Optimal Sampling-Based Motion Planning under Differential Constraints: the Drift Case with Linear Affine Dynamics

Edward Schmerling, Lucas Janson, Marco Pavone

Abstract— **In this paper we provide a thorough, rigorous theoretical framework to assess optimality guarantees of sampling-based algorithms for drift control systems: systems that, loosely speaking, can not stop instantaneously due to momentum. We exploit this framework to design and analyze a sampling-based algorithm (the Differential Fast Marching Tree algorithm) that is asymptotically optimal, that is, it is guaranteed to converge, as the number of samples increases, to an optimal solution. In addition, our approach allows us to provide concrete bounds on the rate of this convergence. The focus of this paper is on mixed time/control energy cost functions and on linear affine dynamical systems, which encompass a range of models of interest to applications (e.g., double-integrators) and represent a necessary step to design, via successive linearization, sampling-based and provably-correct algorithms for non-linear drift control systems. Our analysis relies on an original perturbation analysis for two-point boundary value problems, which could be of independent interest.**

I. Introduction

A key problem in robotics is how to compute an obstacle-free and dynamically-feasible trajectory that a robot can execute [1]. The problem, in the simplest setting where the robot does not have kinematic/dynamical (in short, differential) constraints on its motion and the problem becomes one of finding an obstacle-free "geometric" path, is reasonably well-understood and sound algorithms exist for most practical scenarios. However, robotic systems *do* have differential constraints (e.g., momentum), which most often cannot be neglected. Despite the long history of robotic motion planning, the inclusion of differential constraints in the planning process is currently considered an open challenge [2], in particular with respect to guarantees on the quality of the obtained solution and class of dynamical systems that can be addressed. Arguably, the most common approach in this regard is a decoupling approach, whereby the problem is decomposed in steps of computing a collision-free path (neglecting the differential constraints), smoothing the path to satisfy the motion constraints, and finally reparameterizing the trajectory so that the robot can execute it [2]. This approach, while oftentimes fairly computationally efficient, presents a number of disadvantages, including computation of trajectories whose cost (e.g., length or control effort) is far from the theoretical optimum or even failure in finding any solution trajectory due to the decoupling scheme itself [2]. For these reasons, it has been advocated that there is a need for planning algorithms that solve the differentially-constrained motion planning problem (henceforth referred to as the DMP problem) *in one shot*, i.e., without decoupling.

Broadly speaking, the DMP problem can be divided into two categories: (i) DMP for driftless systems, and (ii) DMP for drift systems. Intuitively, systems with drift constraints are systems where from some states it is impossible to stop instantaneously (this is typically due to momentum). More rigorously, a system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$ is a drift system if for some state $\mathbf{x}$ there does not exist any admissible control $\mathbf{u}$ such that $f(\mathbf{x}, \mathbf{u}) = 0$ [1]. For example the basic, yet representative, double integrator system $\ddot{\mathbf{x}} = \mathbf{u}$ (modeling the motion of a point mass under controlled acceleration) is a drift system. From a planning perspective, DMP for drift systems is notoriously more challenging than its driftless counterpart, due, for example, to the inherent lack of symmetries in the dynamics and the presence of regions of inevitable collision (that is, sets of states from which obstacle collision will eventually occur, regardless of applied controls) [1].

To date, the state of the art for one-shot solutions to the DMP problem (both for driftless and drift systems) is represented by sampling-based techniques, whereby an explicit construction of the configuration space is avoided and the configuration space is probabilistically "probed" with a sampling scheme [1]. Arguably, the most successful algorithm for DMP to date is the rapidly-exploring random tree algorithm (RRT) [3], which incrementally builds a tree of trajectories by randomly sampling points in the configuration space. However, the RRT algorithm lacks optimality guarantees, in the sense that one can prove that the cost of the solution returned by RRT converges to a suboptimal cost as the number of sampled points goes to infinity, almost surely [4]. An asymptotically-optimal version of RRT for the geometric (i.e., without differential constraints) case has been recently presented in [4]. This version, named RRT*, essentially adds a rewiring stage to the RRT algorithm to counteract its greediness in exploring the configuration space. Prompted by this result, a number of works have proposed extensions of RRT* to the DMP problem [5, 6, 7, 8, 9], with the goal of retaining the asymptotic optimality property of RRT*. Care must be taken in arguing optimality for drift systems in particular, as the control asymmetry requires a consideration of both forward-reachable and backward-reachable trajectory approximations. Even in the driftless case, the matter of assessing optimality is quite subtle, and hinges upon a careful characterization of a system's locally reachable sets in order to ensure that a planning algorithm examines "enough volume" in its operation, and thus enough sample points, to ensure asymptotic optimality [10]. Another approach to asymptotically optimal DMP planning is given by STABLE SPARSE RRT which achieves optimality through random control propagation instead of connecting sampled points using a steering subroutine [11]. This paper, like the RRT* variations, is based on a steering function,

Edward Schmerling is with the Institute for Computational & Mathematical Engineering, Stanford University, Stanford, CA 94305, schmrlng@stanford.edu.

Lucas Janson is with the Department of Statistics, Stanford University, Stanford, CA 94305, ljanson@stanford.edu.

Marco Pavone is with the Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, pavone@stanford.edu.

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although it may be considered less general, as it is our view that leveraging as much knowledge as possible of the differential constraints while planning is necessary for the goal of planning in real-time. In our related work [10] we provide a theoretical framework to study optimality guarantees of sampling-based algorithms for the DMP problem by focusing on *driftless* control-affine dynamical systems of the form $\dot{\mathbf{x}}(t) = \sum_{i=1}^{m} g_i(\mathbf{x}(t))\mathbf{u}_i(t)$. While this model is representative for a large class of robotic systems (e.g., mobile robots with wheels that roll without slipping and multi-fingered robotic hands), it is of limited applicability in problems where momentum (i.e., drift) is a key feature of the problem setup (e.g., for a spacecraft or a helicopter).

