Analog and Digital Communications
EE 179

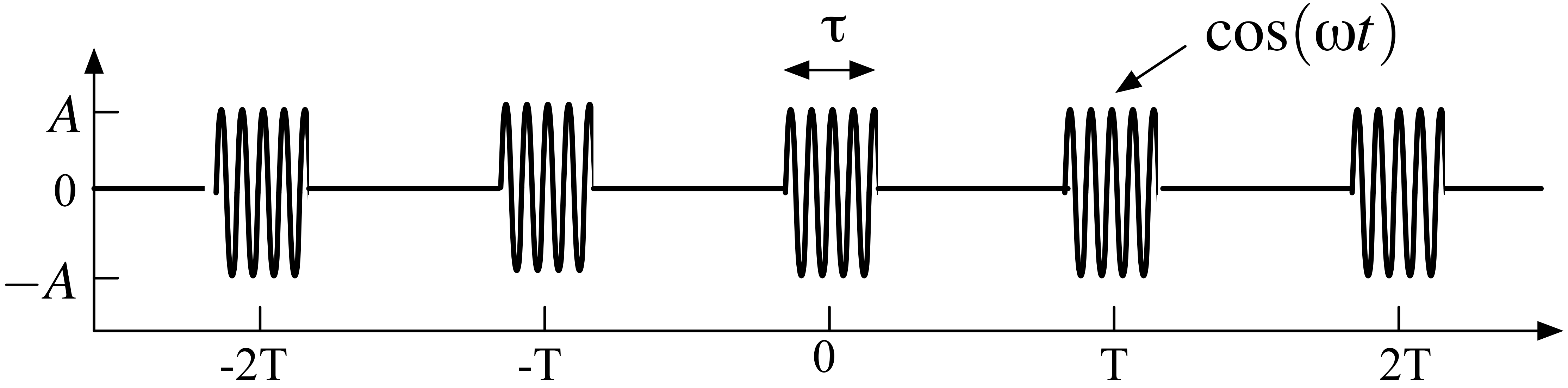
2. Time Domain Signals and the $2\pi f$ *Fourier Transform*

Today

- Repesenting signals in the time domain
- Basic signals
- Operations on signals
- Complex signals
- Power and Energy
- Decibels
- 2π Fourier Transforms

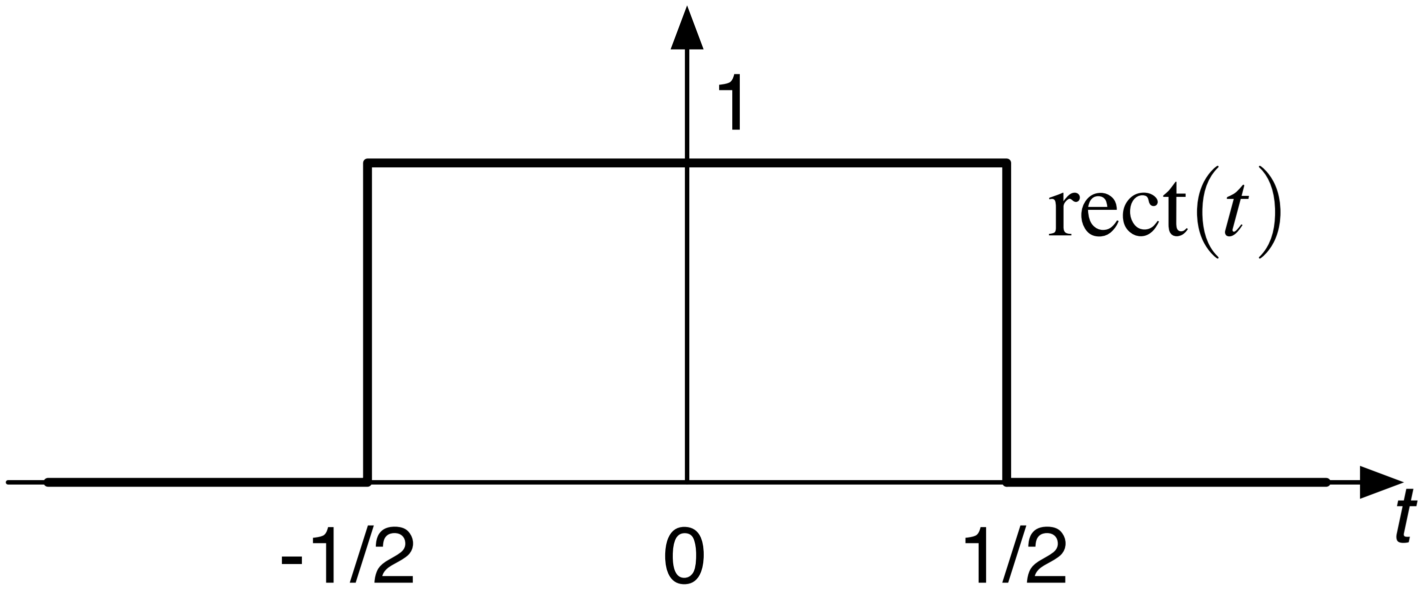
L&D Chapter 1

Representing Signals



- This is a pulse train, that could come up in radar or communications
- How do we represent it?
- What are its properties: spectral width, power etc
- Build it up from simpler components

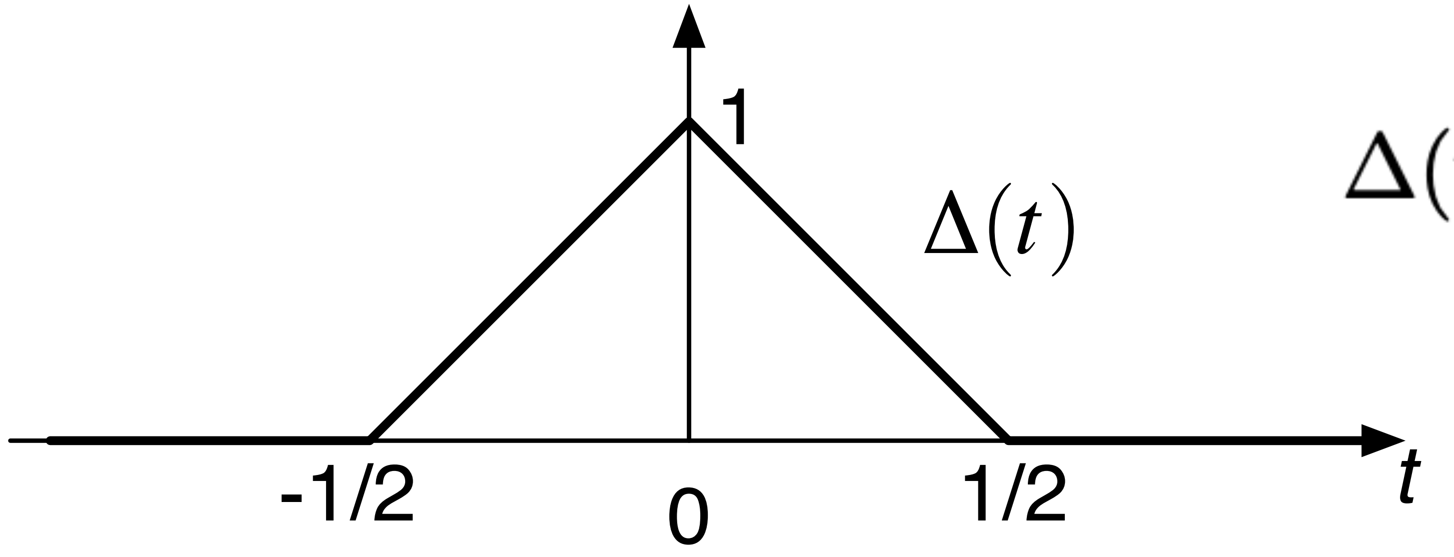
Unit Rectangle



$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

- Unit width, and unit area
- Also written as $\Pi(t)$

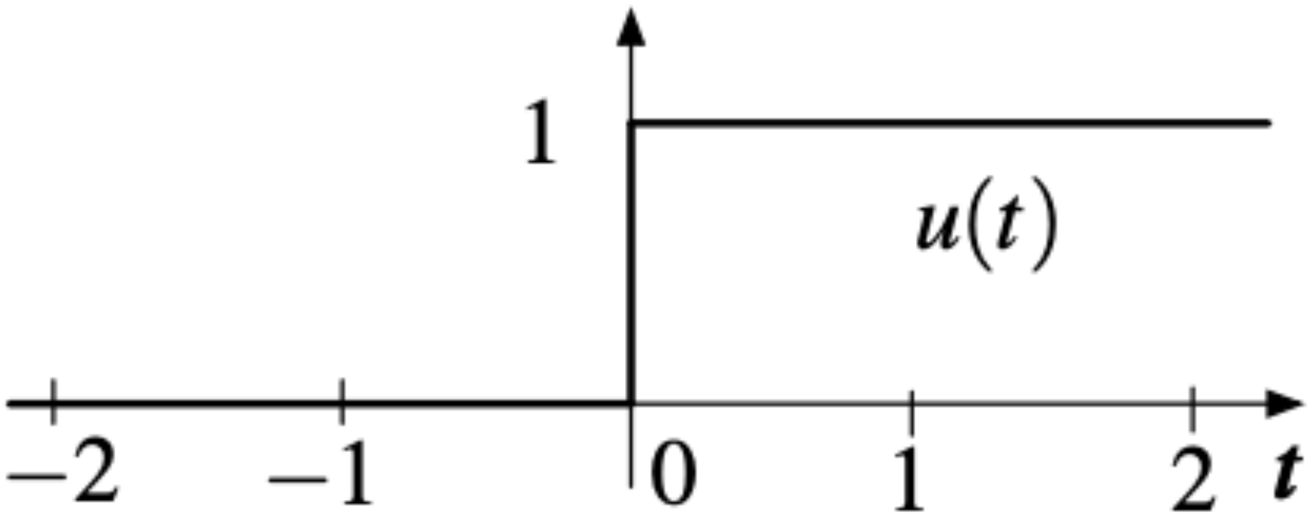
Unit Triangle



$$\Delta(t) = \begin{cases} 1 - 2|x| & |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

- Unit width, and and area 1/2
- Also written as $\Lambda(t)$ or $\mathrm{tri}(t)$
- This is a non-standard definition, which is twice as wide
- The conventional definition gives $\mathrm{rect}(t) * \mathrm{rect}(t) = \Delta(t)$

