Lecture 6: Angle Modulation

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Modulation

- Modulation encodes a real message $m(t)$ on a carrier $\cos(2\pi f_c t)$
- There are many ways to do this.
- So far we've looked at various amplitude modulation methods such as AM, SSB, or QAM. Here a carrier is multiplied by a *real* envelope.
- We can also encode information in the phase or frequency of the carrier.
- This can be described as a carrier multiplied by a *complex* envelope.

Based on lecture notes from John Gill

Amplitude Modulation

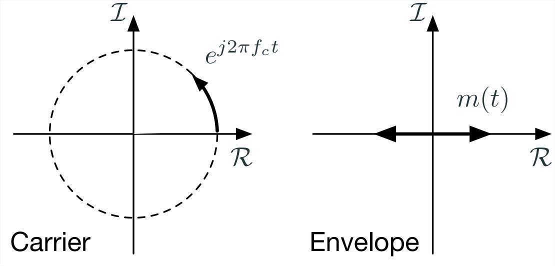
The DSB-SC modulated signal is

$$2m(t)\cos(2\pi f_c t) = m(t)e^{j2\pi f_c t} + m(t)e^{-j2\pi f_c t}$$

If we just focus on the positive frequency term we have

- The envelope which is the message $m(t)$ multiplied by
- The carrier $e^{j2\pi f_c t}$

If we plot these in the complex plane at some time t, this looks like



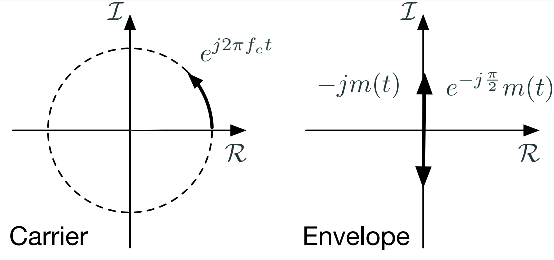
If we modulate a sine instead,

$$2m(t)\sin(2\pi f_c t) = -jm(t)e^{j2\pi f_c t} + jm(t)e^{-j2\pi f_c t}$$

Again we just focus on the positive frequency term, which we can consider to be

- The envelope $-jm(t)$ which includes the message $m(t)$
- The same carrier $e^{j2\pi f_c t}$

Note that the fact that we multiplied by a sine is reflected in the $-j$ in the *envelope*. The carrier is the same. This looks like

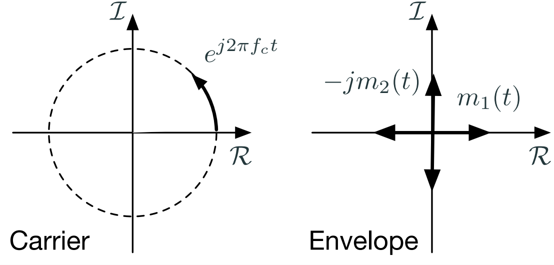


With QAM we have two messages, one modulated by a cosine, and a second with a sine

$$
\begin{aligned}
2m_1(t)\cos(2\pi f_c t) \quad &+ \quad 2m_2\sin(2\pi f_c t) \\
&= \quad (m_1 - jm_2(t))e^{j2\pi f_c t} + (m_1(t) + jm_2(t))e^{-j2\pi f_c t}
\end{aligned}
$$

The positive frequency term is then

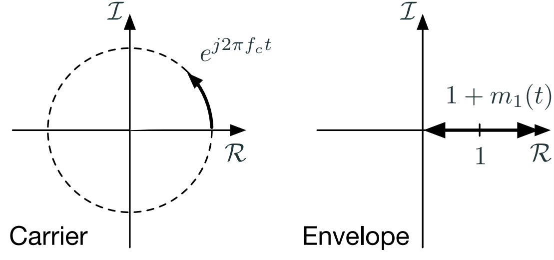
- The complex envelope $m_1(t) - jm_2(t)$ which has $m_1(t)$ as the real part, and $m_2(t)$ as the imaginary part.
- The same carrier $e^{j2\pi f_c t}$.



Broadcast AM is like DSB-SC with an additional bias which makes the envelope always positive

$$2(1+m(t))\cos(2\pi f_c t) = (1+m(t))e^{j2\pi f_c t} + (1+m(t))e^{-j2\pi f_c t}$$

This looks like



We are transmitting the AM bias term all the time.

Is there something else we could do with it?

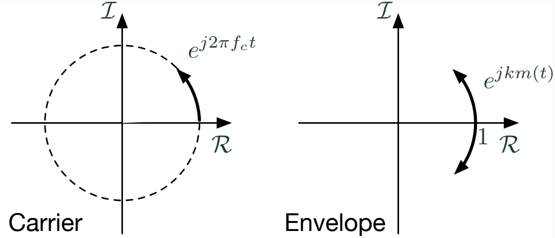
Instead of mutiplying the carrier by the message, we can include the message as the phase of the carrier

$$\begin{aligned} 2\cos(2\pi f_c t + km(t)) &= e^{j(2\pi f_c t + km(t))} + e^{j(2\pi f_c t + km(t))} \\ &= e^{jkm(t)} e^{j2\pi f_c t} + e^{-jkm(t)} e^{j2\pi f_c t} \end{aligned}$$

Now we have

- A complex envelope $e^{jkm(t)}$, which has unit magnitude and phase $km(t)$
- The same carrier $e^{j2\pi f_c t}$.

This looks like



Angle Modulation

- We can encode information as either
  - Time varying phase
  - Time varying frequency
- Both result in angle modulation
- Both are very closely related
- Easy to do with an sdr, just another complex envelope!

Instantaneous Frequency

- In general, the frequency of a signal at an instant in time depends on the entire signal.
- For generalized sinusoids, we can use a simpler approach. Suppose

  $$\phi(t) = A\cos\theta(t)\,.$$

  Then $\theta(t)$ is the *generalized angle*. For a true sinusoid,

  $$\theta(t) = 2\pi f_c t + \theta_0\,,$$

  linear with slope ω_c and offset θ_0
- The generalized angle is *not* limited to $[0, 2\pi]$. Wrapping introduces discontinuities.
- Phase unwrapping is easy at an IF, where the phase changes are small from one sample to the next.

Instantaneous Frequency (cont.)