Statement of Contributions: The objective of this paper is to provide a theoretical framework to study optimality guarantees of sampling-based algorithms for the DMP problem with *drift*. Specifically, as in [9], we focus on linear affine systems of the form,

$$\dot{\mathbf{x}}[t] = A\mathbf{x}[t] + B\mathbf{u}[t] + \mathbf{c}, \quad \mathbf{x}[t] \in \mathcal{M},\, \mathbf{u}[t] \in \mathcal{U},$$

where $\mathcal{M}$ and $\mathcal{U}$ are the configuration and control spaces, respectively, and it is of interest to find an obstacle-free trajectory π that minimizes the mixed time/energy criterion

$$c[\pi] = \int_0^T \left(1 + \mathbf{u}[t]^T R\mathbf{u}[t]\right) \mathrm{d}t,$$

where R is a positive definite matrix that weights control energy expenditure versus traversal time. Henceforth, we will refer to a DMP problem involving linear affine dynamics and a mixed time/energy cost criterion as Linear Quadratic DMP (LQDMP). The LQDMP problem is relevant to applications for two main reasons: (i) it models the "essential" features of a number of robotic systems (e.g., spacecraft in deep space, helicopters, or even ground vehicles), and (ii) its theoretical study forms the backbone for sampling-based approaches that rely on linearization of more complex underlying dynamics. From a theoretical and algorithmic standpoint, the LQDMP problem presents two challenging features: (i) dynamics are not symmetric [1], which makes forward and backward reachable sets different and requires a more sophisticated analysis of sampling volumes to prove asymptotic optimality, and (ii) not all directions of motion are equivalent, in the sense that some motions incur dramatically higher cost than others due to the algebraic structure of the constraints. Indeed, these are the very same challenges that make the DMP problem with drift difficult in the first place, and they make approximation arguments (e.g., those needed to prove asymptotic optimality) more involved. Fortunately, for LQDMP an explicit characterization for the optimal trajectory connecting two sampled points in the absence of obstacles is available, which provides a foothold to begin the analysis. Specifically, the contribution of this paper is threefold. First, we show that *any* trajectory in an LQDMP problem may be "traced" arbitrarily well, with high probability, by connecting randomly distributed points from a sufficiently large sample set covering the configuration space. We will refer to this property as *probabilistic exhaustivity*, as opposed to probabilistic completeness [1], where the requirement is that *at least* one trajectory is traced with a sufficiently large sample set. Second, we introduce a sampling-based algorithm for solving the LQDMP problem, namely the Differential Fast Marching Tree algorithm (DFMT*), whose design is enabled by our analysis of the notion of probabilistic exhaustivity. In particular, we are able to give a precise characterization of neighborhood radius, an important parameter for many asymptotically optimal motion planners, in contrast with previous work on LQDMP [9]. Third, by leveraging probabilistic exhaustivity, we show that DFMT* is asymptotically optimal. This analysis framework builds upon [10], and elements of our approach are inspired by [9].

Organization: This paper is structured as follows. In Section II we formally define the DMP problem we wish to solve. In Section III we review known results about the problem of optimally connecting fixed initial and terminal states under linear affine dynamics with a quadratic cost function. Furthermore, we provide some simple, yet novel (to the best of our knowledge) asymptotic bounds on the spectrum of the weighted controllability Gramian, which are instrumental to our analysis. In Section IV we prove the aforementioned probabilistic exhaustivity property for drift systems with linear affine dynamics. In Section V we present the DFMT* algorithm, and in Section VI we prove its asymptotic optimality (together with a convergence rate characterization). Finally, in Section VIII we discuss several features of our analysis, we draw some conclusions, and we discuss directions for future work.

II. Problem Formulation

In this section we state the problem we wish to solve. Let $\mathcal{M} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ be the configuration space and control space, respectively, of a robotic system. Within this space let us assume the dynamics of the robot are given by the linear affine system:

$$\dot{\mathbf{x}}[t] = A\mathbf{x}[t] + B\mathbf{u}[t] + \mathbf{c}, \quad \mathbf{x}[t] \in \mathcal{M},\, \mathbf{u}[t] \in \mathcal{U}, \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $\mathbf{c} \in \mathbb{R}^n$ are constants.

A tuple $\pi = (\mathbf{x}[], \mathbf{u}[], T)$ defines a *dynamically feasible* trajectory, alternatively path, if the state evolution $\mathbf{x} : [0,T] \to \mathcal{M}$ and control input $\mathbf{u} : [0,T] \to \mathcal{U}$ satisfy equation (1) for all $t \in [0,T]$. We define the cost of a trajectory π by the function:

$$c[\pi] = \int_0^T \left(1 + \mathbf{u}[t]^T R\mathbf{u}[t]\right) \mathrm{d}t \qquad (2)$$

where $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite, constant, and given. We may rewrite this cost function as $c[\pi] = T + c_u[\pi]$, where $c_u[\pi] = \int_0^T \mathbf{u}[t]^T R\mathbf{u}[t]\,\mathrm{d}t$, with the interpretation that this cost function penalizes both trajectory duration T and control effort c_u. The matrix R determines the relative costs of the control inputs, as well as their costs relative to the duration of the trajectory. We denote this linear affine dynamical system with cost by $\Sigma = (A, B, \mathbf{c}, R)$.

Let $\mathcal{M}_{\text{obs}} \subset \mathcal{M}$ be the obstacle region within the configuration space, such that $\mathcal{M} \setminus \mathcal{M}_{\text{obs}}$ is an open set, and denote the obstacle-free space as $\mathcal{M}_{\text{free}} = \mathcal{M} \setminus \mathcal{M}_{\text{obs}}$. The starting configuration $\mathbf{x}_{\text{init}}$ is an element of $\mathcal{M}_{\text{free}}$, and the goal region $\mathcal{M}_{\text{goal}}$ is an open subset of $\mathcal{M}_{\text{free}}$. The trajectory planning problem is denoted by the tuple $(\Sigma, \mathcal{M}_{\text{free}}, \mathbf{x}_{\text{init}}, \mathcal{M}_{\text{goal}})$. A dynamically feasible trajectory $\pi = (\mathbf{x}, \mathbf{u}, T)$ is *collision-free* if $\mathbf{x}[t] \in \mathcal{M}_{\text{free}}$ for all $t \in [0,T]$. A trajectory π is said to be *feasible* for the trajectory planning problem

$(\Sigma, \mathcal{M}_{\text{free}}, \mathbf{x}_{\text{init}}, \mathcal{M}_{\text{goal}})$ if it is dynamically feasible, collision-free, $\mathbf{x}[0] = \mathbf{x}_{\text{init}}$, and $\mathbf{x}[T] \in \mathcal{M}_{\text{goal}}$.