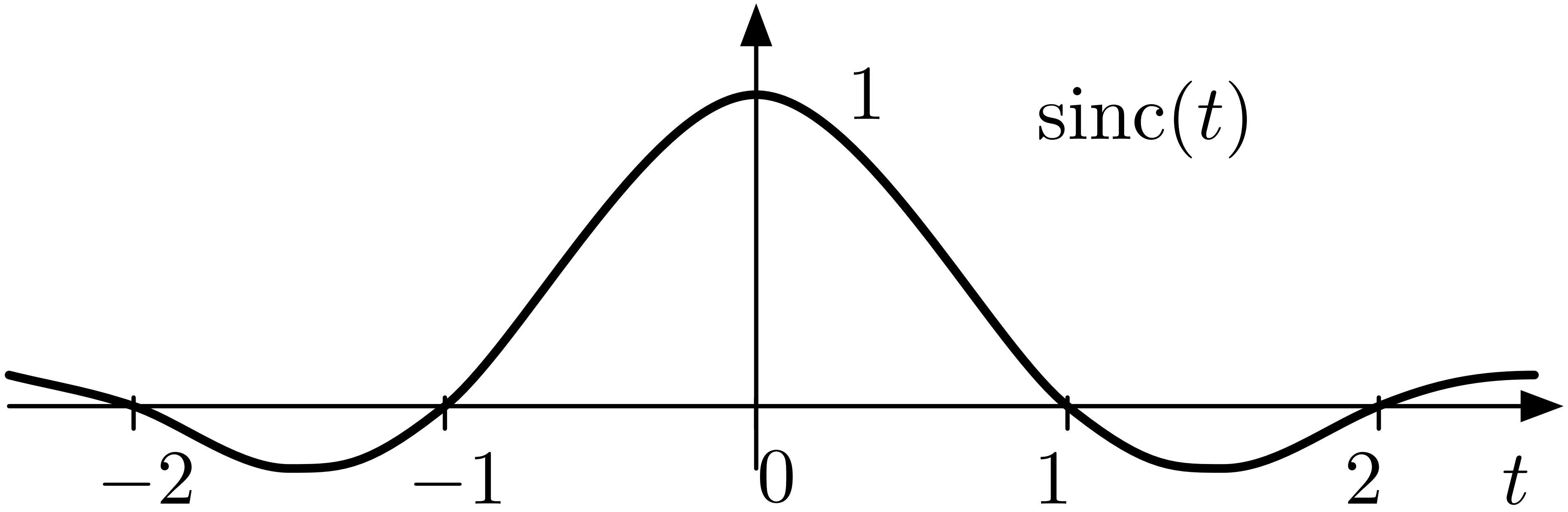
Unit Step



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

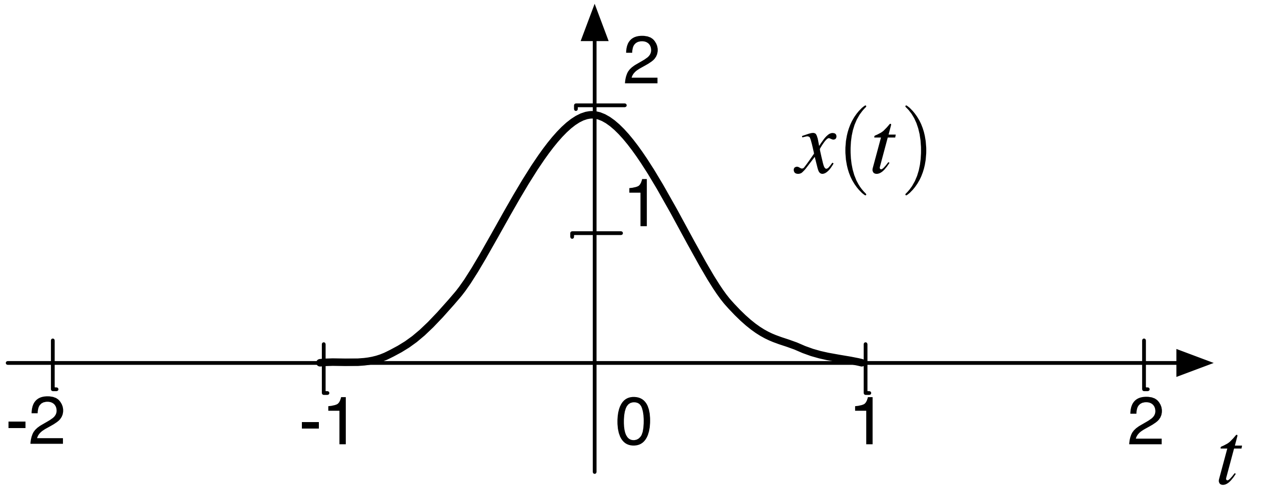
- This is 1 for positive time
- Very useful in communications
- When used in frequency, this is a single sideband filter

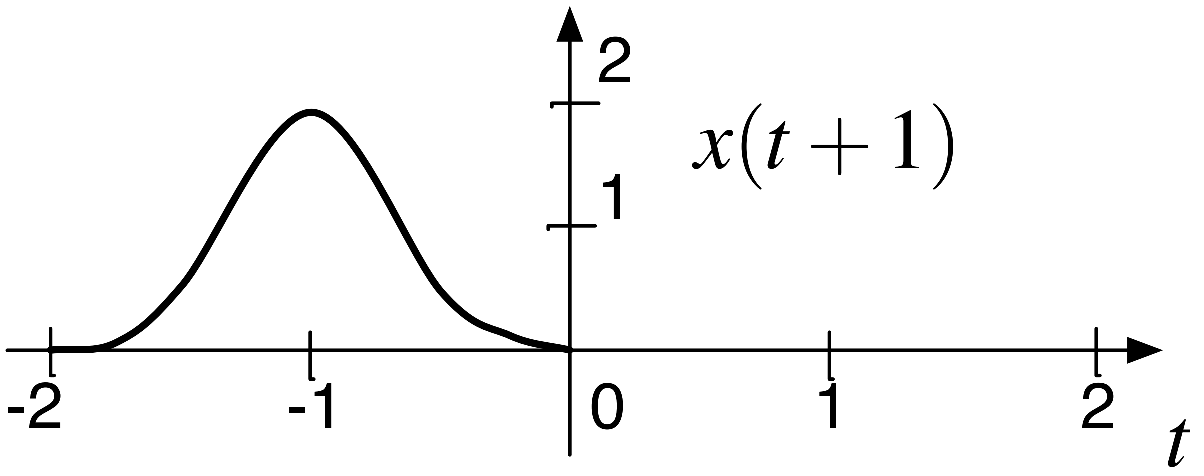
Sinc Function

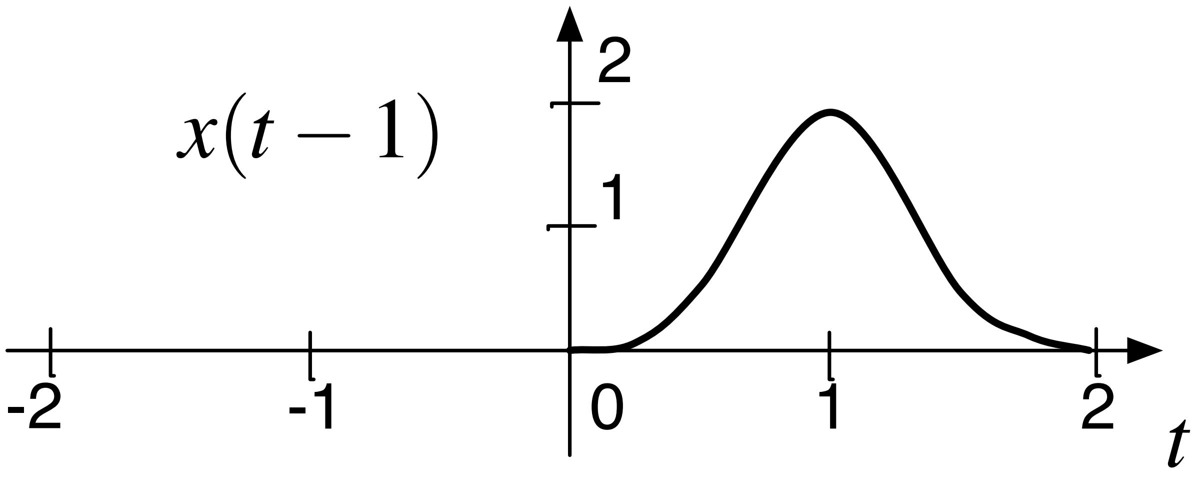


- This is the unusual definition, zeros are on the integers
- Less common is $\mathrm{sinc}(t) = \dfrac{\sin(t)}{t}$, with zeros on multiples of π. Watch out for this!

Operations: Time Shift

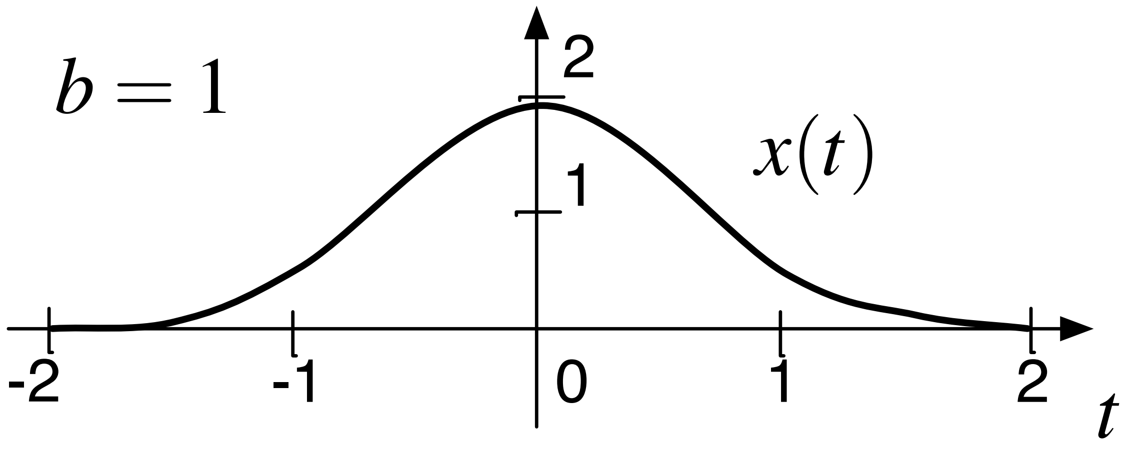


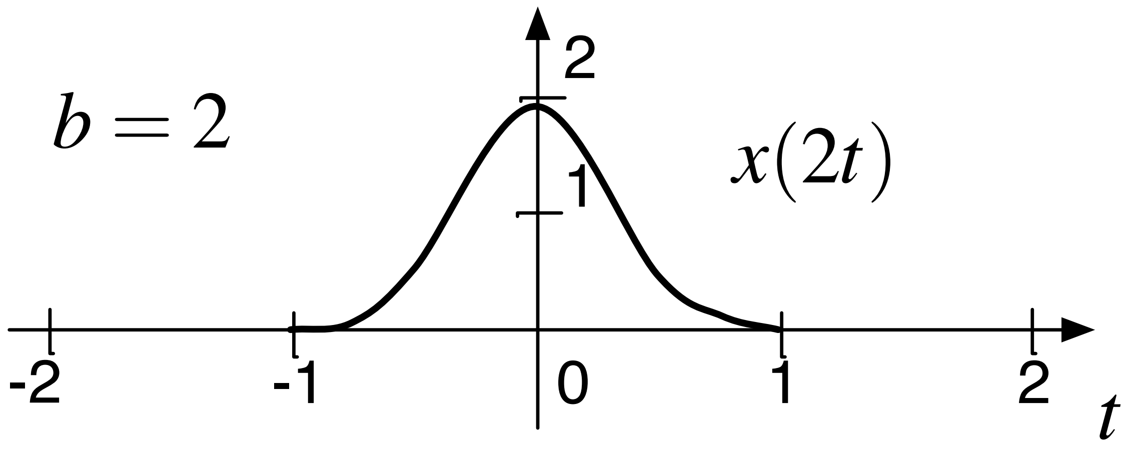


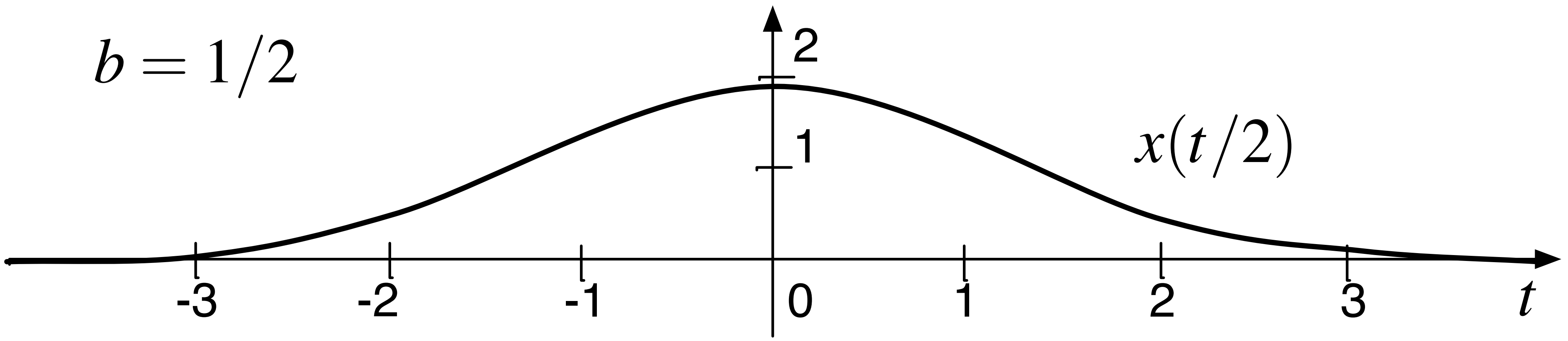


- Think “where does the argument equal 0”?

