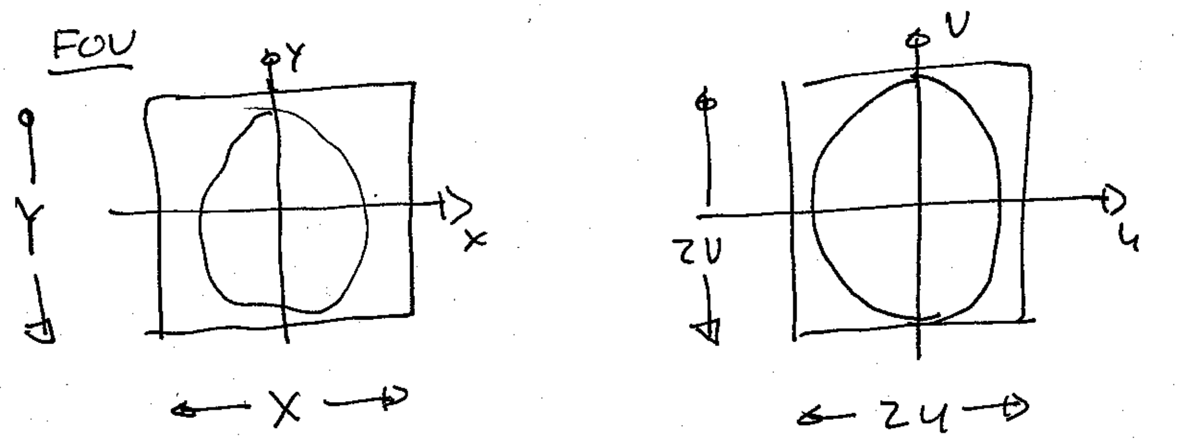


IMAGE DOMAIN FREQUENCY DOMAIN

NYQUIST SAMPLING

$$\Delta X = \frac{1}{2u} \qquad \Delta Y = \frac{1}{2V}$$

NUMBER OF SAMPLES

$$N_s = \left(\frac{X}{\Delta X}\right)\left(\frac{Y}{\Delta Y}\right) = (2uX)(2VY)$$