Let Π be the set of all feasible paths. The objective is to find the feasible path with minimum associated cost. The optimal trajectory planning problem is then defined as follows:

> **LQDMP problem**: Given a trajectory planning problem $(\Sigma, \mathcal{M}_{\text{free}}, \mathbf{x}_{\text{init}}, \mathcal{M}_{\text{goal}})$ with cost function $c : \Pi \to \mathbb{R}_{\geq 0}$ given by equation (2), find a feasible path π^* such that $c[\pi^*] = \min\{c[\pi] \mid \pi \text{ is feasible}\}$. If no such path exists, report failure.

Our analysis will rely on two key sets of assumptions, relating, respectively, to the system Σ and the problem-specific parameters $\mathcal{M}_{\text{free}}, \mathbf{x}_{\text{init}}, \mathcal{M}_{\text{goal}}$.

Assumptions on system: We assume that the system Σ is controllable, (i.e., the pair (A, B) is controllable) [12] so that there even exist dynamically feasible trajectories between states. Also, we assume that the control space is unconstrained, i.e. $\mathcal{U} = \mathbb{R}^m$, and that the cost weight matrix R is symmetric positive definite, so that every control direction has positive cost. These assumptions will be collectively referred to as A_Σ.

Assumptions on problem parameters: We require that the configuration space is a compact subset of $\mathbb{R}^m$. Furthermore, we require that the goal region $\mathcal{M}_{\text{goal}}$ has *regular boundary*, that is there exists $\xi > 0$ such that $\forall y \in \partial\mathcal{M}_{\text{goal}}$, there exists $z \in \mathcal{M}_{\text{goal}}$ with $B[z, \xi] \subseteq \mathcal{M}_{\text{goal}}$ and $y \in \partial B[z, \xi]$, where B denotes the Euclidean 2-norm ball. This requirement that the boundary of the goal region has bounded curvature ensures that a point sampling procedure may expect to select points in the goal region near any point on the region's boundary. Define a trajectory π to be *piecewise optimal* if it is the concatenation of optimal trajectories between successive states $\mathbf{x}_0 = \mathbf{x}[0], \mathbf{x}_1, \ldots, \mathbf{x}_J = \mathbf{x}[T] \in \mathcal{M}_{\text{free}}$ (we call these states the *pivot nodes* of π). We assume that the optimum for the LQDMP problem $(\Sigma, \mathcal{M}_{\text{free}}, \mathbf{x}_{\text{init}}, \mathcal{M}_{\text{goal}})$ is piecewise optimal, an assumption also made in [9]. This ensures, in particular, that the optimal control policy is piecewise continuous. We also make requirements on the *clearance* of the optimal trajectory, i.e., its "distance" from $\mathcal{M}_{\text{obs}}$ [10]. For a given $\delta > 0$, the δ-interior of $\mathcal{M}_{\text{free}}$ is defined as the set of all states that are at least a Euclidean distance δ away from any point in $\mathcal{M}_{\text{obs}}$. A collision-free path π is said to have strong δ-clearance if its state trajectory $\mathbf{x}$ lies entirely inside the δ-interior of $\mathcal{M}_{\text{free}}$. A collision-free path π is said to have weak δ-clearance if there exists a path π' that has strong δ-clearance and there exists a homotopy ψ, with $\psi[0] = \pi$ and $\psi[1] = \pi'$ that satisfies the following three properties: (a) $\psi[\alpha]$ is a dynamically feasible, piecewise-optimal trajectory for all $\alpha \in (0, 1]$, (b) $\lim_{\alpha \to 0} c[\psi[\alpha]] = c[\pi]$, and (c) for all $\alpha \in (0, 1]$ there exists $\delta_\alpha > 0$ such that $\psi[\alpha]$ has strong δ_α-clearance. Properties (a) and (b) are required since pathological obstacle sets may be constructed that squeeze all optimum-approximating homotopies into undesirable motion. In practice, however, as long as $\mathcal{M}_{\text{free}}$ does not contain any passages of infinitesimal width, the fact that Σ is controllable will allow every trajectory to be weak δ-clear. We claim that the assumptions about the problem parameters are very mild (they can be regarded as "minimum" regularity assumptions).

All trajectories discussed in this paper are dynamically feasible unless otherwise noted. The symbol $\|\cdot\|$ denotes the 2-norm, induced or otherwise, and $\sigma_{\min}$ denotes the minimum singular value of a matrix.

III. Optimal Control in the Absence of Obstacles

The goal of this section is twofold: to review results about two-point boundary value problems for linear affine systems, and to present simple, yet novel asymptotic bounds on the spectrum of the controllability Gramian. Both results will be instrumental to our analysis of LQDMP.

A. Fixed-Time 2BVP

The material in this section is standard, we provide it to make the paper self-contained. Our presentation follows the treatment in [13, 9]. Specifically, this section is concerned with local steering between states in the absence of environment boundaries and obstacles. Given a start state $\mathbf{x}_0 \in \mathcal{M}$, an end state $x_1 \in \mathcal{M}$, and a travel time $\tau > 0$, the *fixed-time two point boundary value problem* (fixed-time 2BVP) is to find a trajectory between $\mathbf{x}[0] = \mathbf{x}_0$ and $\mathbf{x}[\tau] = \mathbf{x}_1$ that satisfies the system Σ and minimizes its cost function (2). Denote this trajectory and its cost as $\pi^*_\tau[\mathbf{x}_0, \mathbf{x}_1]$ and $c^*_\tau[\mathbf{x}_0, \mathbf{x}_1]$ respectively: $\pi^*_\tau[\mathbf{x}_0, \mathbf{x}_1] = \operatorname{argmin}\{\pi = (\mathbf{x}, \mathbf{u}, T) \mid \mathbf{x}[0] = \mathbf{x}_0 \wedge \mathbf{x}[T] = \mathbf{x}_1 \wedge T = \tau\}\ c(\pi)$, and $c^*_\tau[\mathbf{x}_0, \mathbf{x}_1] = \min\{\pi = (\mathbf{x}, \mathbf{u}, T) \mid \mathbf{x}[0] = \mathbf{x}_0 \wedge \mathbf{x}[T] = \mathbf{x}_1 \wedge T = \tau\}\ c(\pi)$.