Operations: Time Scaling

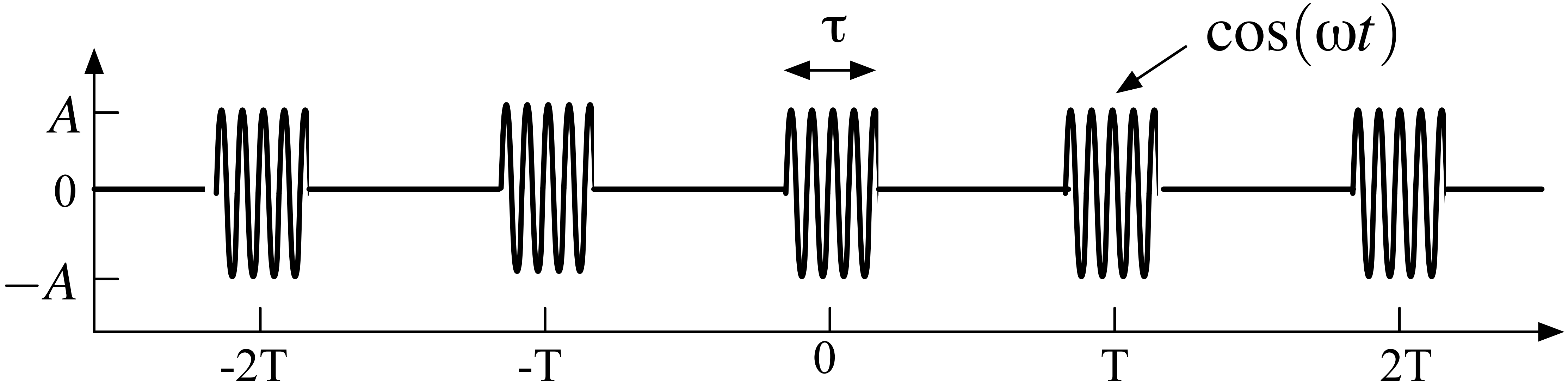






- Think “for what *t* does the argument equal 1”?

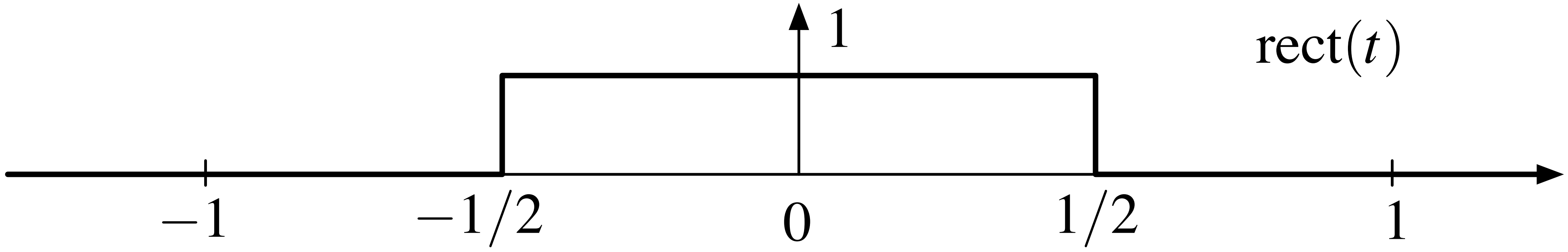
Back to the Pulse Train



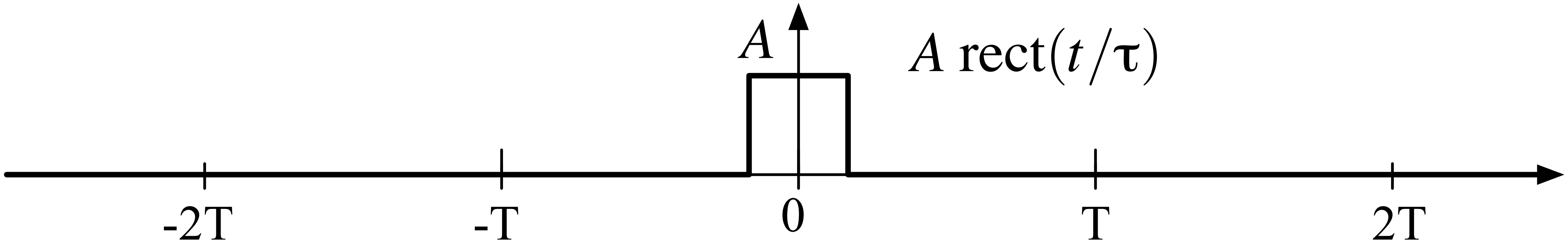
- This is:
    - a carrier at frequency ω
    - gated on for τ seconds
    - repeated every T seconds for N pulses
- Used for
    - Radar (Pulsed Doppler, Doppler Ultrasound, SAR)
    - If the pulses have variable amplitudes, this is digital communications waveform

Pulse Train

- Start with a rect() pulse



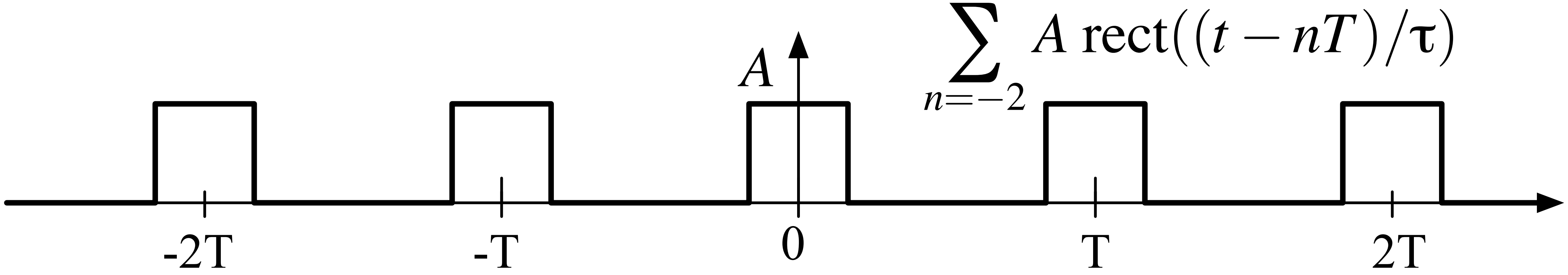
- Scale to the right length and amplitude



Pulse Train

- Combine shifted copies

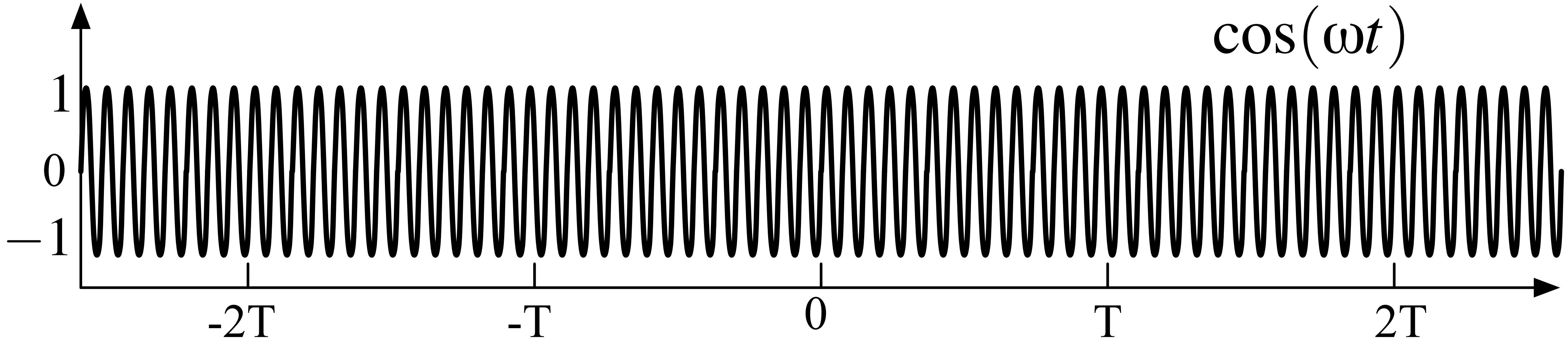
$$\sum_{n=-2}^{2} A\,\mathrm{rect}((t-nT)/\tau)$$



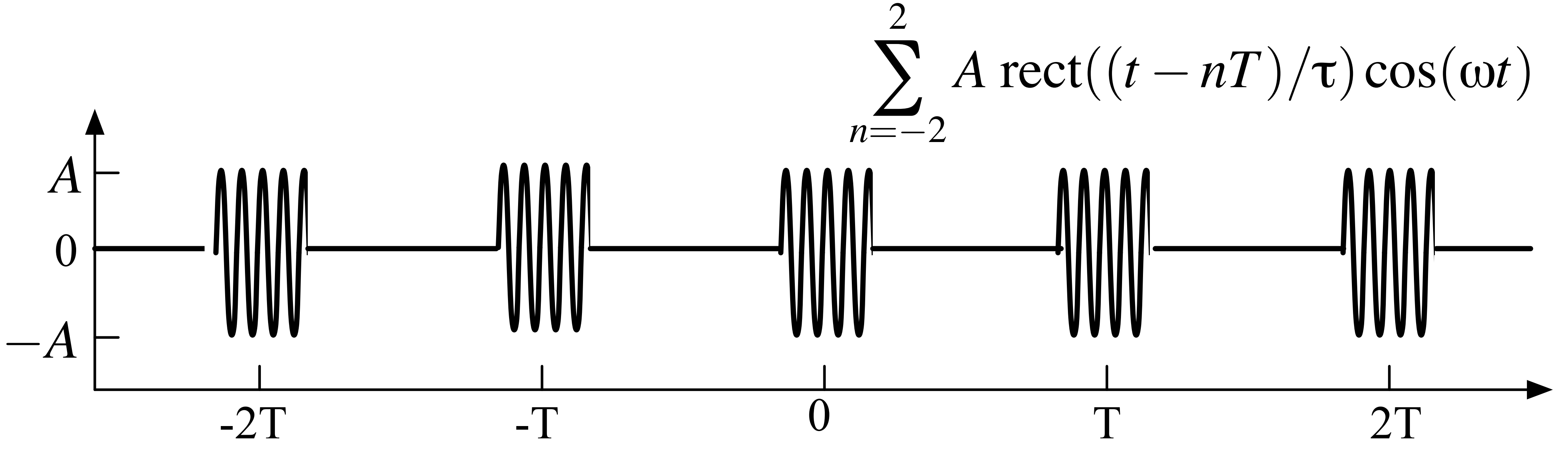
- This is the *envelope* of the signal
  - Also called the *baseband* signal
  - This describes all of the information that is encoded in the signal

Pulse Train

- Then multiply by a carrier



- This is gives the pulse train waveform



Signal Energy and Power

- The energy of a signal $g(t)$ is

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt$$

- This is finite for
  - Bounded, finite signals
  - Decaying signals
- Most signal were concerned with won’t be finite energy

Signal Energy and Power

- The power of a signal $g(t)$ is

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt$$

- This is a limit
  - Energy per unit time
  - If g(t) can often be complex
  - For periodic signals, only need to integrate over one period

Units of Power

- If $g(t)$ was a voltage applied to a resistor

$$P = \frac{g(t)^2}{R}$$

- This would be watts. We don't care about the load and let $R = 1$
- For many cases, the effect of the signal varies with the log of the signal: human hearing and sight
- Often convenient to express power in terms of logs

Decibels

- The decibel (dB) is defined as a ratio, relative to a reference

$$P_{dB} = 10\log_{10}\frac{P}{P_{ref}}$$

- If P is measured in watts, then
  - dBW is $10\log_{10}(\text{power relative to 1W})$
  - dBm is $10\log_{10}(\text{power relative to 1mW})$
- Lots of other examples
  - dBA for hearing, compared to the threshold of hearing
  - dBi for antennas, gain relative to an isotropic antenna

dB in Communications

- Communications systems have a wide range of powers
  - 1 kW transmitted
  - 1 μW received
- Many components have multiplicative effects
  - Antenna gain
  - Path loss
  - Signal processing gain

dB Example

- Communications systems link budget
  - Transmit 10W (+40 dBm)
  - Transmit antenna gain of 10 (+10 db)
  - Path loss of 10^9 (-90 dB)
  - Receive antenna gain of 10 (+10 dB)
- Total is 40 dBm + 10 dB + -90 dB +10 dB = -30 dBm
- This is $10^{-30/10} = 10^{-3}\,\mathrm{mW}$ or 1 μW, which is a strong signal
- The nice thing is this allows you to trade off parts of the system
  - Reduce transmit power for a better antenna, for example

Fourier Transform

- In 102A the Fourier Transform was

$$G(j\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

- In 102A the Fourier Transform was

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

- We used ω so that all of the transforms were of similar form (Fourier, Laplace, DTFT, etc).