- Instananeous frequency is derivative of generalized angle:

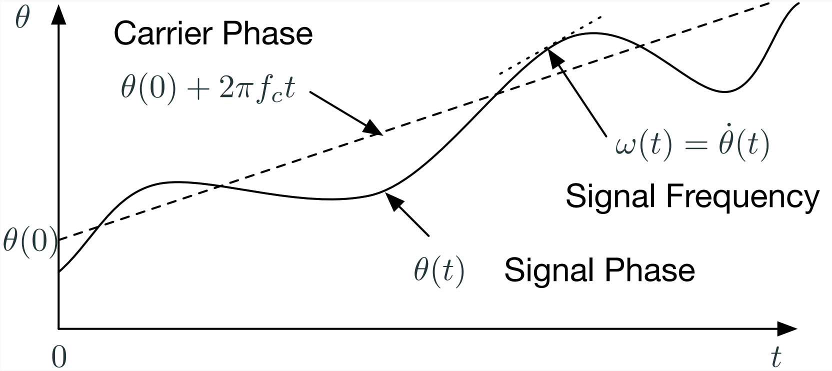
$$\omega_i(t) = \frac{d\theta}{dt} = \theta'(t)$$

- The phase is just the integral of the frequency

$$\theta(t) = \int_{-\infty}^{t} \omega_i(u)\ du = \theta(0) + \int_{0}^{t} \omega_i(u)\ du$$

- We can modulate a generalized sinusoid by using a signal $m(t)$ to vary either $\theta(t)$ or $\omega_i(t)$.
- In either case, the frequency of the modulated signal changes as a function of $m(t)$.

Instantaneous Frequency (cont.)



- Dashed line is the carrier phase
- Solid line is the phase of the transmitted signal
- Slope of the solid line is the instantaneous frequency of the transmitted signal.

Phase Modulation (PM)

- In PM, *phase* is varied *linearly* with $m(t)$:

$$\theta(t) = 2\pi f_c t + k_p m(t)$$

which produces a transmitted signal

$$\phi_{PM}(t) = \cos\big(2\pi f_c t + k_p m(t)\big)$$

- The instantaneous frequency is

$$\omega_i(t) = \frac{d\theta}{dt} = 2\pi f_c + k_p \dot{m}(t)$$

- If $m(t)$ varies rapidly, then the frequency deviations are larger.
- The bandwidth of the signal is determined by $\dot{m}(t)$, similar to AM.

Frequency Modulation (FM)

- In FM, *frequency* is is varied *linear* in $m(t)$:

$$\omega_i(t) = 2\pi f_c + k_f m(t)$$

which produces a signal

$$\phi_{FM}(t) = \cos\big((2\pi f_c + k_f m(t))t\big)$$

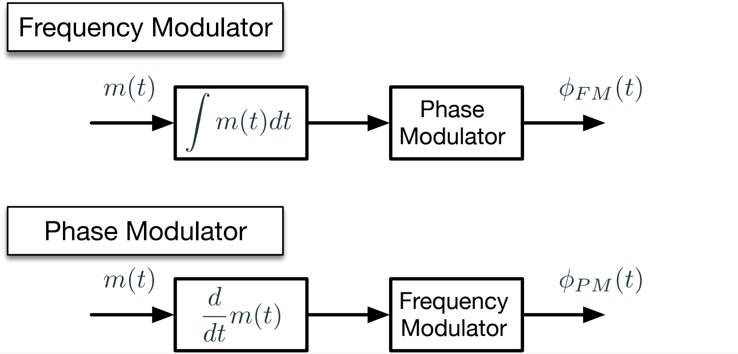
- The bandwidth of the signal is determined by the *amplitude* of $k_f m(t)$ (obvious), and also the bandwidth of $m(t)$ (less obvious).
- The angle is

$$\theta(t) = \int_{-\infty}^{t} \big(2\pi f_c + k_f m(u)\big)\, du = 2\pi f_c t + k_f \int_{-\infty}^{t} m(u)\, du$$

- We could apply this $\theta(t)$ to a phase modulator, and get exactly the same effect as applying $\omega_i(t)$ to a frequency modulator.

Relationship Between FM and PM

- Phase modulation of $m(t)$ = frequency modulation of $\dot{m}(t)$.
- Frequency modulation of $m(t)$ = phase modulation of $\int m(u)\, du$.



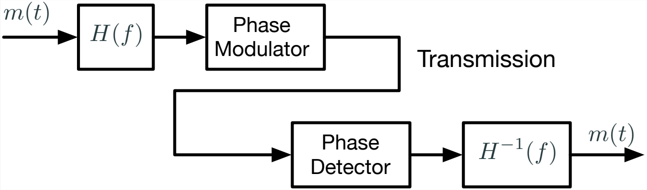
- We can produce both types of modulation with either modulator.
- Direct digital synthesis (DDS) chips will do this for you.

Generalized Angle Modulation

- We can generalize modulation by convolving the message signal with an impulse response $h(t)$

$$\phi_{EM}(t) = A\cos\left(2\pi f_c t + h(t) * m(t)\right)$$

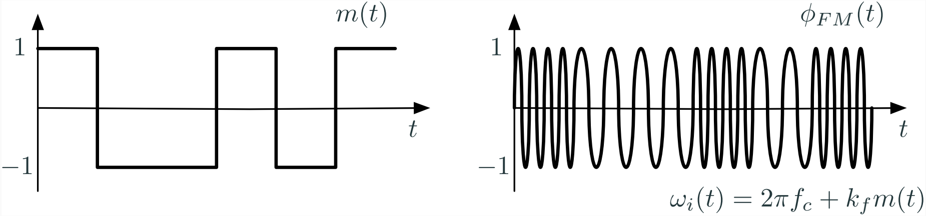
  This is a filter with a transfer function $H(f)$.
- We recover $m(t)$ from phase by using inverse filter $H^{-1}(f)$.
  E.g., for FM the inverse of integration is differention.



- PM ($h(t) = k_p\delta(t)$) and FM ($h(t) = k_f u(t)$) are special cases.
- Also used for pre-emphasis to improve noise characteristics, and pulse shaping to reduce signal bandwidth.

Frequency Shift Keying (FSK)

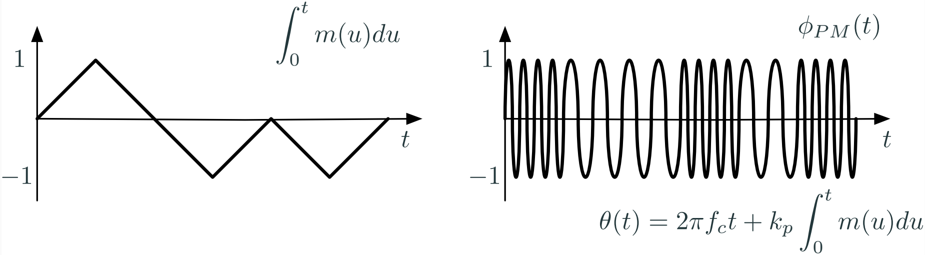
- Basic idea is to send a string of bits as two different frequencies
- These are encoded in $m(t)$ as a value of 1 for a bit 1, and a value of -1 for a bit 0.
- We then transmit a frequency $2\pi f_c t + k_f$ for a 1, and $2\pi f_c - k_f$ for a zero.