Let us define the weighted controllability Gramian $G[t]$ as the solution of the Lyapunov equation:

$$\dot{G}[t] = AG[t] + G[t]A^T + BR^{-1}B^T, \quad G[0] = 0$$

which has the closed form expression:

$$G[t] = \int_0^t \exp[At']BR^{-1}B^T \exp[A^T t']\,\mathrm{d}t'. \tag{3}$$

Under the assumptions A_Σ (in particular, system (1) is controllable), we have that $G[t]$ is symmetric positive definite for all $t > 0$. This fact allows us to define the weighted norm $\|\cdot\|_{G^{-1}}$ for $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_{G[t]^{-1}} = \sqrt{\mathbf{x}^T G[t]^{-1} \mathbf{x}}.$$

Let $\bar{\mathbf{x}}[t]$ be the zero input response of system (1), that is the solution of the differential equation:

$$\dot{\bar{\mathbf{x}}}[t] = A\bar{\mathbf{x}}[t] + \mathbf{c}, \quad \bar{\mathbf{x}}[0] = \mathbf{x}_0,$$

which has the closed form expression:

$$\bar{\mathbf{x}}[t] = \exp[At]\mathbf{x}_0 + \int_0^t \exp[A(t-s)]\mathbf{c}\,\mathrm{d}s. \tag{4}$$

Then the optimal control policy for the fixed-time 2BVP is given by [13]:

$$\mathbf{u}[t] = R^{-1}B^T \exp[A^T(\tau - t)]G[\tau]^{-1}(\mathbf{x}_1 - \bar{\mathbf{x}}[\tau]), \tag{5}$$

which corresponds to the minimal cost (as a function of travel time τ):

$$c[\tau] = \tau + \|\mathbf{x}_1 - \bar{\mathbf{x}}[\tau]\|^2_{G[\tau]^{-1}}. \tag{6}$$

The state trajectory $\mathbf{x}[t]$ that evolves from this control policy may be computed explicitly as:

$$\mathbf{x}[t] = \bar{\mathbf{x}}[t] + G[t]\exp[A^T(\tau - t)]G[\tau]^{-1}(\mathbf{x}_1 - \bar{\mathbf{x}}[\tau]). \tag{7}$$

Let $\pi^*_\tau[\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_K]$ denote the concatenation of the trajectories $\pi^*_\tau[\mathbf{x}_k, \mathbf{x}_{k+1}]$ between successive states $\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_K \in \mathcal{M}$ (so that the total path has travel time τK).

B. Asymptotic Bounds on the Spectrum of the Controllability Gramian

Our convergence results require an asymptotic (i.e., for small t) characterization of the spectrum of the controllability Gramian, or, in other words, of the small-time controllability ellipsoid. Let us define the *controllability index* of a controllable system as [1]: $s[A,B] = \min\left\{k \in \{0,\ldots,n-1\} \mid \text{rank}\left([B\ AB\ \cdots\ A^kB]\right) = n\right\}$ which is well-defined for a controllable pair (A,B) since $\text{rank}\left([B\ AB\ \cdots\ A^{n-1}B]\right) = n$. The following result characterizes the minimum eigenvalue of the controllability Gramian, i.e., $\lambda_{\min}[G[t]]$, as $t \to 0$.

Lemma III.1 (Small-time minimum eigenvalue of controllability Gramian)**.** *Assume that the pair (A,B) has controllability index s, then $\lambda_{\min}[G[t]] = \Omega(t^{2s+1})$ as $t \to 0$, or, equivalently, $\|G[t]^{-1}\| = O(t^{-2s-1})$.*

Proof. From equation (3) we have:

$$\begin{aligned}\lambda_{\min}[G[t]] &= \min_{\mathbf{x}\in\mathbb{R}^n, \|x\|=1} \mathbf{x}^T G[t]\mathbf{x} \\ &= \min_{\mathbf{x}\in\mathbb{R}^n, \|x\|=1} \int_0^t \mathbf{x}^T \exp[At']BR^{-1}B^T\exp[A^Tt']\mathbf{x}\,\mathrm{d}t' \\ &\geq \int_0^t \left(\min_{\mathbf{x}\in\mathbb{R}^n, \|x\|=1} \left\|\mathbf{x}^T\exp[At']BJ^{-T}\right\|^2\right)\mathrm{d}t',\end{aligned}$$

where $R = JJ^T$ is the Cholesky factorization of R. By representing the matrix exponential as a series, one can write:

$$\begin{aligned}\exp[At']B &= \sum_{j=0}^{s} \frac{t'^j}{j!}A^jB + \sum_{j=s+1}^{\infty}\frac{t'^j}{j!}A^jB \\ &= \sum_{j=0}^{s}\frac{t'^j}{j!}A^jB + O(t'^{s+1}).\end{aligned}$$

Note that $\sum_{j=0}^{s}\frac{t'^j}{j!}A^jB$ has full row rank; otherwise if $\mathbf{y}^T\left(\sum_{j=0}^{s}\frac{t'^j}{j!}A^jB\right) = 0$ for some $\mathbf{y} \neq 0$ then $\mathbf{y}^TA^jB = 0$ for all $0 \leq j \leq s$ (successively differentiate with respect to t' and plug in $t' = 0$), contradicting the definition of s. Then:

$$\begin{aligned}&\min_{\mathbf{x}\in\mathbb{R}^n, \|x\|=1}\left\|\mathbf{x}^T\exp[At']BJ^{-T}\right\| \\ &\geq \sigma_{\min}\left[\sum_{j=0}^{s}\frac{t'^j}{j!}A^jB\right]\lambda_{\min}[R^{-1}]^{1/2} - O(t'^{s+1}) = \Omega(t'^s),\end{aligned}$$

so that $\lambda_{\min}[G[t]] = \Omega(t^{2s+1})$. □

Lemma III.1 has two immediate corollaries. The first is a lower bound for the determinant of $G[t]$, a result that will prove useful for estimating the volumes of reachable sets; the second is a bound for the Cholesky factorization of the Gramian.