Fourier Transform

- In this class we'll use $2\pi f$ instead of ω. If we replace $2\pi f$ with ω, the Fourier transform is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt\,.$$

- Noting that $d\omega = 2\pi\, df$. the inverse transform is

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df\,.$$

- Note that these are almost completely symmetric, only the sign of the exponent changes
- This simplifies a lot of the transforms and theorems

Fourier Transform Example

- The one-sided exponential decay is

$$g(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t > 0 \end{cases}$$

- The Fourier transform is

$$G(f) = \int_0^\infty e^{-j2\pi ft} e^{-at}\,dt = \int_0^\infty e^{-(a+j2\pi f)t}\,dt$$

$$= \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)}\right]_{t=0}^{t=\infty} = \frac{1}{a+j2\pi f}$$

- Same result as in 102A with ω replaced by $2\pi f$

Duality

- This is much easier and more useful with $2\pi f$
- The Fourier transform pair is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df$$

- The inverse transform differs only in exponent sign
- If we consider $G(f)$ to be a function of t, and apply the transform again

$$\int_{-\infty}^{\infty} G(t) e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} G(t) e^{j2\pi(-f)t}\, dt = g(-f)$$

Duality

- We can summarize this as

$$\mathcal{F}\{g(t)\} = G(f) \;\Rightarrow\; \mathcal{F}\{G(t)\} = g(-f)$$

- Assume we know a transform pair

$$g(t) \rightleftharpoons G(f)$$

- Then we immediately know another pair

$$G(t) \rightleftharpoons g(-f)$$

- This is much more convenient

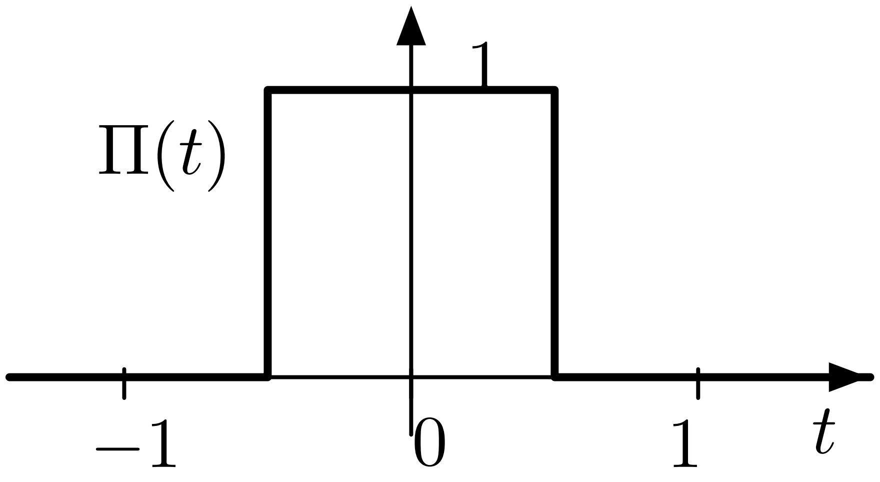
Duality Example

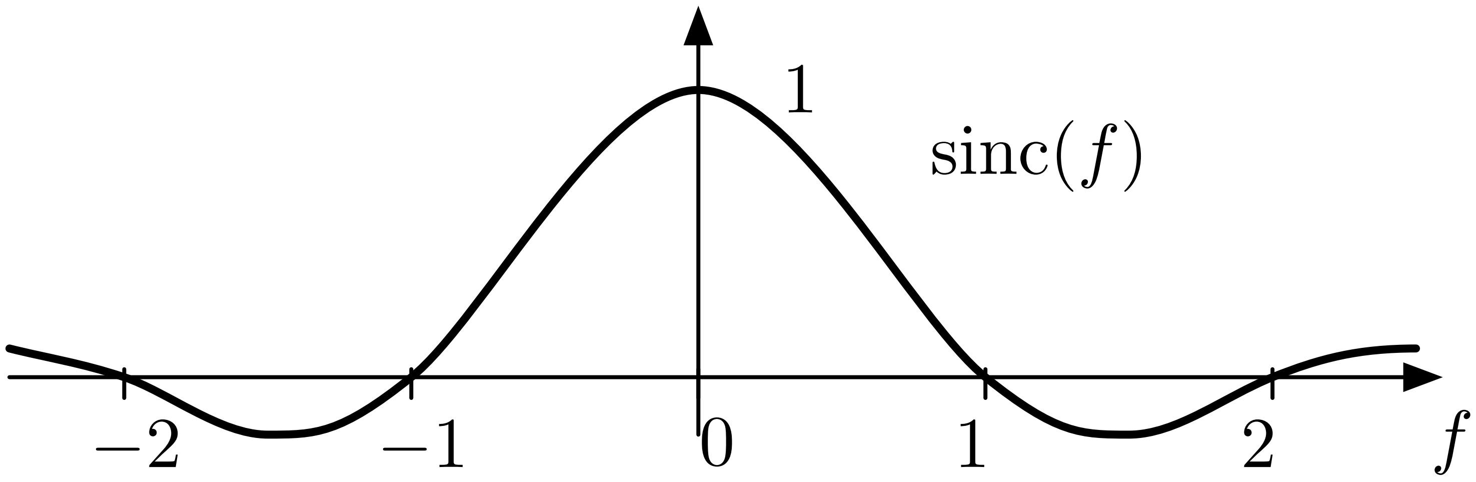
- The unit rectangle is

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

- Its transform is

$$\mathcal{F}\left\{\Pi(t)\right\} = \int_{-\infty}^{\infty} e^{-i2\pi ft}\Pi(t)\,dt = \int_{-1/2}^{1/2} e^{-i2\pi ft}\,dt = \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(f)$$



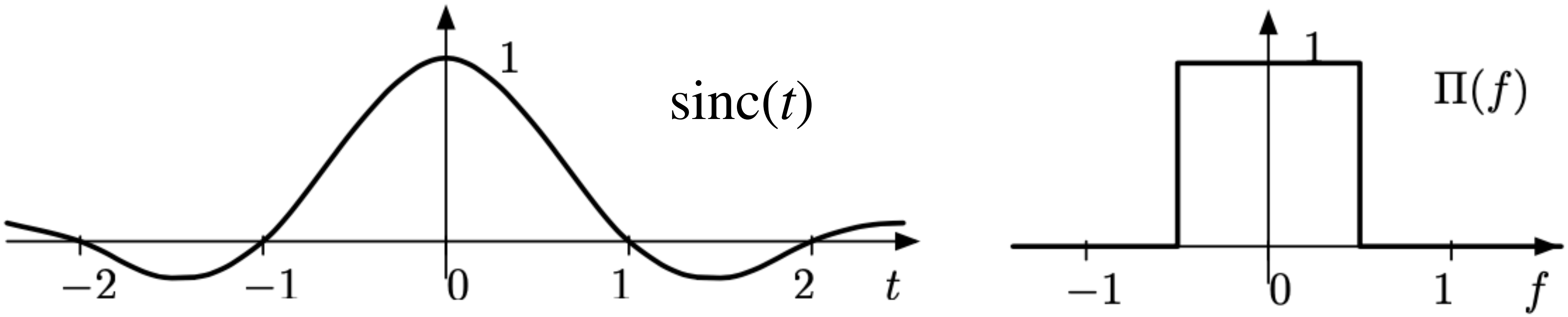


Duality Example

- By duality, we also have

$$\text{sinc}(t) \leftrightharpoons \Pi(f)$$

since $\Pi(t)$ is even, so $\Pi(f) = \Pi(-f)$



Impulses

- The Fourier transform of an impulse follows directly from its definition

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt = e^{-j2\pi f t}\Big|_{t=0} = 1$$

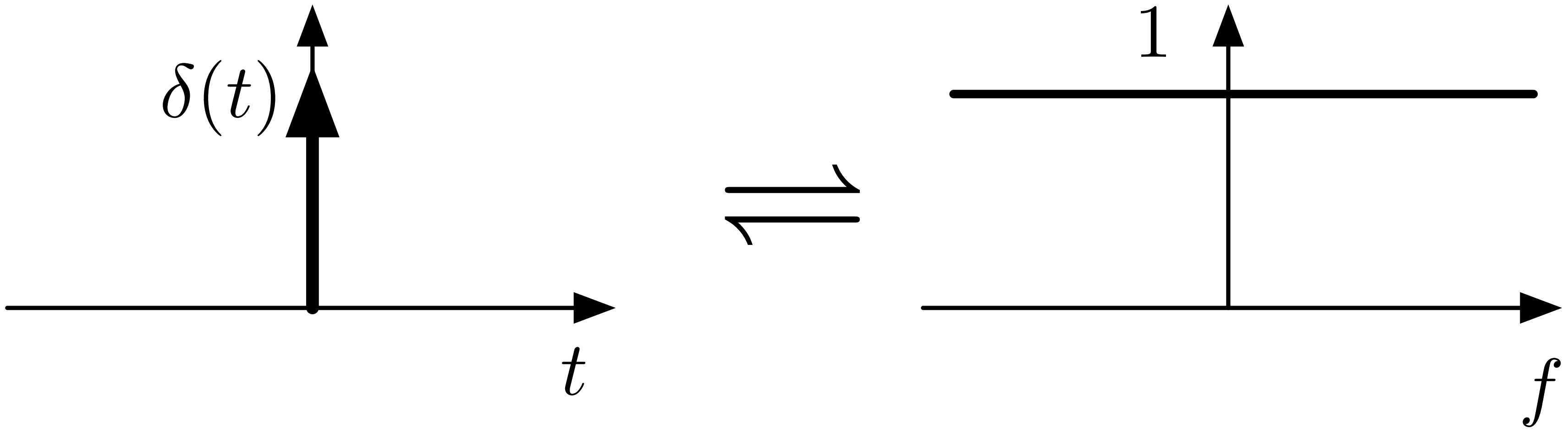
$$\delta(t) \rightleftharpoons 1$$

- Then, by duality

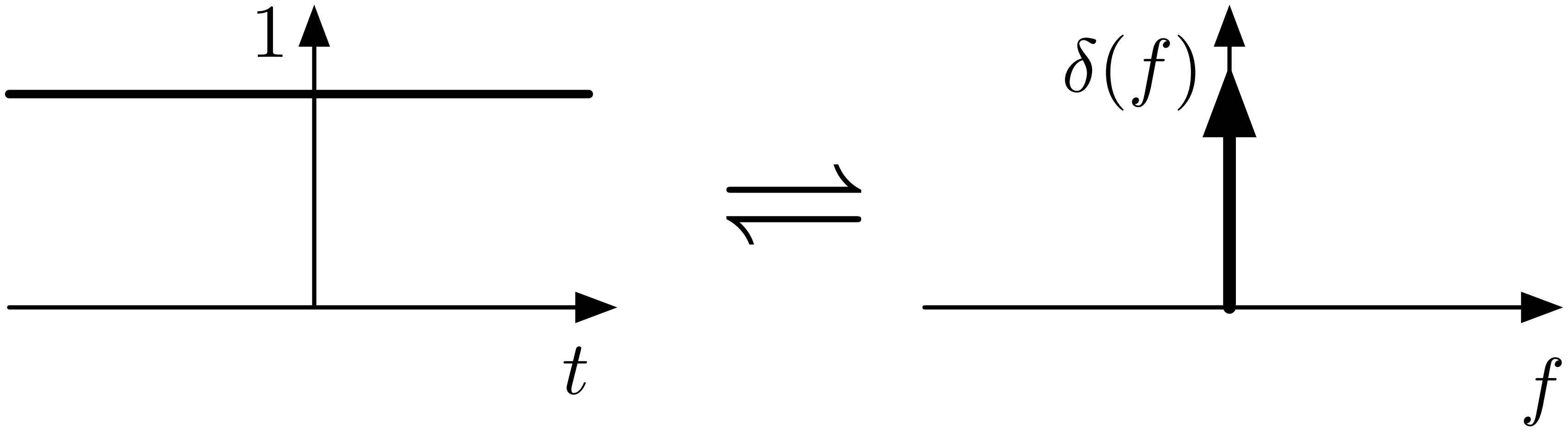
$$1 \rightleftharpoons \delta(f)$$

- Note, no 2π here.