- This type of modulation is very common in modems, and also digital radio
- We'll see this later in one of the labs.

FSK with a Phase Modulator

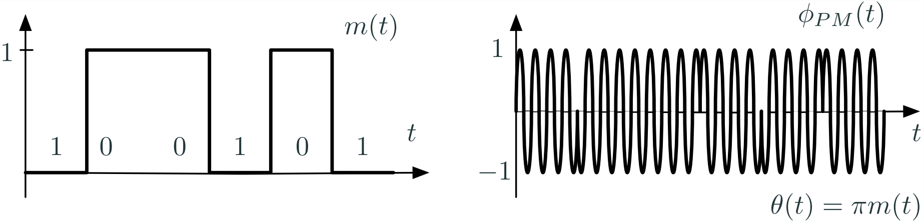
- We could achieve the same effect with a phase modulator.
- The input is now $\int_o^t m(u)du$



- This results in exactly the same waveform given k_p is properly scaled.

Phase Shift Keying (PSK)

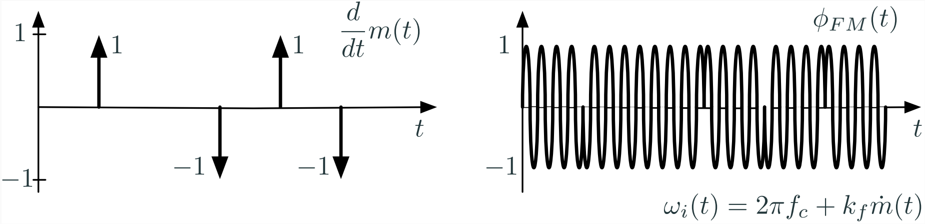
- Basic idea is to send a string of bits as two (or more) phases of a carrier
- These are encoded in $m(t)$ as a value of 0 for a bit 1, and a value of 1 for a bit 0, which is then scaled to 0 and π.
- This inverts the carrier for zero bits



- This is also common for digital radio and modems
- Some car key fobs used this. We will also see this later.

PSK with a Frequency Modulator

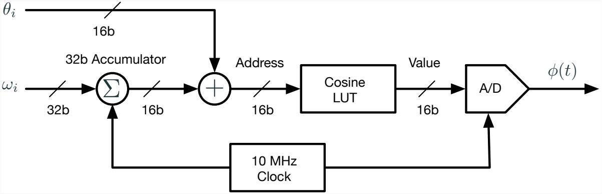
- We could achieve the same effect with a frequency modulator
- The input is now $\frac{d}{dt}m(t)$.



- Again, this results in exactly the same waveform given k_f is properly scaled.

Direct Digital Synthesis

Direct Digital Synthesis (DDS) system block diagram



- LUT is 2^{16} samples of a cosine. It takes an index and outputs the value
- Generating the FM or PM waveform consists of stepping through this array at the right speed
- The 10 MHz clock sets the speed. Every 0.1 μ s there is a new output
- The step size is set by ω_i. For the current $\omega(t)$, how much does the address advance in 0.1 μ
- The phase θ_i is just an offset into the $[0..2^{16}]$ samples for $[0..2\pi]$.

Narrowband and Wideband FM

- FM bandwidth and Carson's rule
- Spectral analysis of FM
- Narrowband FM Modulation
- Wideband FM Modulation

Bandwidth of Angle-Modulated Waves

Angle modulation is nonlinear and complex to analyze.

Early developers thought that bandwidth could be reduced to 0.
They were wrong. FM has infinite bandwidth.

Two approximations for FM:

- Narrowband approximation (NBFM)
- Wideband approximation (WBFM)

This depend on if the FM modulation is larger than the signal bandwidth.
If we define

$$a(t) = \int_{-\infty}^{t} m(u)\, du$$

Then the frequency modulated signal is

$$\phi FM(t) = \cos(2\pi f_c t + k_f a(t))$$

since phase modulation is the integral of frequency modulation.

For the narrow band case

$$|k_f a(t)| \ll 1$$

We will show that the NBFM bandwidth is the same as the signal bandwidth

$$2B_s \approx 2B_m$$

and

$$\phi_{FM}(t) \approx A\big(\cos(2\pi f_c t) - k_f a(t)\sin(2\pi f_c t)\big)$$

This is what we saw last time. The NBFM case looks like a small modulation in quadrature with the carrier.

For the wideband case, if the peak frequency modulation is

$$\Delta f = \max |k_f m(t)|$$

then the bandwidth of the WBFM signal is

$$2B_s = 2\Delta f + 2B_m$$

This is known as known as Carson's rule. (Carson, Proc. IRE, 1922.)

Narrowband FM

Recall that

$$a(t) = \int_{-\infty}^{t} m(u)\ du$$

The *complex FM signal* from last time is

$$\begin{aligned} \hat{\phi}_{FM}(t) &= Ae^{j(2\pi f_c t + k_f a(t))} \\ &= Ae^{jk_f a(t)} e^{j2\pi f_c t} \end{aligned}$$

The transmitted signal is just the real part, $\phi_{FM}(t) = \Re\big(\hat{\phi}_{FM}(t)\big)$.

Using the Maclaurin power series for the exponential $e^{jk_f a(t)}$ in $\hat{\phi}_{FM}(t)$:

$$\hat{\phi}_{FM}(t) = A\left(1 + jk_f a(t) - \frac{k_f^2}{2!}a^2(t) + \cdots + j^n \frac{k_f^n}{n!}a^n(t) + \cdots\right) e^{j2\pi f_c t}$$

If $a(t)$ has a bandwidth $2B$ Hz, then the nth term has a bandwidth $n2B$!

This expansion for $\hat{\phi}_{FM}(t)$ shows that the bandwidth is infinite.

However, things aren't quite that bad ...

Since $k_f^n/n! \to 0$, all but a small amount of power is in a finite band.