Lemma III.2 (Small-time determinant of controllability Gramian)**.** *Assume that the pair (A,B) has controllability index s, then $\det[G[t]] = \Omega\left(t^{(2s+1)n}\right)$ as $t \to 0$.*

Remark III.3. *This bound is quite conservative. For a system Σ, we define $D[A,B]$ as the least exponent satisfying $\det[G[t]] = \Omega(t^D)$. Clearly, $D \leq (2s+1)n$.*

[1]Note that the definition of controllability index given here is different (although related) from the one given in [12, Page 150]

Lemma III.4 (Small-time norm of Cholesky factor of controllability Gramian)**.** *Assume that the pair (A,B) has controllability index s, and consider the Cholesky factorization $G[t] = L[t]L[t]^T$. Then:*

$$\|L[t]^{-1}\| = O(t^{-s-1/2}) \quad \text{as } t \to 0.$$

Finally, we note that for any symmetric positive definite $G = LL^T$ and $\mathbf{x} \in \mathbb{R}^n$, we have the norm-equivalence bounds:

$$\frac{\|\mathbf{x}\|}{\|L\|} \leq \|\mathbf{x}\|_{G^{-1}} \leq \|L^{-1}\|\|\mathbf{x}\|. \tag{8}$$

IV. Probabilistic Exhaustivity

In this section we prove a key result characterizing random sampling schemes for the LQDMP problem: *any* feasible trajectory through the configuration space $\mathcal{M}$ is "traced" arbitrarily well by connecting randomly distributed points from a sufficiently large sample set covering the configuration space. We will refer to this property as probabilistic exhaustivity. The same notion of probabilistic exhaustivity (clearly much stronger than the usual notion of probabilistic completeness) is also introduced in our related paper [10] in the context of DMP for *driftless* systems. The result proven in that work does not carry over to the drift case as it relies on the metric inequality to bound the cost of approximate paths; the drift case lacks the control symmetry to make such estimates. Thus in order to prove probabilistic exhaustivity in the case of linear affine systems, we first provide a result analogous to the metric inequality characterizing the effect that perturbations of the endpoints of a path have on its cost and state trajectory. The idea, then, is that tracing waypoints may be selected as small perturbations of points along the trajectory to be approximated, provided the sample density is sufficiently high.

Lemma IV.1 (Fixed-Time Local Trajectory Approximation)**.** *Let $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{M}$, $\tau > 0$, and $\pi = \pi^*_\tau[\mathbf{x}_0, \mathbf{x}_1] = (\mathbf{x}, \mathbf{u}, \tau)$. Consider start and end state perturbations $\boldsymbol{\delta}_0, \boldsymbol{\delta}_1 \in \mathbb{R}^n$ and denote $d = \|\exp[A\tau]\boldsymbol{\delta}_0\|_{G[\tau]^{-1}} + \|\boldsymbol{\delta}_1\|_{G[\tau]^{-1}}$. Then the optimal fixed-time trajectory $\sigma = \pi^*_\tau[\mathbf{x}_0 + \boldsymbol{\delta}_0, \mathbf{x}_1 + \boldsymbol{\delta}_1] = (\mathbf{y}, \mathbf{v}, \tau)$ between $\mathbf{x}_0 + \boldsymbol{\delta}_0$ and $\mathbf{x}_1 + \boldsymbol{\delta}_1$ satisfies:*

$$\|\mathbf{y}[t] - \mathbf{x}[t]\| = O(\tau^{-s+1/2}d),$$

as $\tau \to 0$. Additionally we have the cost bound:

$$c[\sigma] \leq c[\pi]\left(1 + \frac{d}{2\tau}\left(d + \sqrt{d^2 + 4\tau}\right)\right).$$

Proof. Consider the optimal fixed-time trajectory $\rho = (\mathbf{z}, \mathbf{w}, \tau)$ connecting $\mathbf{z}[0] = \boldsymbol{\delta}_0$ and $z[\tau] = \boldsymbol{\delta}_1$ subject to the homogeneous system:

$$\dot{\mathbf{z}}[t] = A\mathbf{z}[t] + B\mathbf{w}[t]$$

with cost function consisting only of the control effort:

$$c_U[\rho] = \int_0^\tau \mathbf{w}[t]^T R\mathbf{w}[t]\,\mathrm{d}t.$$

We may solve for ρ as a special case of the results presented in Section III. In this case $\bar{\mathbf{z}}[t] = \exp[At]\boldsymbol{\delta}_0$, and therefore $c_U[\rho] = \|\boldsymbol{\delta}_1 - \exp[A\tau]\boldsymbol{\delta}_0\|^2_{G[\tau]^{-1}}$. This cost may be bounded as:

$$c_U[\rho] \leq \left(\|\boldsymbol{\delta}_1\|_{G[\tau]^{-1}} + \left\|e^{A\tau}\boldsymbol{\delta}_0\right\|_{G[\tau]^{-1}}\right)^2 = d^2. \tag{9}$$

We also note the explicit form of the state trajectory:

$$\mathbf{z}[t] = e^{At}\boldsymbol{\delta}_0 + G[t]e^{A^T(\tau-t)}G[\tau]^{-1}(\boldsymbol{\delta}_1 - e^{At}\boldsymbol{\delta}_0),$$

so we may bound the extent of $\mathbf{z}$ as:

$$\begin{aligned}\|\mathbf{z}[t]\| &\le \|e^{At}\boldsymbol{\delta}_0\| + \|G[t]\|e^{\|A\|(\tau-t)}\|G[\tau]^{-1}(\boldsymbol{\delta}_1 - e^{At}\boldsymbol{\delta}_0)\| \\ &\le \|L[\tau]\| \left\|e^{A\tau}\boldsymbol{\delta}_0\right\|_{G[\tau]^{-1}} + \|G[\tau]\|e^{\|A\|\tau}\|L[\tau]^{-1}\|d \\ &= O(\tau^{-s+1/2}d) \quad \text{ as } \tau \to 0,\end{aligned} \tag{10}$$

since the second term dominates and $\|G[\tau]\| = O(\tau)$, $\exp[\|A\|\tau] \to 1$, and $\|L[\tau]^{-1}\| = O(\tau^{-s-1/2})$ as $\tau \to 0$ by Lemma III.4.