$$\delta(t) \rightleftharpoons 1$$



$$1 \rightleftharpoons \delta(f)$$



Shifted Impulses

- Shifted impulse to $\delta(t - t_0)$

$$\mathcal{F}\{\delta(t - t_0)\} = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi f t}\, dt = e^{-j2\pi f t_0}$$

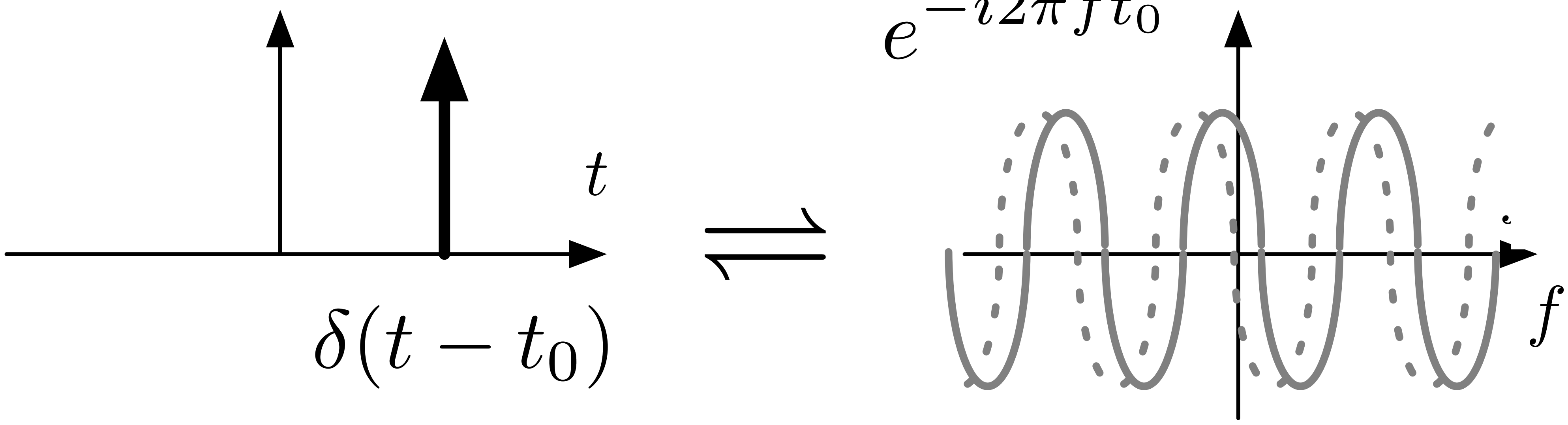
- This is a complex exponential in frequency

$$\delta(t - t_0) \rightleftharpoons e^{-j2\pi f t_0}$$

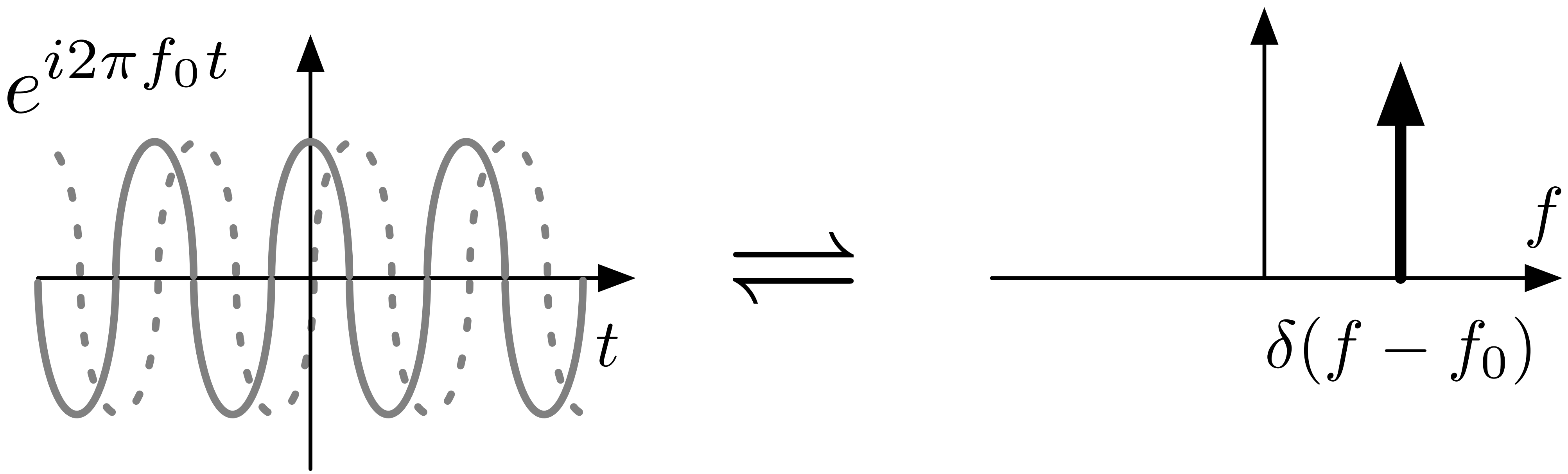
- Then by duality

$$e^{j2\pi f_0 t} \rightleftharpoons \delta(f - f_0)$$

$$\delta(t - t_0) \rightleftharpoons e^{-j2\pi f t_0}$$



$$e^{j2\pi f_0 t} \rightleftharpoons \delta(f - f_0)$$



Sine and Cosine

- Since

$$\cos(2\pi f_0 t) = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

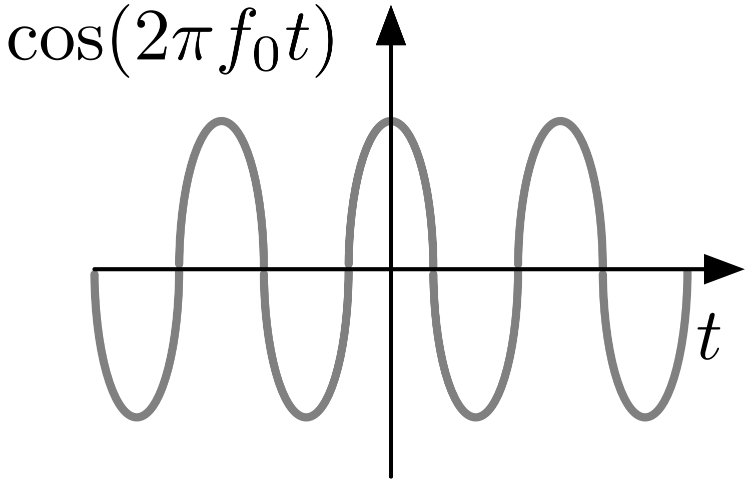
$$\sin(2\pi f_0 t) = \frac{1}{2j}(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t})$$

- These have the Fourier transforms

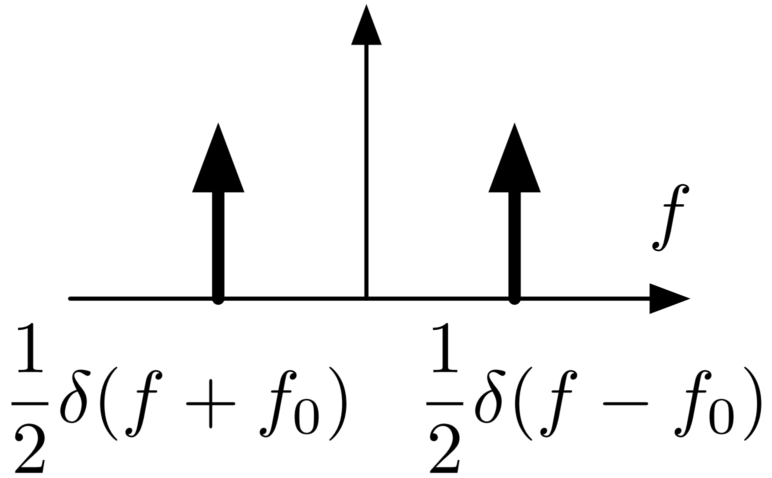
$$\cos(2\pi f_0 t) \rightleftharpoons \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

$$\sin(2\pi f_0 t) \rightleftharpoons \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$$

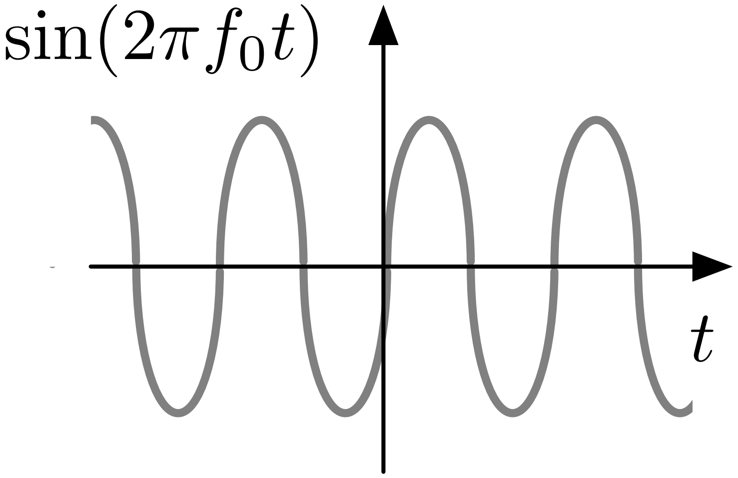
$$\cos(2\pi f_0 t) \rightleftharpoons \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$



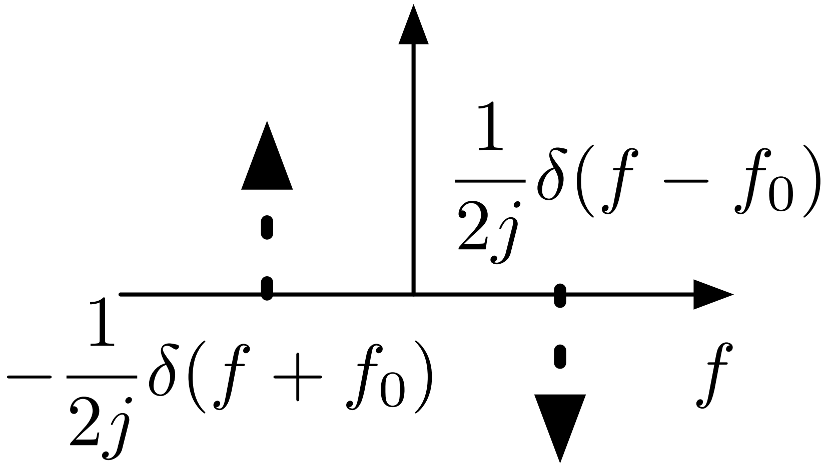
$\rightleftharpoons$



$$\sin(2\pi f_0 t) \rightleftharpoons \frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$$



$\rightleftharpoons$



Convolution in Time

- Convolution

$$g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(u) g_2(t-u)\, du$$

- This has the Fourier transform

$$g_1(t) * g_2(t) \rightleftharpoons G_1(f) G_2(f)$$

- Note, there is no factor of $\frac{1}{2\pi}$ here.