Using $\phi_{FM}(t) = \mathcal{R}\left\{\hat{\phi}_{FM}(t)\right\}$, the FM signal is

$$\begin{aligned}\phi_{FM}(t) &= \mathcal{R}\left\{A\left(1 + jk_f a(t) - \frac{k_f^2}{2!}a^2(t) + \cdots +\right)(\cos(2\pi f_c t) + j\sin(2\pi f_c t)\right\} \\ &= A\Big(\cos 2\pi f_c t - k_f a(t)\sin 2\pi f_c t - \frac{k_f^2}{2!}a^2(t)\cos 2\pi f_c t + \cdots\Big)\end{aligned}$$

If $|k_f a(t)| \ll 1$ then all but first two terms are negligible.

The narrowband FM approximation is

$$\phi_{FM}(t) \approx A\big(\cos(2\pi f_c t) - k_f a(t)\sin(2\pi f_c t)\big)$$

NBFM signal has bandwidth $2B$, same as bandwidth of AM.

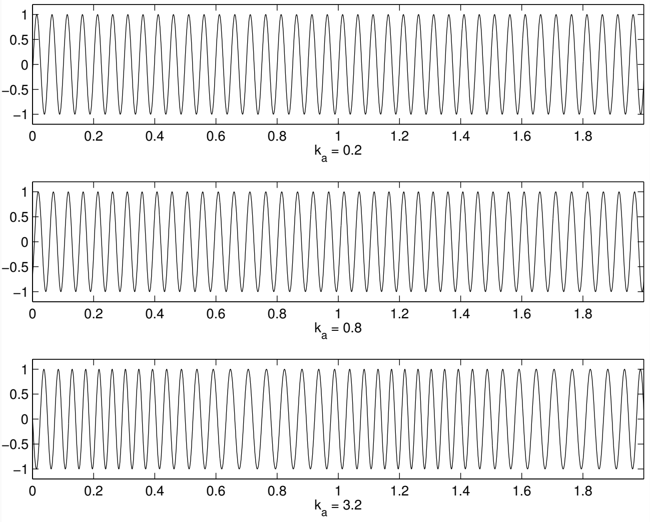
NBFM has power $\frac{1}{2}A^2$, which does not depend directly on $m(t)$.

A narrowband argument for phase modulation gives is similar result:

$$\phi_{PM}(t) \approx A\big(\cos(2\pi f_c t - k_p m(t)\sin(2\pi f_c t)\big)$$

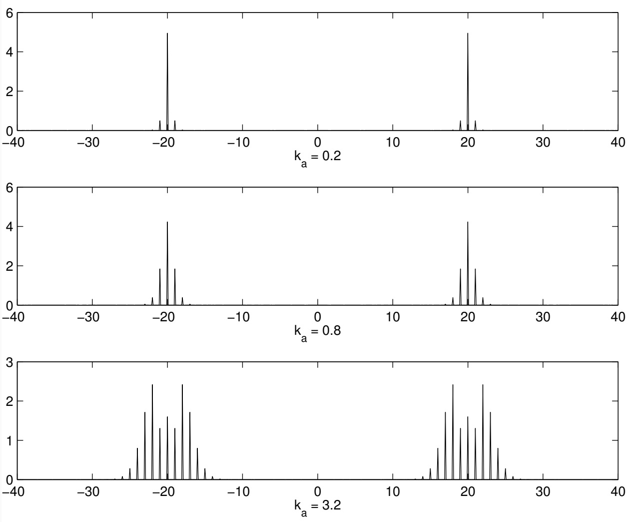
Tone Frequency Modulation, $f_c = 20$, $f_m = 1$

$$\varphi_{\mathrm{FM}}(t) = \cos\left(2\pi f_c t + k_a a(t)\right), |k_f a(t)| = 0.2, 0.8, 3.2$$



Fourier Transforms of Tone FM

$$\varphi_{\text{FM}}(t) = \cos\left(2\pi f_c t + k_a a(t)\right), |k_f a(t)| = 0.2, 0.8, 3.2$$



Wideband FM (WBFM) Bandwidth

For wideband FM, the frequency deviation contributes to the FM bandwidth. If the message signal is $m(t)$, and the FM signal is

$$\phi_{FM}(t) = \cos(2\pi f_c t + k_f a(t))$$

where again

$$a(t) = \int_{\infty}^{t} m(\tau) d\tau.$$

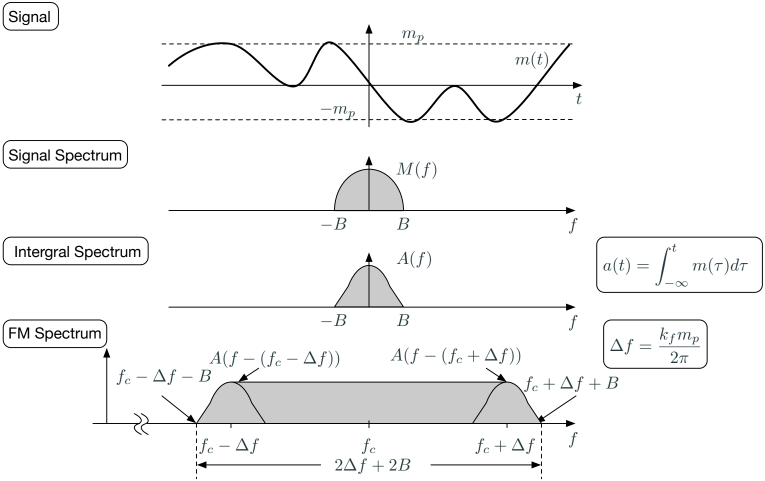
If $m(t)$ has a bandwidth $2B$ Hz, then $a(t)$ also has a bandwidth of $2B$ Hz.

What is the bandwidth of $\phi_{FM}(t)$? This is a difficult question in general. There are explicit solutions for only a few signals, such as sinusoids.

In practice, there are two contributors to the bandwidth

- Signal bandwidth $2B$
- FM deviation frequency $\Delta f = \frac{k_f m_p}{2\pi}$

This leads to Carson's rule.



Carson's Rule for the FM bandwidth is then

$$B_{FM} = 2\Delta f + 2B$$

where B_{FM} is the total signal bandwidth (not the half bandwidth).

Frequency Modulation of Tone

Spectral analysis of FM is difficult/impossible for general signals.

The special case of a sinusoidal input $m(t) = \cos(2\pi f_m t)$ is tractable. In this case $B_m = f_m$, and

$$a(t) = \int_{-\infty}^{t} m(u)\ du = \frac{1}{2\pi f_m} \sin(2\pi f_m t)$$

assuming $a(-\infty) = 0$. Then

$$\hat{\phi}_{FM} = Ae^{j2\pi f_c t}\, e^{j\beta \sin 2\pi f_m t}$$

where $\beta = k_f \,/\, 2\pi f_m$ is frequency deviation ratio (also called FM modulation index).