Now we note that the trajectory $\sigma = (\mathbf{y} = \mathbf{x} + \mathbf{z}, \mathbf{v} = \mathbf{u} + \mathbf{w}, \tau)$ satisfies the original system (1). In addition, we can see from equation (5) that $\mathbf{u} + \mathbf{w}$ is the optimal control policy for connecting the two states $\mathbf{x}_0 + \boldsymbol{\delta}_0$ and $\mathbf{x}_1 + \boldsymbol{\delta}_1$ in time τ. That is, the optimal trajectory for a perturbed fixed-time 2BVP is the sum of the unperturbed trajectory and the optimal trajectory spanning the perturbation. We bound its cost:

$$\begin{aligned}c[\rho] &= \int_0^\tau \left(1 + (\mathbf{u}[t] + \mathbf{v}[t])^T R(\mathbf{u}[t] + \mathbf{v}[t])\right) \mathrm{d}t \\ &= c[\pi] + c_U[\sigma] + 2\int_0^\tau \mathbf{u}[t]^T R\mathbf{v}[t]\,\mathrm{d}t \\ &\le c[\pi] + d^2 + 2c_U[\pi]^{1/2}d \\ &= c[\pi]\left(1 + \frac{d^2 + 2c_U[\pi]^{1/2}d}{\tau + c_U[\pi]}\right),\end{aligned}$$

where in the second line we have applied the Cauchy-Schwarz inequality. This bound depends on the breakdown of $c[\pi]$ between its time penalty component τ and control effort component $c_U[\pi]$. We may apply elementary calculus to compute

$$\max_{c>0}\left(\frac{d^2 + 2c^{1/2}d}{\tau + c}\right) = \frac{d}{2\tau}\left(d + \sqrt{d^2 + 4\tau}\right),$$

and therefore we may express a bound irrespective of the cost breakdown:

$$c[\rho] \le c[\pi]\left(1 + \frac{d}{2\tau}\left(d + \sqrt{d^2 + 4\tau}\right)\right).$$

□

Remark IV.2. *For $d = \eta\sqrt{\tau}$ such that $\eta \le 2$, the cost bound reduces to:*

$$c[\rho] \le c[\pi]\left(1 + \eta\left(\eta + \sqrt{\eta^2 + 4}\right)/2\right) \le c[\pi](1 + 3\eta).$$

Motivated by Lemma IV.1, we define the fixed-time perturbation "ball":

$$\Delta[\mathbf{x}, \tau, r] = \left\{\mathbf{z} \mid \|\mathbf{x} - \mathbf{z}\|_{G[\tau]^{-1}} \le r \wedge \left\|e^{A\tau}(\mathbf{x} - \mathbf{z})\right\|_{G[\tau]^{-1}} \le r\right\}.$$

This set represents perturbations of $\mathbf{x}$ with limited effects on both incoming and outgoing trajectories (depending on whether a point is viewed as an end state or start state perturbation respectively). In order to expect the sample points of a motion planning algorithm to lie within $\Delta[\mathbf{x}, \tau, r]$, we must lower bound its volume.

Lemma IV.3 (Lower Bound on Volume of Perturbation Ball)**.** *Let Σ be a controllable system. Then there exists a threshold $\tau_{vol} > 0$ such that for all $0 < \tau \le \tau_{vol}$, $\mathbf{x} \in \mathcal{M}$, and $r > 0$ the volume of $\Delta[\mathbf{x}, \tau, r]$ is lower-bounded as:*

$$\mu[\Delta[\mathbf{x}, \tau, r]] \ge (\zeta_n/2)\det[L[\tau]]r^n \ge (\zeta_n/2)C_{\det}\tau^{D/2}r^n,$$

where $C_{\det}$ is a constant determined by the asymptotics of $\det[G[\tau]] = \det[L[\tau]]^2$ in Lemma III.2.

Proof. Note that as $\tau \to 0$, $\exp[A\tau] = I + O(\tau)$. Then:

$$\|\exp[A\tau](\mathbf{x} - \mathbf{z})\|_{G[\tau]^{-1}} = \|(\mathbf{x} - \mathbf{z})\|_{G[\tau]^{-1}} + O(\tau^{-s+1/2}),$$

which implies that there exists $\tau_{\text{vol}} > 0$ such that for all $0 < \tau \le \tau_{\text{vol}}$, if $\|(\mathbf{x} - \mathbf{z})\|_{G[\tau]^{-1}} \le (1/\sqrt{2})^n r \le r$, then $\|\exp[A\tau](\mathbf{x} - \mathbf{z})\|_{G[\tau]^{-1}} \le r$ as well. That is, for all $\tau \le \tau_{\text{vol}}$:

$$\left\{\mathbf{z} \mid \|\mathbf{x} - \mathbf{z}\|_{G[\tau]^{-1}} \le (1/\sqrt{2})^n r\right\} \subset \Delta[\mathbf{x}, \tau, r].$$

To complete the proof, we note that the inequality $\|\mathbf{x} - \mathbf{z}\|_{G[\tau]^{-1}} \le (1/\sqrt{2})^n r$ defines an ellipsoid with volume $(\zeta_n/2)\det[L[\tau]]r^n$. □

We now give a definition of what it means for a series of states to closely approximate a given trajectory. Let $\pi = (\mathbf{x}, \mathbf{u}, T)$ be a dynamically feasible trajectory. Given a set of waypoints $\{\mathbf{y}_k\}_{k=0}^K \subset \mathcal{M}$, we associate the trajectory $\sigma = \pi^*_\tau[\mathbf{y}_0, \ldots, \mathbf{y}_K] = (\mathbf{y}, \mathbf{v}, K\tau)$. We consider the $\{\mathbf{y}_k\}$ to $(\tau, \varepsilon, d, p)$*-trace* the trajectory x if: a) the cost of σ is bounded as $c[\sigma] \le (1 + \varepsilon)c[\pi]$, b) $c^*_\tau[\mathbf{y}_k, \mathbf{y}_{k+1}] \le d$ for all k, and c) the maximum distance from any point of $\mathbf{y}$ to $\mathbf{x}$ is no more than p, i.e.