Multiplication in Time

- Multiplication

$$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)\,d\lambda$$

- This has the Fourier transform

$$g_1(t)g_2(t) \rightleftharpoons G_1(f) * G_2(f)$$

- Convolution in one domain corresponds exactly to multiplication in the other! For example:

$$x(t) * (y(t)z(t)) \rightleftharpoons X(f)\,(Y(f) * Z(f))$$

Shift Theorems

- Using impulses, convolution, and multiplication:

$$g(t - t_0) = g(t) * \delta(t - t_0)$$

$$g(t - t_0) \rightleftharpoons G(f)e^{-j2\pi f t_0}$$

- Similarly

$$G(f - f_0) = G(f) * \delta(f - f_0)$$

$$g(t)e^{i2\pi f_0 t} \rightleftharpoons G(f - f_0)$$

Modulation

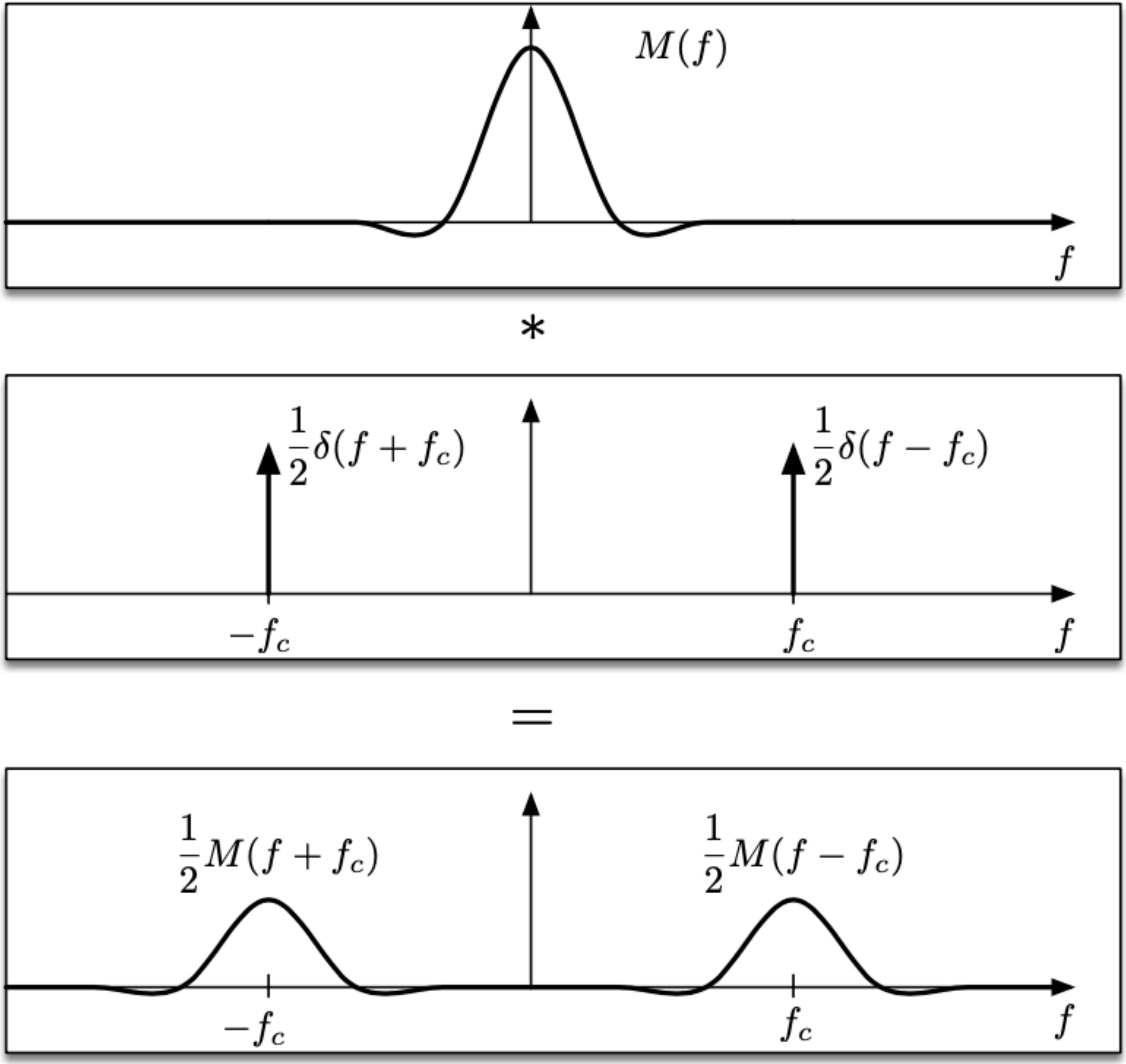
- Modulation is multiplication in time. Since

$$\cos(2\pi f_0 t) \rightleftharpoons \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

- Then

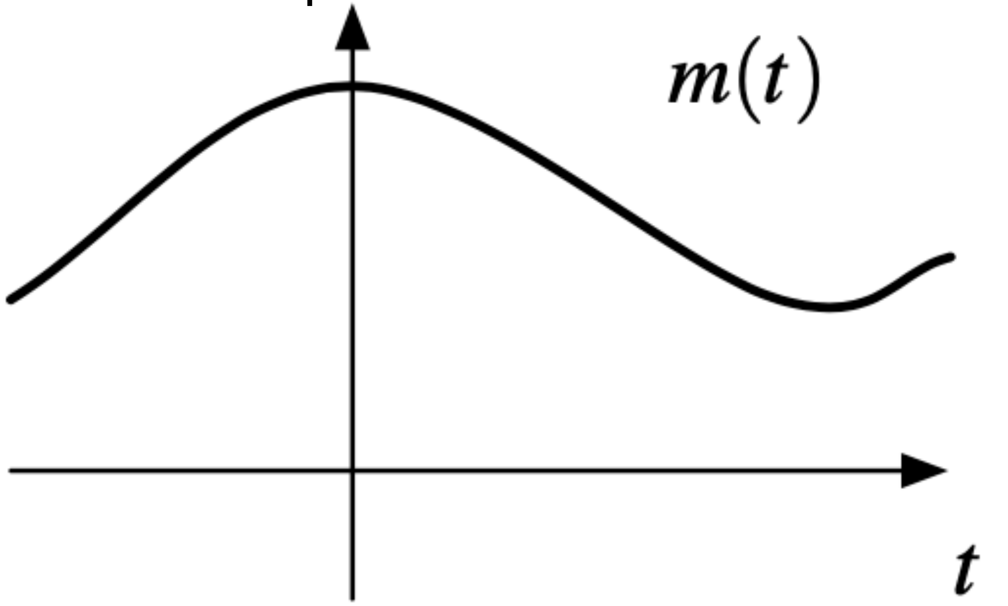
$$\mathcal{F}\left[m(t)\cos(2\pi f_c t)\right] = M(f) * \left(\frac{1}{2}\delta(f + f_c) + \frac{1}{2}\delta(f - f_c)\right)$$

- Modulation corresponds to convolution in frequency
- Convolution with an impulse is just a shift
- Modulation lets us move signals around in frequency.


$M(f)$
f
$*$
$\frac{1}{2}\delta(f+f_c)$
$\frac{1}{2}\delta(f-f_c)$
$-f_c$
f_c
f
$=$
$\frac{1}{2}M(f+f_c)$
$\frac{1}{2}M(f-f_c)$
$-f_c$
f_c
f

Airband AM Signal

Audio Envelope



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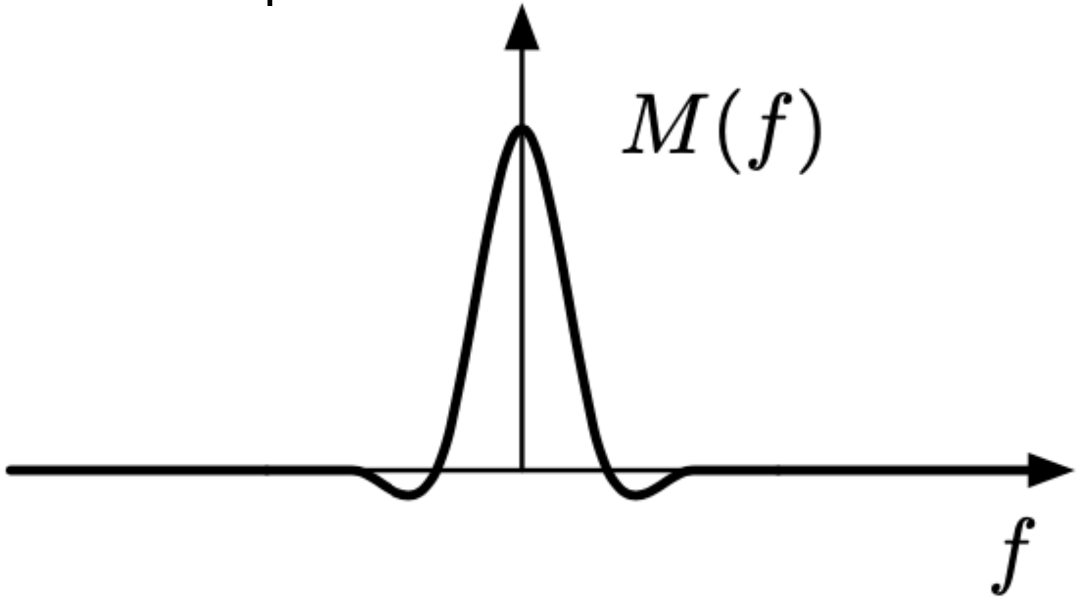
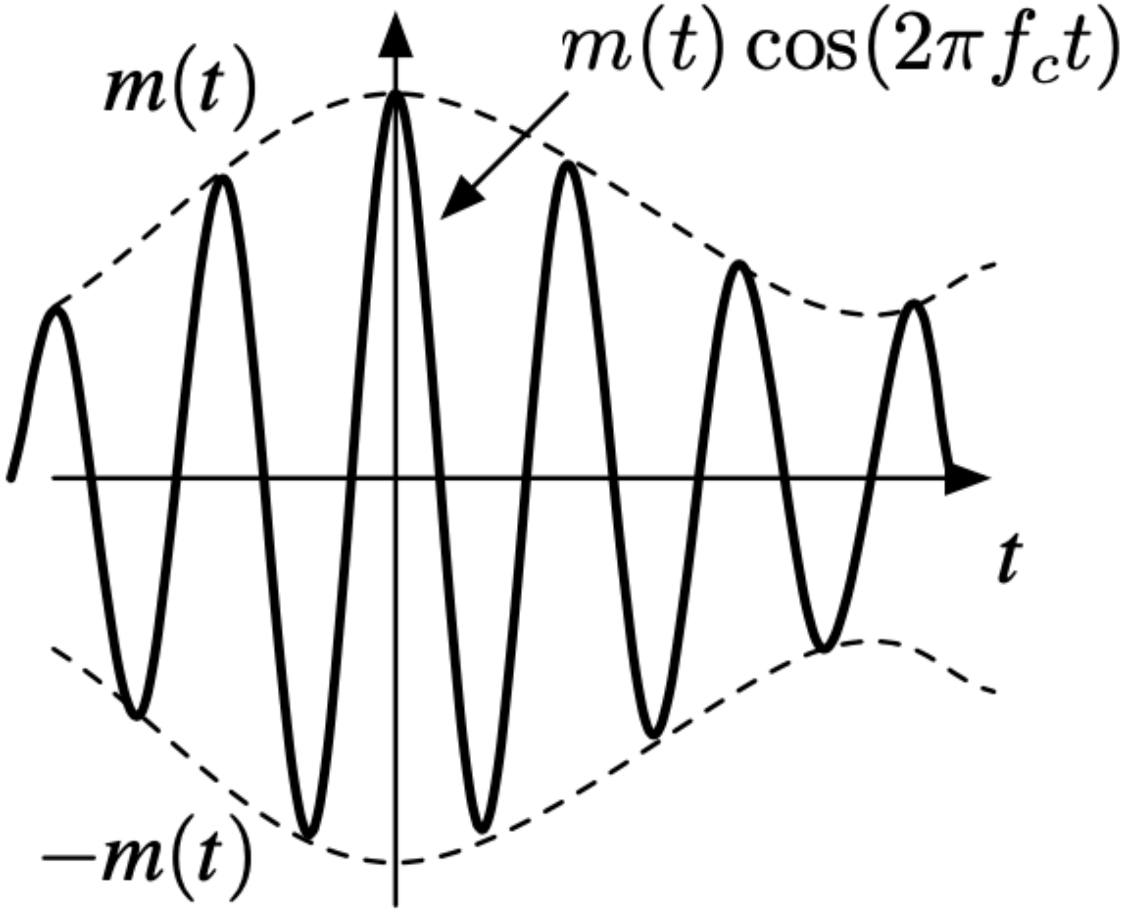
Audio Spectrum