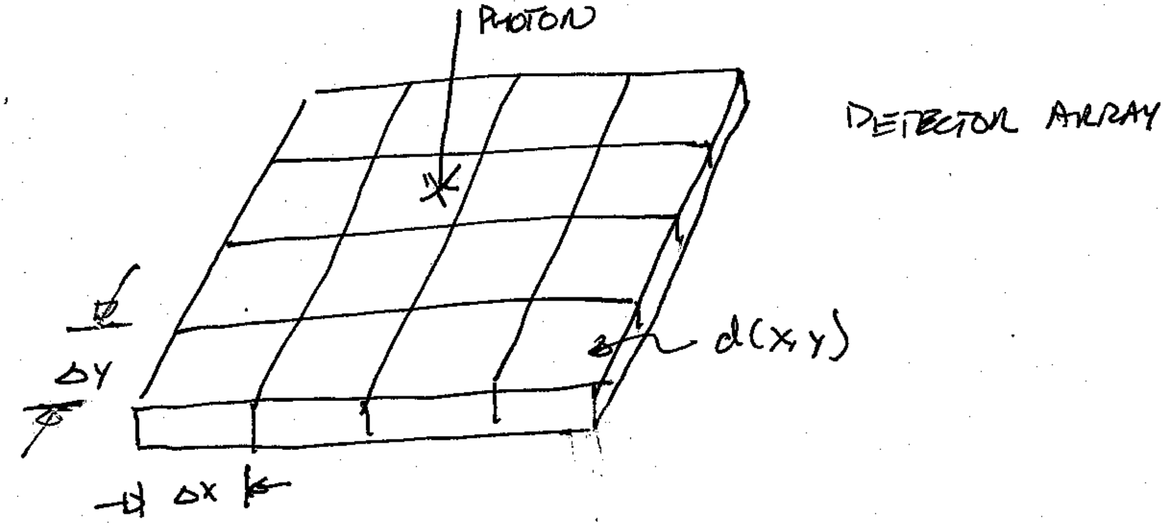
BANDWIDTH SPATIAL EXTENT

THIS IS THE SPACE BANDWIDTH PRODUCT

HOW MUCH INFORMATION THERE IS IN THE SIGNAL

EXAMPLE

PHYSICAL SAMPLING AND AREA DETECTORS



SIGNAL IS INTEGRAL OVER AREA $(\Delta x)(\Delta y)$

MODEL AS

1) CONVOLUTION WITH

$$d(x,y) = \mathrm{rect}\left(\frac{x}{\Delta x}, \frac{y}{\Delta y}\right)$$

REPRESENTS INTEGRATION

2) MULTIPLICATION BY

$$\delta_s(x,y;\Delta x,\Delta y)$$

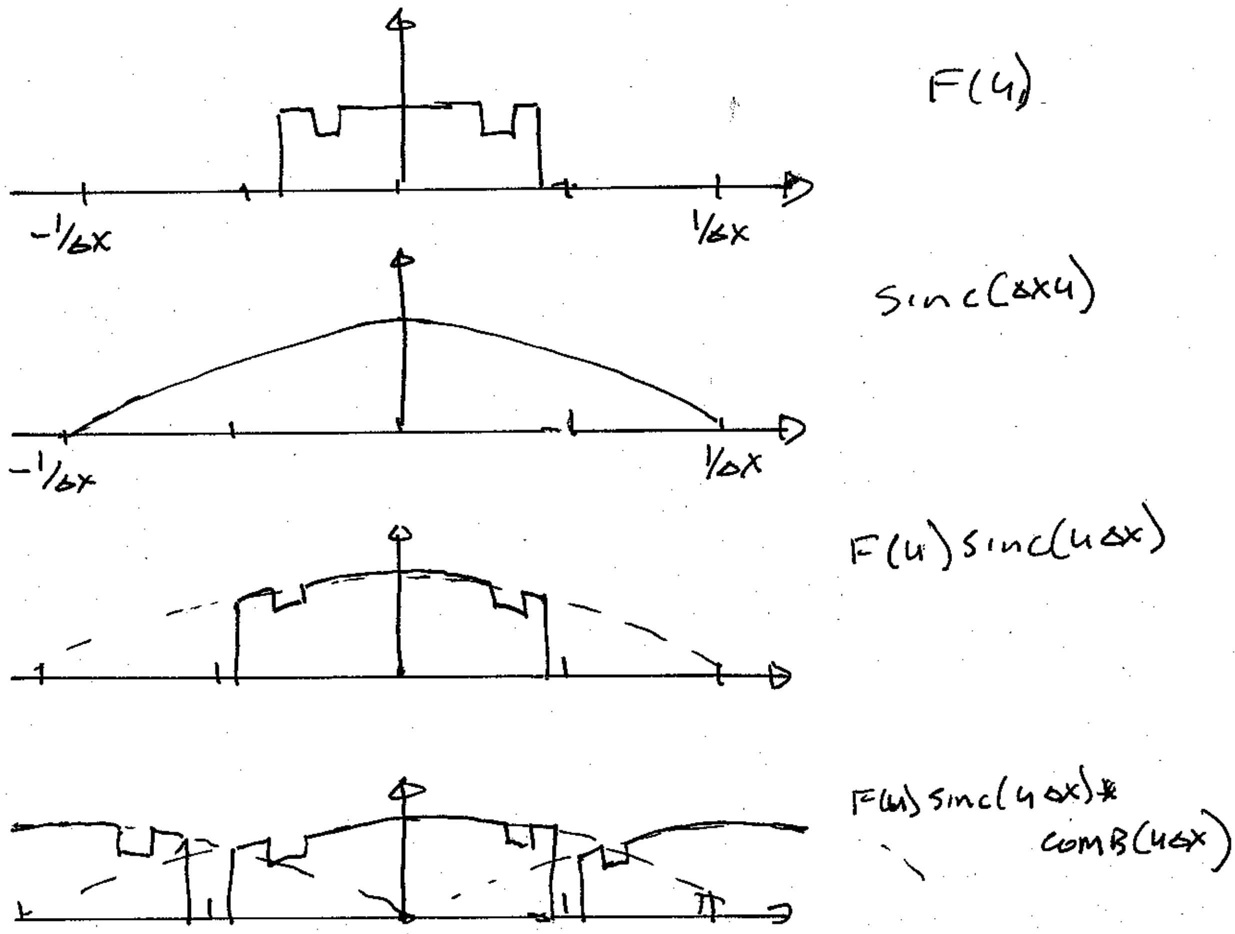
REPRESENTS SAMPLING ONCE PER INTEGRATION AREA

$$f_{sa}(x,y) = \left(f(x,y) * d(x,y)\right) \cdot \delta_s(x,y;\Delta x,\Delta y)$$
$$= \left(f(x,y) * \mathrm{rect}\left(\tfrac{x}{\Delta x},\tfrac{y}{\Delta y}\right)\right) \cdot \delta_s(x,y;\Delta x,\Delta y)$$

THE SPECTRUM IS

$$F_{sa}(u,v) = \left[F(u,v) \cdot \mathcal{F}_{2D}\left\{\mathrm{rect}\left(\tfrac{x}{\Delta x},\tfrac{y}{\Delta y}\right)\right\}\right] * \mathcal{F}_{2D}\left\{\delta_s(x,y;\Delta x,\Delta y)\right\}$$
$$= \underbrace{\left[\underbrace{F(u,v)(\Delta x \Delta y)\,\mathrm{sinc}(u\Delta x)\,\mathrm{sinc}(v\Delta y)}_{\text{APODIZED } F(u,v)}\right] * \mathrm{COMB}(u\Delta x, v\Delta y)}_{\text{REPLICATED}}$$

IN 1D

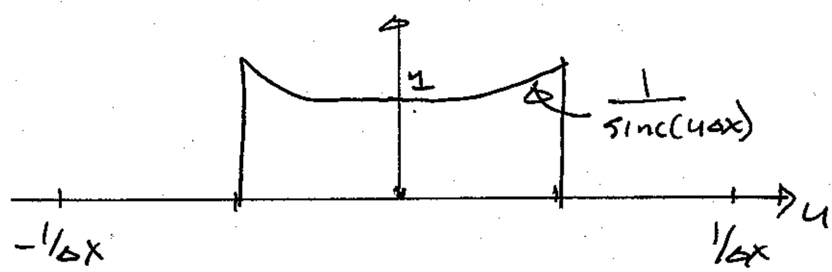


LG;

CAN I RECOVER $f(x,y)$?

YES!

$$H(u,v) = \frac{\mathrm{rect}(u\Delta x, v\Delta y)}{\mathrm{sinc}(u\Delta x)\,\mathrm{sinc}(v\Delta y)} \qquad |u| < \frac{1}{2\Delta x} \quad |v| < \frac{1}{2\Delta y}$$



CORRECTS FOR FREQUENCY DOMAIN APODIZATION

WHAT IF $f(x,y)$ IS <u>NOT</u> BANDLIMITED?