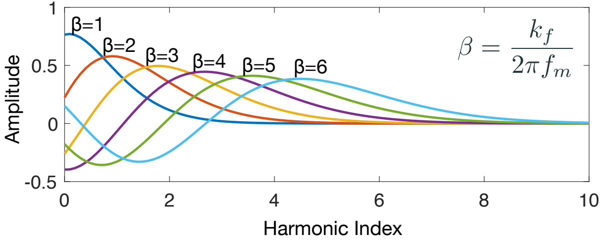
Since $e^{j\beta\sin(2\pi f_m t)}$ is periodic with a fundamental frequency f_m we can compute it's Fourier series as

$$e^{j\beta\sin(2\pi f_m t)} = \sum_n J_n(\beta) e^{jn2\pi f_m t}$$

where the coefficients are

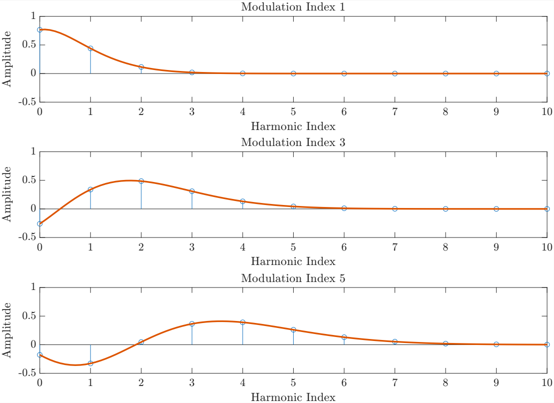
$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin t - nt)}\, dt$$

Each frequency component n has a coefficient that is $J_n(\beta)$, where n is the order of the Bessel function, and β is the frequency deviation ratio. $J_n(\beta)$ is negligible if $n > \beta + 1$.



FM Harmonic Amplitudes

The spectrum is the discrete values of the bessel function, sampled at the harmonic index. This looks like



The harmonics fall off above the modulation index value.

US Broadcast FM

- Frequency range: $88.0 - 108.0$ MHz
- Channel width: 200 KHz (100 channels)
- Channel center frequencies: $88.1, 88.3, \ldots, 107.9$
- Frequency deviation: ± 75 KHz
- Signal bandwidth: high-fidelity audio requires ± 20 KHz, so bandwidth is available for other applications:
    - Muzak (elevator music) (1936)
    - Stock market quotations
    - Interactive games
- Stereo uses sum and difference of L/R audio channels

FM radio was assigned the 42–50 MHz band of the spectrum in 1940. In 1945, at the behest of RCA (David Sarnoff CEO), the FCC moved FM to 88–108 MHz, obsoleting all existing receivers.

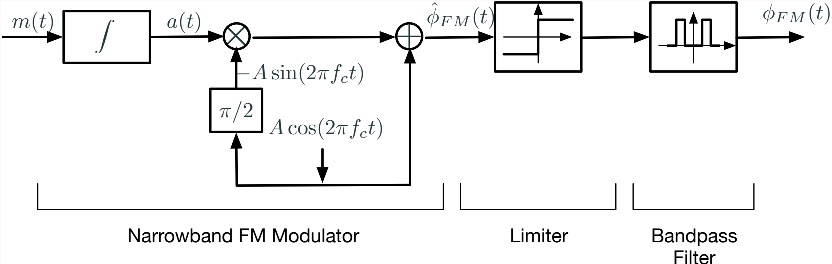
NBFM Modulation

For narrowband signals, $|k_f a(t)| \ll 1$ and $|k_p m(t)| \ll 1$,

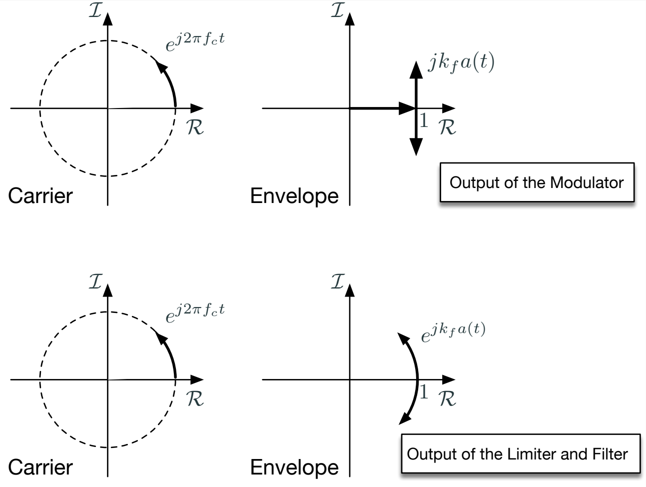
$$\phi_{NBFM} \approx A(\cos(2\pi f_c t) - k_f a(t)\sin(2\pi f_c t))$$

We can use a DSB-SC modulator with a phase shifter. In practice, this modulation will not be perfect, and there will be some amplitude modulation remaining.

To fix this up, follow with a limiter and a bandpass filter. For the case of FM,



NBFM Envelope and Carrier

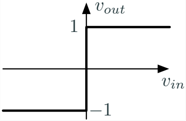


NBFM: Bandpass Limiter

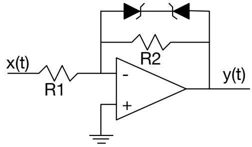
The input-output diagram for an ideal hard limiter is

$$v_o(t) = \begin{cases} +1 & v_i(t) > 0 \\ -1 & v_i(t) < 0 \end{cases}$$

This is a signum function, the output of a comparison against 0.



A hard limiter can be implemented by an op amp inverting amplifier, with back-to-back zener diodes to limit the output amplitude.



Input to bandpass limiter is

$$v_i(t) = A(t)\cos\theta(t)\,,\ \text{where } \theta(t) = 2\pi f_c t + k_f a(t)$$

Ideally, $A(t)$ is constant, but it may vary slowly. We assume $A(t) > 0$. The input to the bandpass filter is

$$v_o(\theta) = \begin{cases} +1 & \cos\theta > 0 \\ -1 & \cos\theta < 0 \end{cases}$$

which is periodic in θ with period 2π. Its Fourier series is

$$\begin{aligned} v_o(\theta) &= \frac{4}{\pi}\left(\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta + \cdots\right) \\ &= \frac{4}{\pi}\Big(\cos\Big(2\pi f_c t + k_f a(t)\Big) - \tfrac{1}{3}\cos 3\Big(2\pi f_c t + k_f a(t)\Big) + \cdots\Big) \end{aligned}$$

The bandpass filter eliminates all but the first term.