RF Waveform



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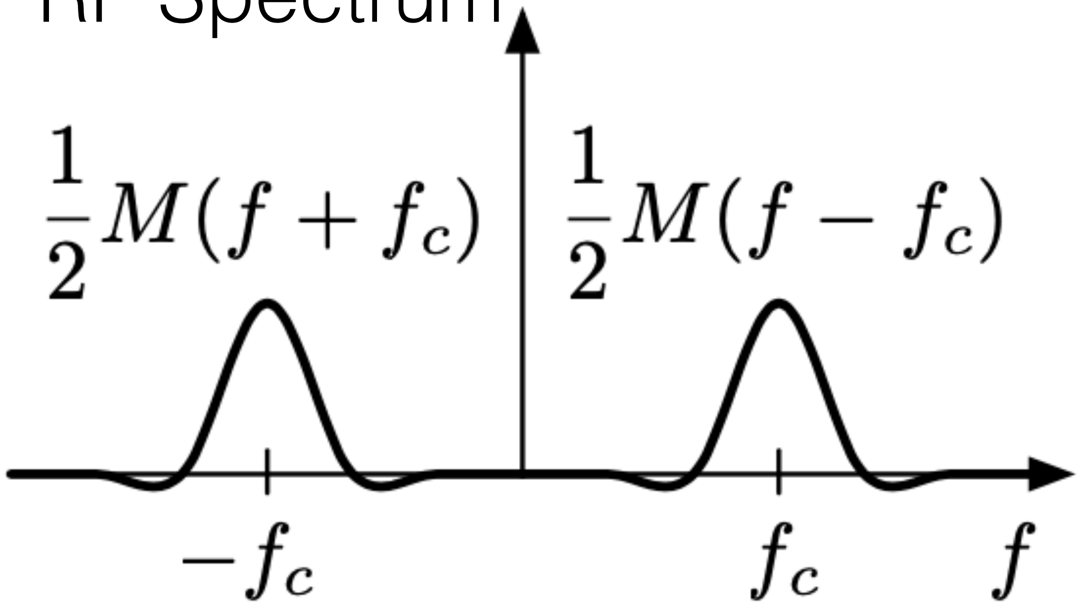
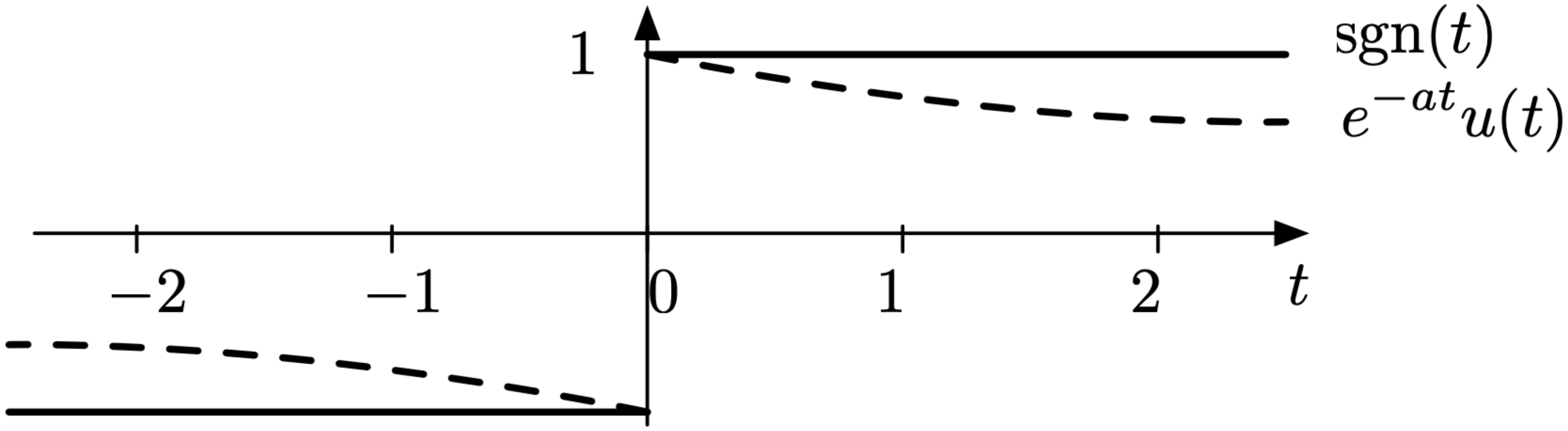
RF Spectrum



Signum Signal

- The signum signal can be approximated as

$$\operatorname{sgn}(t) = \lim_{a\to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right)$$



- This has the Fourier transform

$$\begin{aligned}\mathcal{F}\left\{\operatorname{sgn}(t)\right\} &= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right)\\ &= \frac{1}{j\pi f}\end{aligned}$$

Signum Signal

- The transform pair is

$$\mathrm{sgn}(t) \rightleftharpoons \frac{1}{j\pi f}$$

- Then, by duality

$$\frac{j}{\pi t} \rightleftharpoons \mathrm{sgn}(f)$$

Step Signal

- The unit step can be written as

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

- This has the Fourier transform

$$u(t) \rightleftharpoons \tfrac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

- By duality

$$\tfrac{1}{2}\delta(t) + \frac{j}{2\pi t} \rightleftharpoons u(f)$$

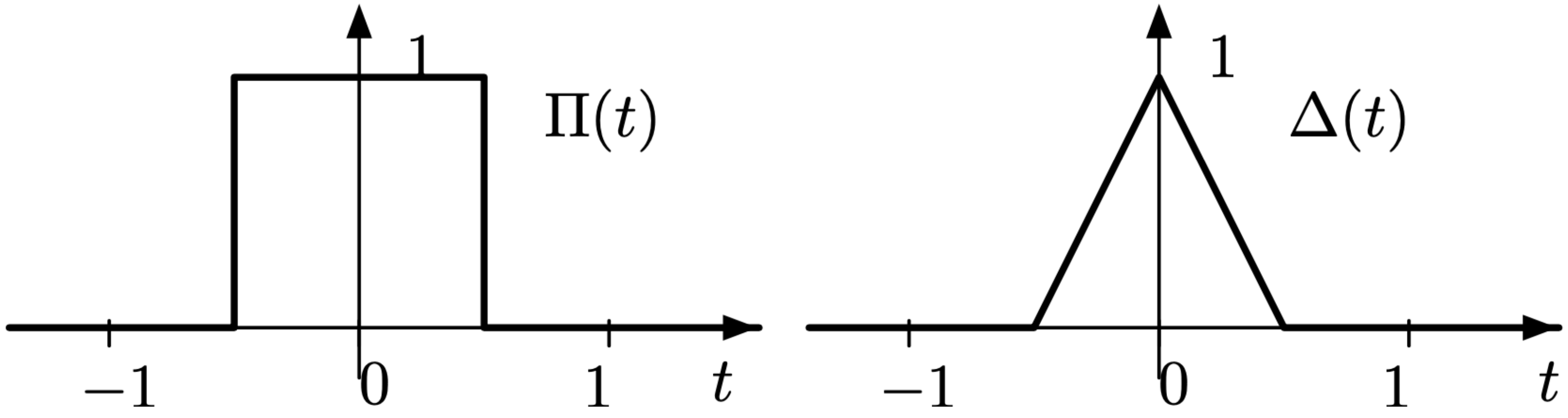
- This will be important for single sideband signals (SSB)

$\Delta(t)$

- The triangle function is

$$\Delta(t) = \begin{cases} 1 - 2|x| & |x| \leq \frac{1}{2} \\ 0 & \textsf{otherwise} \end{cases}$$

- It is the convolution of two scaled $\Pi(t)$'s



- Each $\Pi(t)$ is scaled to half the length,

$$\Delta(t) = 2\ \Pi(2t) * \Pi(2t)$$

- The factor of 2 makes the area work out to 1

$\Delta(t)$ Fourier Transform

- The Fourier transform is then

$$\begin{aligned}\mathcal{F}\left\{\Delta(t)\right\} &= \mathcal{F}\left\{2\Pi(2t)*\Pi(2t)\right\}\\ &= 2\mathcal{F}\left\{\Pi(2t)\right\}^2\\ &= 2\left(\frac{1}{2}\mathrm{sinc}\left(\frac{f}{2}\right)\right)^2\\ &= \frac{1}{2}\mathrm{sinc}^2\left(\frac{f}{2}\right)\end{aligned}$$

- The transform pair is

$$\Delta(t) \leftrightharpoons \frac{1}{2}\mathrm{sinc}^2\left(\frac{f}{2}\right)$$

Bandpass Signals

- A signal is bandlimited if

$$G(f) = 0 \text{ if } |f| > B$$

- Modulating $g(t)$ by a cosine at a carrier frequency f_c

$$x(t) = g(t)\cos(2\pi f_c t)$$

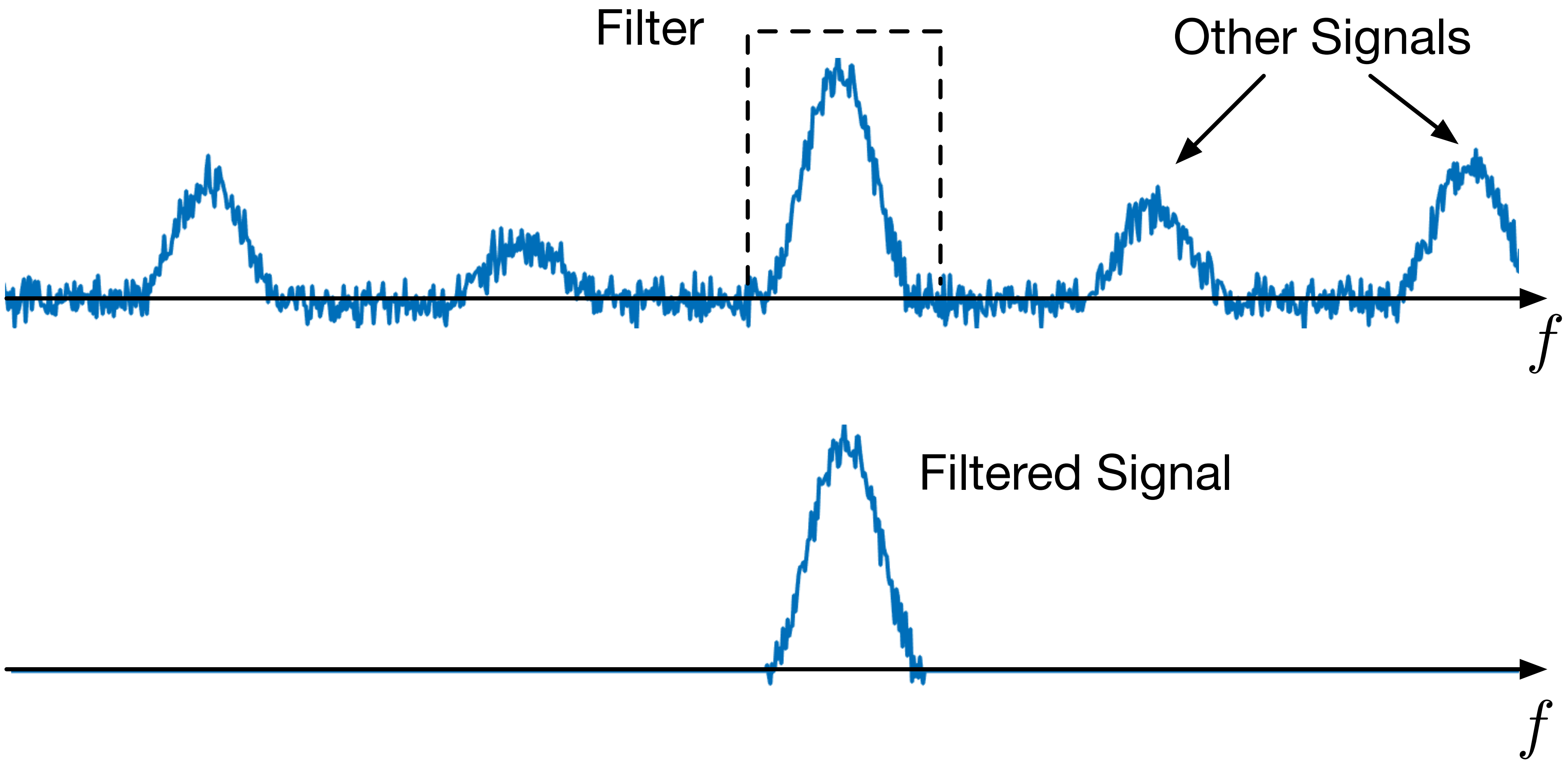
- This is a bandpass signal, with

$$f_c - B < |f| < f_c + B$$

- Many of the signals in the course will be bandpass
- The AM signal earlier is an example