$$\max_{t\in[0,K\tau]}\left(\min_{t'\in[0,T]}\|\mathbf{y}[t] - \mathbf{x}[t']\|\right) \le p.$$

The combination of these three properties is what makes σ, if approximating a near-globally-optimal trajectory π, amenable to recovery by the path planning algorithms we propose in the next section. Additionally define the maximum instantaneous control effort over a piecewise-optimal path π:

$$M_U[\pi] = \max_{t\in[0,T]}\mathbf{u}[t]^T R\mathbf{u}[t].$$

This maximum is well defined since $\mathbf{u}$ is piecewise-continuous. In Theorem IV.7 we show that suitable waypoints may be found with high probability as a subset of a set of randomly sampled nodes, the proof of which requires the following three technical lemmas.

Lemma IV.4 (Waypoint Trajectory Approximation)**.** *Let $\pi = (\mathbf{x}, \mathbf{u}, T)$ be a piecewise-optimal trajectory and consider a fixed partitioning time $\tau < T$. Denote $\mathbf{x}[k\tau] = \mathbf{x}[k\tau]$ for each $k \in \{0, 1, \ldots, K = \lfloor T/\tau \rfloor\}$. Suppose that $\{\mathbf{y}_k\}_{k=1}^K \subset \mathcal{M}$ satisfy:*

(a) $\mathbf{y}_k \in \Delta[\mathbf{x}[k\tau], \tau, \sqrt{\tau}]$ for all $k \in \{0, 1, \ldots, K = \lfloor T/\tau \rfloor\}$,
(b) more than a $(1 - \alpha)$ fraction of the $\mathbf{y}_k$ satisfy $\mathbf{y}_k \in \Delta[\mathbf{x}[k\tau], \tau, \beta\sqrt{\tau}]$,

for fixed parameters $\alpha, \beta \in (0, 1)$. Then $c^*_\tau[\mathbf{y}_{k-1}, \mathbf{y}_k] \le 7(1 + M_U)\tau$ for all k. Additionally, the trajectory $\sigma = \pi^*_\tau[\mathbf{y}_0, \ldots, \mathbf{y}_K] = (\mathbf{y}, \mathbf{v}, K\tau)$ satisfies

$$c[\sigma] \le c[\pi]\left(1 + 6\left((1 + M_U)\alpha + \beta\right)\right).$$

Finally, the maximum distance deviation σ takes from π is $O(\tau^{-s})$, that is:

$$\max_{t\in[0,K\tau]}\left(\min_{t'\in[0,T]}\|\mathbf{y}[t] - \mathbf{x}[t']\|\right) = O((1 + \sqrt{M_U})\tau^{-s+1/2}).$$

Proof. The proof of this lemma is rather technical and is provided in the Appendix of the extended version of this paper [14]. □

Let $\texttt{SampleFree}(n)$ denote a set of n points sampled independently and identically from the uniform distribution on $\mathcal{M}_{\text{free}}$. The following to lemmas provide basic probabilistic bounds. Proofs can be found in our related work [10].

Lemma IV.5 (Lemma IV.3, [10])**.** *Fix $N \in \mathbb{N}$, $\alpha \in (0,1)$, and let $S_0, \dots, S_K$ be disjoint subsets of $\mathcal{M}_{free}$ with*

$$\mu(S_k) = \mu(S_1) \geq (2 + \log(1/\alpha))e^2 \left(\frac{1}{N}\right) \mu(\mathcal{M}_{free}),$$

for each k. Let $V = \texttt{SampleFree}[N]$; then the probability that more than an α fraction of the sets S_k contain no point of V is bounded as:

$$\mathbb{P}\left(\#\{k \in \{0,\dots,K\} : S_k \cap V = \emptyset\} \geq \alpha K\right) \leq 2e^{-\alpha K}.$$

Lemma IV.6 (Lemma IV.4, [10])**.** *Fix $N \in \mathbb{N}$ and let $T_0, \dots, T_K$ be subsets of $\mathcal{M}_{free}$, possibly overlapping, with*

$$\mu(T_k) = \mu(T_1) \geq \kappa \left(\frac{\log N}{N}\right) \mu(\mathcal{M}_{free})$$

for each k and some constant $\kappa > 0$. Let $V = \texttt{SampleFree}[N]$; then the probability that there exists a T_k that does not contain a point of V is bounded as:

$$\mathbb{P}\left(\bigvee_{k=0}^{K} \{T_k \cap V = \emptyset\}\right) \leq K N^{-\kappa}.$$

We are now in a position to prove the main result of this section.

Theorem IV.7 (Probabilistic exhaustivity)**.** *Let Σ be a controllable system and suppose $\pi = (\mathbf{x}, \mathbf{u}, T)$ is a piecewise-optimal trajectory. Let $N \in \mathbb{N}$, $\varepsilon > 0$, and consider a set of sample nodes $V = \{\mathbf{x}[0]\} \cup \texttt{SampleFree}[N]$. Define the event E_N that there exist waypoints $\{\mathbf{y}_k\}_{k=0}^{K} \subset V$ which $(\tau_N, \varepsilon, d_N, p_N)$-trace x, where*

$$\tau_N = \left(\frac{2\mu[\mathcal{M}_{free}]}{C_{\det}\zeta_n}\right)^{1/\tilde{D}} \left(\frac{1+\eta}{\tilde{D}}\right)^{1/\tilde{D}} \left(\frac{\log N}{N}\right)^{1/\tilde{D}},$$

for a parameter $\eta \geq 0$, $\tilde{D} = (D+n)/2$, $d_N = 7(1+M_U)\tau_N$, and $p_N = C_p(1+\sqrt{M_U})\tau_N^{-s+1}$ for some constant C_p depending only on Σ (determined from the full asymptotics of Lemma IV.4). Then, as $N \to \infty$, the probability that no such waypoint set exists is asymptotically bounded as:

$$1 - \mathbb{P}(E_N) = O\left(N^{-\eta/\tilde{D}} \log^{-1/\tilde{D}} N\right).$$