Note that the angle modulation for the third term is three times greater. We'll return to this next time.

WBFM Modulation: Direct Generation Using VCO

A voltage controlled oscillator generates a signal whose instantaneous frequency proportional to an input $m(t)$:

$$f_i(t) = f_c + \frac{k_f}{2\pi} m(t)$$

The signal with frequency $f_i(t)$ is bandpass filtered, then used in a modulator.

VCO can be constructed by using input voltage to control one or more circuit parameters in an oscillator

One example is using a reverse-biased diode as a variable capacitor in the tank circuit of an oscillator.

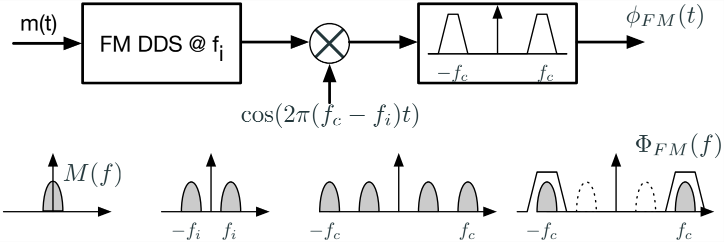
- Input voltage is the message signal, and it modulates the reverse bias potential
- This changes the capacitance, and hence the frequency of the oscillator

FM Direct Digital Synthesis

Currently, it is very common to synthesize the FM waveform digitally either at an intermediate frequency f_i and then mix it up to the desired carrier, or to directly synthesize the waveform at the carrier frequency.

This is Direct Digital Synthesis, or DDS.

Typical intermediate frequencies are a few MHz, so that the calculations are accurate but manageable, and the undesired sidebands can be suppressed easily.



FM Demodulation

There are many ways to demodulation the WBFM signal. Most use differentiation one way or another.

Differentiator

Slope detection

Frequency-selective filter

- RC high-pass filter
$$H(f) = \frac{j2\pi RCf}{1 + j2\pi RCf} \approx j2\pi RCf \quad (2\pi RC \ll 1)$$
- RLC circuit with carrier frequency $\omega_c < \omega_0 = 1/\sqrt{LC}$

Zero-crossing detectors

Phase-locked loop

Derivative Theorem for Fourier Transform

If $G(f)$ is the Fourier transform of $g(t)$, then

$$\frac{dg(t)}{dt} \leftrightharpoons j2\pi f\, G(f)$$

and

$$\int_{-\infty}^{t} g(\tau)\, d\tau \leftrightharpoons \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$$

We can show this by differentiating the inverse transform

$$\begin{aligned} \frac{dg(t)}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty} G(f)e^{j2\pi f t}dt \\ &= \int_{-\infty}^{\infty} \frac{d}{dt}G(f)e^{j2\pi f t}dt = \int_{-\infty}^{\infty} j2\pi f\, G(f)\, e^{j2\pi f t}dt \end{aligned}$$

By the Fourier inversion theorem, $j2\pi f\, G(f)$ is transform of $g'(t)$.

FM Detection by Differentiation

The complex FM signal is

$$\begin{aligned}\hat{\phi}_{FM}(t) &= Ae^{j(2\pi f_c t + k_f a(t))} \\ &= Ae^{jk_f a(t)} e^{j2\pi f_c t}\end{aligned}$$

The transmitted signal is just the real part, $\phi_{FM}(t) = \Re\big(\hat{\phi}_{FM}(t)\big)$.

First we demodulate the signal to baseband, which eliminates the carrier

$$\hat{\phi}_{BB}(t) = Ae^{jk_f a(t)}$$

Then we differentiate

$$\begin{aligned}\dot{\hat{\phi}}_{BB}(t) &= \frac{d}{dt}\left(Ae^{j(k_f a(t))}\right) \\ &= A(jk_f m(t))e^{j(k_f a(t))}\end{aligned}$$

Note that for this to work A must really be constant.

Extracting the Message with and SDR

We can then extract the FM signal by cancelling out the phase with the original signal

$$\begin{aligned}\dot{\hat{\varphi}}_{BB}(t)\hat{\varphi}^*_{BB}(t) &= A(jk_f m(t))e^{j(k_f a(t))}Ae^{-j(k_f a(t))} \\ &= A^2(jk_f m(t))\end{aligned}$$

This is the message with a scale factor A^2jk_f.

A simpler way is to demodulate to just below the carrier,

- The frequency is always positive
- This is the same as an AM signal with a carrier
- Use an envelope detector

WBFM Envelope Detection

After demodulation to an intermediate freqeuncy f_i,

$$\hat{\varphi}_{IF}(t) = Ae^{jk_f a(t)} e^{j2\pi f_i t}$$

Then when we differentiate we get

$$\begin{aligned} \dot{\hat{\varphi}}_{IF}(t) &= \frac{d}{dt}\left(Ae^{j(k_f a(t))} e^{j2\pi f_i t}\right) \\ &= Aj(2\pi f_i + k_f m(t)) e^{j(k_f a(t))} e^{j2\pi f_i t} \end{aligned}$$

As long as $2\pi f_i > k_f m(t))$ the envelope will always be positive, and we can just take the magnitude of the signal, just like we did with AM.

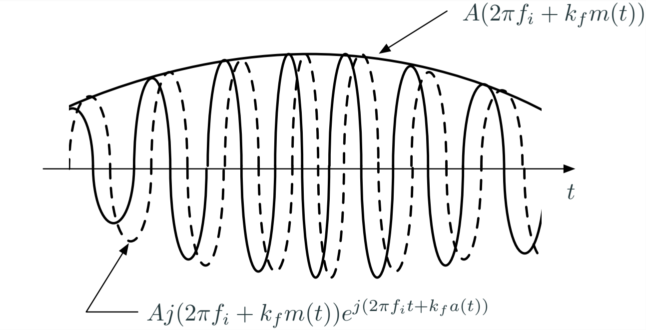
Then

$$\left|\dot{\hat{\varphi}}_{IF}(t)\right| = A(2\pi f_i + k_f m(t))$$

The DC term due to f_i is then filtered out by a highpass filter.

This is a more common way to demodulate WBFM.