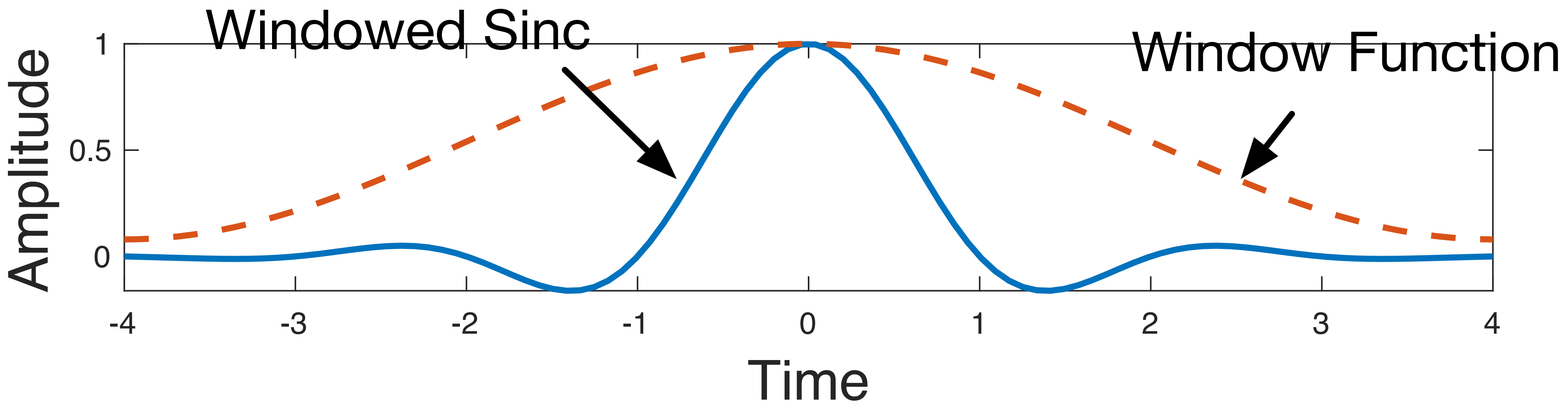
Filters

- Filters allow us to select specific signals by frequency, and suppress noise

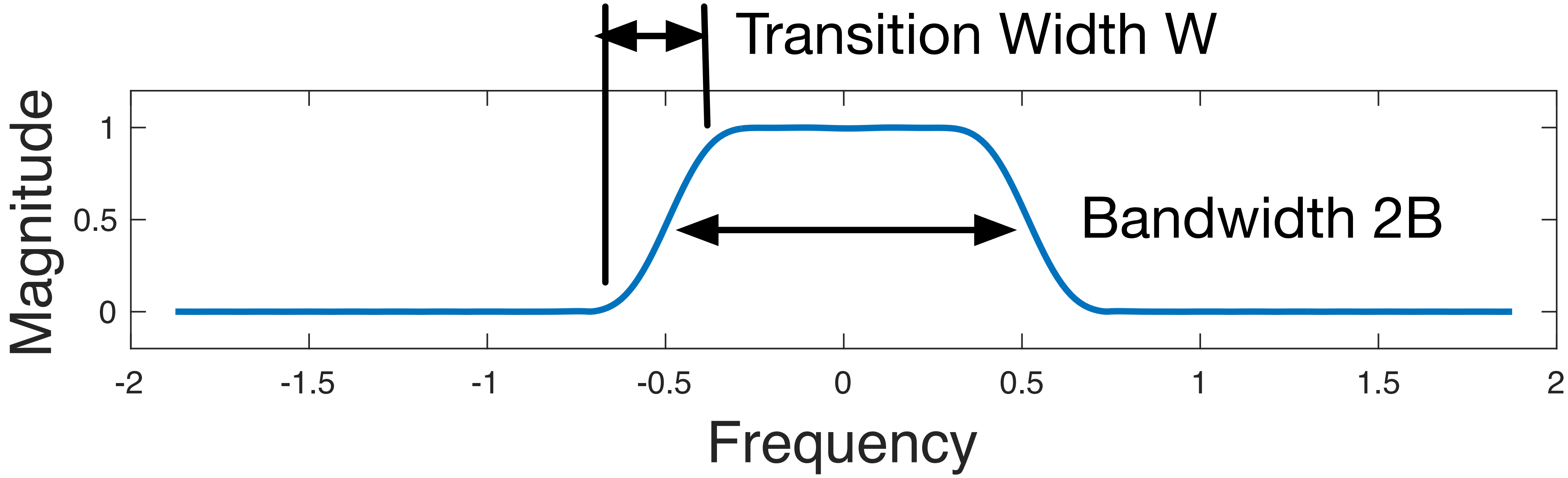


Windowed Sinc Filters

- Windowed sinc filter $h(t) = w(t/n)\,\mathrm{sinc}(t)$, where n is the number of zeros



- Spectral Response



Windowed Sinc Filters

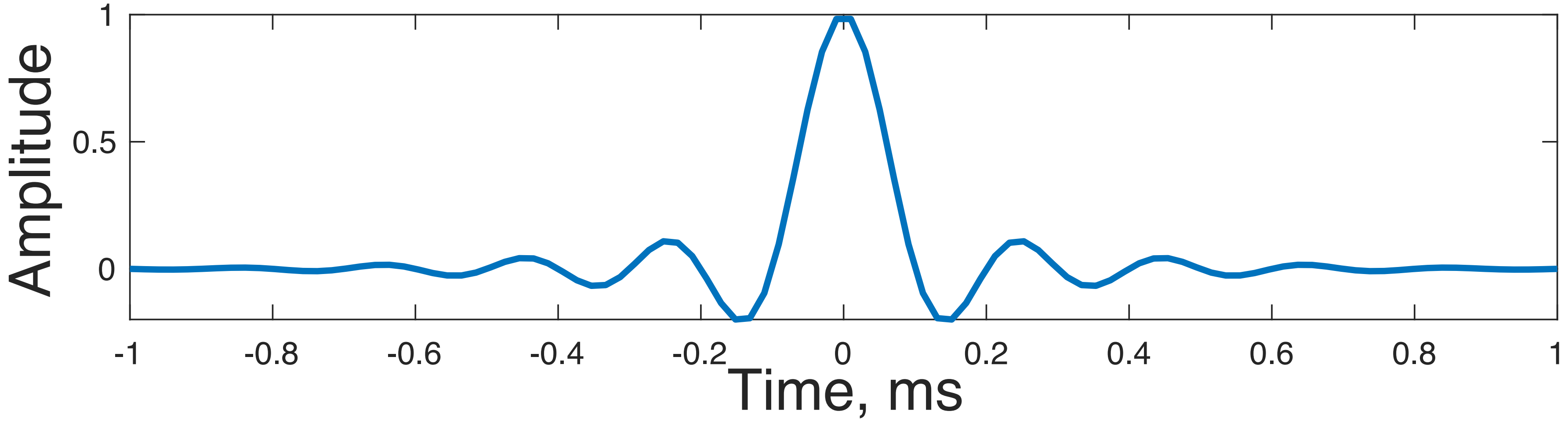
- Scale the duration of $h(t)$ to produce the passband we want for $H(f)$
- The previous filter had a duration of T = n, and a bandwidth 2B = 1
  - Time-Bandwidth Product of (n)(1) = n.
  - Transition width is approximately 2/n
- If we want a bandwidth of $2B = \beta$, we scale $H(f/\beta)$, which corresponds to a filter $\beta h(\beta t)$. The time-bandwidth product stays the same, n.
- The transition width is then $2\beta/n$

Windowed Sinc Filter Example

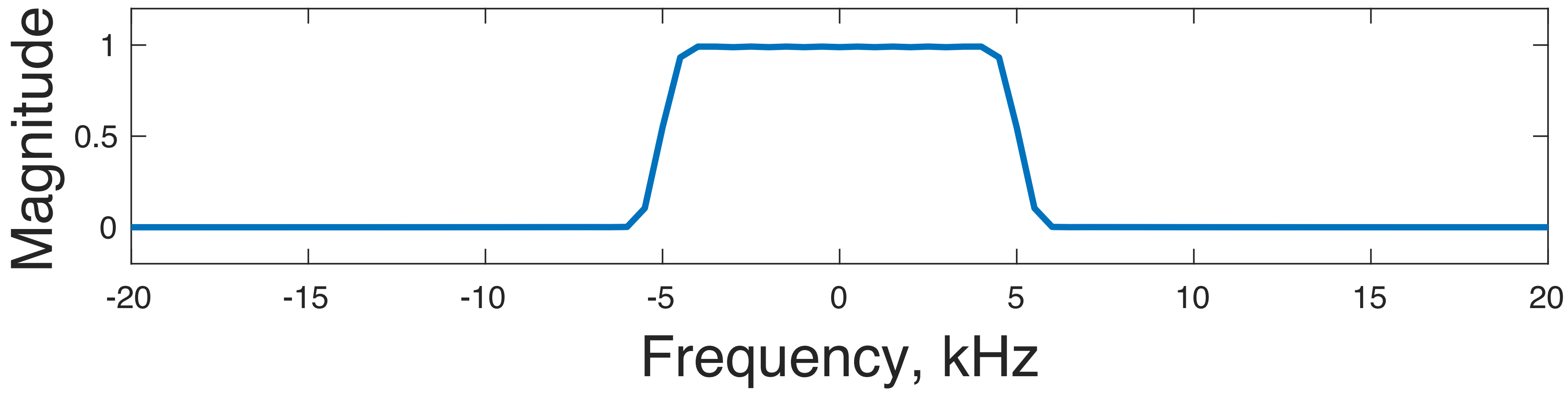
- We want a 10 kHz passband, and a transition width of 1 kHz, so that we can suppress adjacent signals.
- The transition width is
  - 1 kHz = 2*(10 kHz)/n
- That tells us that n = 20, the time-bandwidth product and also the number of sinc zeros
- The bandwidth $2B$ is 10 kHz, so
  - 20 = (10 kHz) * T
- The duration is 2 ms.

Windowed Sinc Filter Example

- The filter looks like this



- Spectral Response



- In matlab, you would sample the filter at the signal sampling rate, and the convolve to do the filtering

More Filters

- Most filters will start with a windowed sinc
- Bandpass filters are just a modulated lowpass windowed sinc
- A Single Sideband (SSB) filter is just a frequency shifted windowed sinc
- A differentiator is just the derivative of a windowed sinc
- We'll see many of these in the coming weeks.

Next Time

- Spectral energy density
- AM modulators and receivers
- Quadrature receivers
- Lab this week: Capturing RF signals with Matlab