FM Demodulator and Differentiator

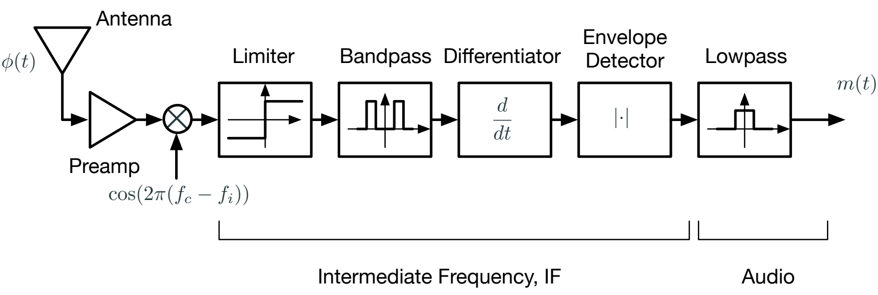


The envelope increases with frequency, and is always positive.

The same envelope detector that we used for AM will work here.

WBFM Envelope Detection

- The block diagram now looks like



- By demodulating to an IF, we can use an envelope detector

Advantages of FM

FM is less susceptible to amplifier nonlinearities. If input is

$$x(t) = A\cos(\omega_c t + \psi(t))$$

and the output is

$$\begin{aligned} y(t) &= a_0 + a_1 x(t) + a_2 x^2(t) + \cdots \\ &= c_0 + c_1 \cos(\omega_c t + \psi(t)) + c_2 \cos(2\omega_c t + 2\psi(t)) + \cdots \end{aligned}$$

The extra terms have spectrum outside the carrier signal band. They will be blocked by bandpass filter.

Nonlinearities in AM cause signal distortion. For $y(t) = ax(t) + bx^3(t)$,

$$\begin{aligned} y(t) &= am(t)\cos\omega_c t + bm^3(t)\cos^3\omega_c t \\ &= (am(t) + \frac{3}{4}bm^3(t))\cos\omega_c t + \frac{1}{4}b\cos 3\omega_c t \end{aligned}$$

FM is preferred for high power applications, such as microwave relay towers.

FM can adjust to rapid fading (change of amplitude) using AGC (automatic gain control)

FM is less vulnerable to signal interference from adjacent channels. Suppose interference is $I\cos(\omega_c+\omega)t$. Then received signal is

$$\begin{aligned} r(t) &= A\cos\omega_c t + I\cos(\omega_c t+\omega)t \\ &= (A + I\cos\omega t)\cos\omega_c t - I\sin\omega t\sin\omega_c t \\ &= E_r(t)\cos(\omega_c t + \psi(t)) \end{aligned}$$

where

$$\psi(t) = \tan^{-1}\left(\frac{I\sin\omega t}{A + I\cos\omega t}\right) \approx \frac{I}{A}\sin\omega t \quad (I \ll A)$$

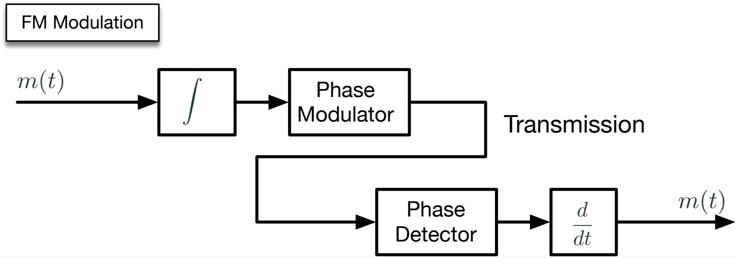
The output of an ideal frequency modulator is $\dot{\psi}(t)$ for FM is

$$y_d(t) = \frac{I\omega}{A}\cos\omega t\,,$$

which is inversely proportional to amplitude A.

Noise and FM

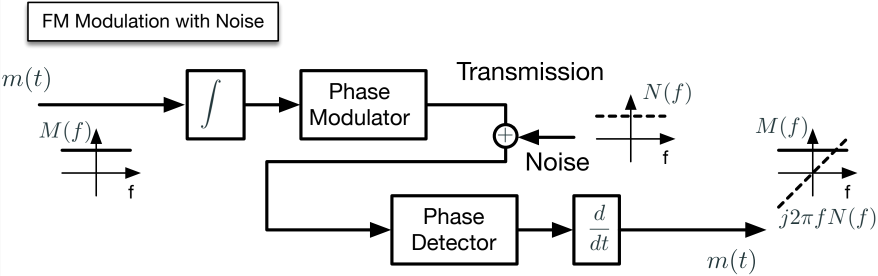
- The FM modulator and demodulator looks like



- The integration and differentiation operations are inverses
- The issue is what happens when noise is added during transmission.

Differentiation Accentuates Noise

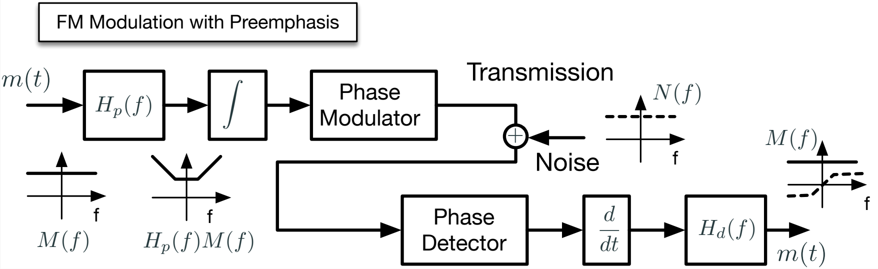
- Noise has a flat spectrum, and is added during the transmission



- After the differentiator, noise has a highpass characteristic, while the message is flat spectrally
- We would like the signal to noise ratio (SNR) to be flat

Preemphasis

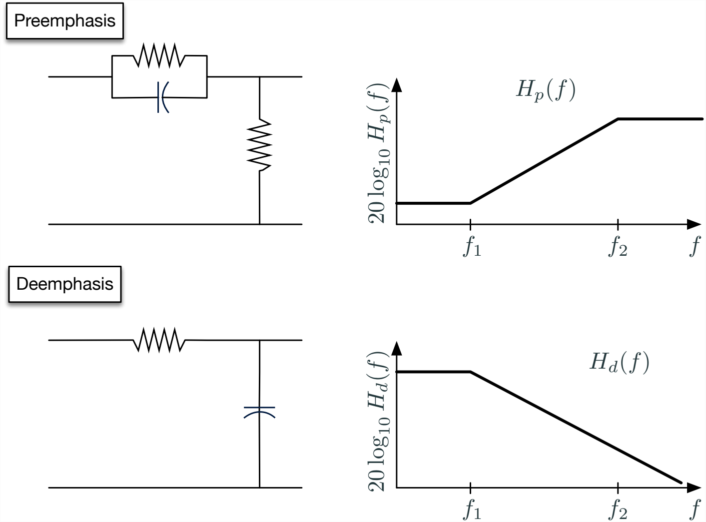
- Solution is to add an preemphasis filter $H_p(f)$ to amplify the high frequency signal



- A deemphasis filter $H_d(f)$ compensates the spectrum so that $H_p(f)H_d(f) = 1$.
- The noise and signal spectra are now flat at the output.
- The preemphasis is approximately a differentiator! Broadcast FM is

FM Preemphasis and Deemphasis

Pre-emphasis: RLC high pass filter. De-emphasis: RC low pass filter.



The linear preemphasis range is $f_1 = 2.1$ kHz to $f_2 = 30$ kHz. The preemphasis filter has transfer function

$$H_p(f) = \frac{f_2}{f_1} \frac{f_1 + j2\pi f}{f_2 + j2\pi f}$$

If $f \ll f_1$ then $H_p(f) \approx 1$.

If $f_1 \ll f \ll f_2$ then

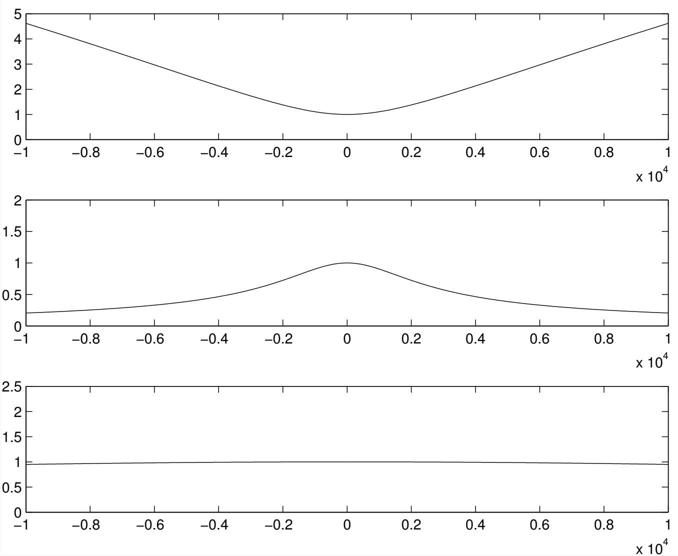
$$H_p(f) \approx \frac{j2\pi f}{f_1}$$

which is a differentiator!

The corresponding deemphasis filter has transfer function.

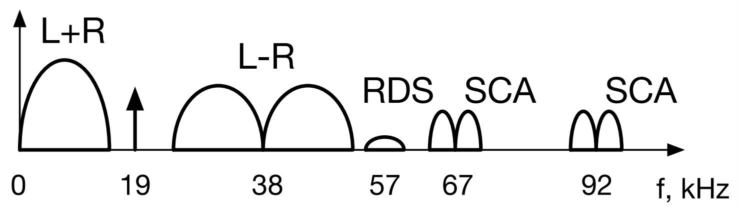
$$H_d(f) = \frac{f_1}{j2\pi f + f_1} \approx \frac{1}{H_p(f)}$$

FM Preemphasis and Deemphasis Filters



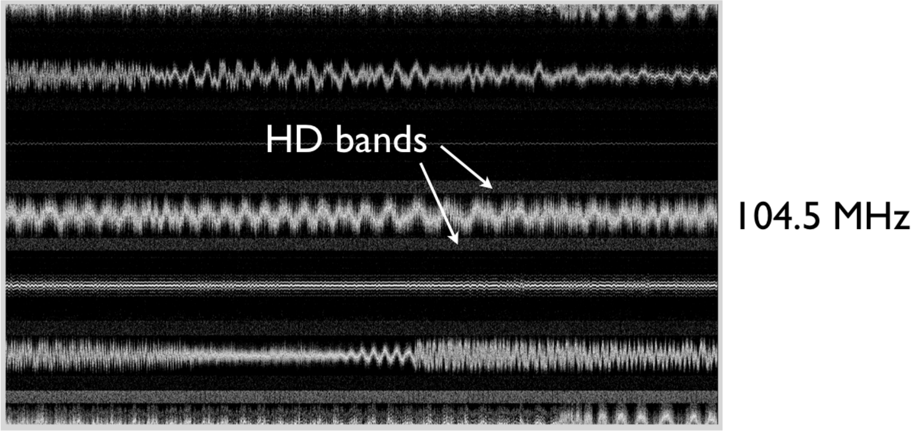
Broadcast FM Signal

- So far we've just been talking about WBFM. The broadcast signal has a lot more in it than the single channel we have so far described.
- Broadcast FM signal is stereo, and has many components
  - Left+Right in the middle of the spectrum
  - Left-Right offset by 38 kHz
  - 19 kHz pilot tone
  - Radio Data System (RDS) digital signal at 57 kHz
  - Two Subsidiary Communications Authorization (SCA) signals at 67 and 92 kHz
- It may also have up to four High Definition (HD) bands



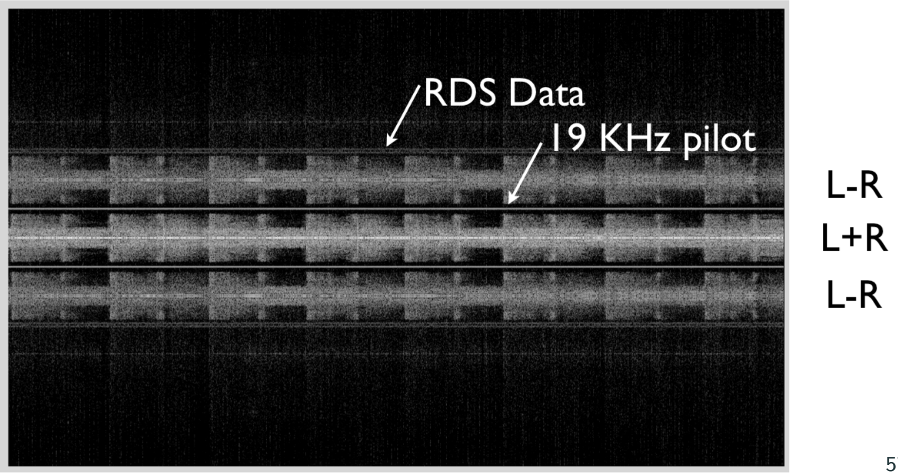
Broadcast FM Signal Spectrogram

This is 2 MHz of the broadcast FM band centered at 104.5 MHz.



Broadcast FM Signal Demodulated

After decimating and demodulating the previous signal, we get the demodulated FM signal.



Next time

- Digital Communications