Proof. Note that in the case $T = 0$ we may pick $\mathbf{y}_0 = \mathbf{x}[0]$ to be the only waypoint and the result is trivial. Therefore assume $T > 0$. We make the identification $\alpha = \varepsilon/12(1+M_U)$, $\beta = \min\{\varepsilon/12, 1/\sqrt{2}\}$ in preparation for the application of Lemma IV.4. Fix N sufficiently large so that $\tau_N \leq \tau_{\text{vol}}$ and also:

$$\log N \geq \beta^{-\tilde{D}} \tilde{D}(2 + \log(1/\alpha))e^2/(1+\eta). \tag{11}$$

Take $K = \lfloor T/\tau_N \rfloor$ and, for $k = 0, \dots, K$, consider the sets $T_k = \Delta[\mathbf{x}[k\tau_N], \tau_N, \sqrt{\tau_N}]$ and $S_k = \Delta[\mathbf{x}[k\tau_N], \tau_N, \beta\sqrt{\tau_N}]$. From Lemma IV.3 we have the volume bound:

$$\mu(T_k) \geq \mu[\mathcal{M}_{\text{free}}] \left(\frac{1+\eta}{\tilde{D}}\right) \left(\frac{\log N}{N}\right), \tag{12}$$

and similarly:

$$\mu(S_k) \geq \beta^{\tilde{D}} \mu[\mathcal{M}_{\text{free}}] \left(\frac{1+\eta}{\tilde{D}}\right) \left(\frac{\log N}{N}\right), \tag{13}$$

for each k. Combining equation (12) and Lemma IV.6, we have that the probability that there exists a T_k that does not contain a sample point (i.e. $T_k \cap V = \emptyset$) is bounded as:

$$\mathbb{P}\left(\bigvee_{m=0}^{K} \{T_k \cap V = \emptyset\}\right) \leq K N^{-(1+\eta)/\tilde{D}}.$$

Note that $\beta \leq 1/\sqrt{2}$ implies that the sets S_k are all disjoint. Then we may combine equations (11) and (13), which together imply that the S_k satisfy the condition of Lemma IV.5, to see that the probability that more than an α fraction of the S_k do not contain a sample point is bounded as:

$$\mathbb{P}\left(\#\{k \in \{0,\dots,K\} : S_k \cap V = \emptyset\} \geq \alpha K\right) \leq 2e^{-\alpha K}.$$

Now, as long as neither of these possibilities holds (i.e. if every T_k and at least a $(1-\alpha)$ fraction of the S_k contains a point of V), we note that the existence of suitable waypoints $\{\mathbf{y}_k\}_{k=0}^{K} \subset V$ is guaranteed by Lemma IV.4. We then union bound the probability of failure:

$$\begin{aligned} 1 - \mathbb{P}(E_N) &\leq K N^{-(1+\eta)/\tilde{D}} + 2e^{-\alpha K} \\ &= O\left(N^{-\eta/\tilde{D}} \log^{-1/\tilde{D}} N\right), \end{aligned}$$

as $N \to \infty$, where we have used the fact that $K = \lfloor T/\tau_N \rfloor = O((N/\log N)^{1/\tilde{D}})$. □

V. DFMT* Algorithm

The algorithm presented here is based on FMT*, from the recent work of [15], which can be thought of as an accelerated version of PRM* [4]. Briefly, PRM* first samples all the vertices, then constructs a fully *locally* connected graph, and then performs shortest path search (e.g., Djikstra's algorithm) on the graph to obtain a solution. FMT* also samples all vertices first, but instead of a graph, lazily builds a tree *via dynamic programming* that very closely approximates the shortest-path tree for PRM*, but saves $O(\log(n))$ collision-checks by not constructing the full graph. The algorithm given by Algorithm 1, DFMT*, is not fundamentally different from the original FMT* algorithm, but mainly changes what "local" means under differential constraints (similar to [10], but now with drift). One more difference of DFMT* presented here, even from the algorithm in [10], is that the edges are now directed, reflecting the fundamental asymmetry of differential constraints with drift.

Specifically, define the fixed-time forward-reachable and backwards-reachable sets respectively:

$$\begin{aligned} R_\tau^+[\mathbf{x}, d] &= \{\mathbf{x}' \in \mathcal{M} \mid c_\tau^*[\mathbf{x}, \mathbf{x}'] < d\} \\ &= \{\mathbf{x}' \in \mathcal{M} \mid \|\mathbf{x}' - \bar{\mathbf{x}}[\tau]\|_{G[\tau]^{-1}}^2 < d - \tau\}, \\ R_\tau^-[\mathbf{x}, d] &= \{\mathbf{x}' \in \mathcal{M} \mid c_\tau^*[\mathbf{x}', \mathbf{x}] < d\} \\ &= \{\mathbf{x}' \in \mathcal{M} \mid \|\mathbf{x} - \bar{\mathbf{x}}'[\tau]\|_{G[\tau]^{-1}}^2 < d - \tau\}. \end{aligned}$$

Membership in either reachable set may be readily checked using the formulation above. Let $\texttt{CollisionFree}_\tau[\mathbf{x}_1, \mathbf{x}_2]$ denote the boolean function which returns true if and only if $\pi_\tau^*[\mathbf{x}_1, \mathbf{x}_2]$ lies within $\mathcal{M}_{\text{free}}$. Given a set of vertices V, a state $\mathbf{x} \in \mathcal{M}$, and a cost threshold $d > 0$, let $\texttt{Near}_\tau^\pm[V, \mathbf{x}, d] = V \cap R_\tau^\pm[\mathbf{x}, d]$. Let (x_1, x_2) denote the directed edge corresponding to $\pi_\tau^*[\mathbf{x}_1, \mathbf{x}_2]$ with edge weight $c_\tau^*[\mathbf{x}_1, \mathbf{x}_2]$. Given a directed graph $G = (V, E)$, where V is the vertex set and E is the edge set, and a vertex $\mathbf{x} \in V$, let $\texttt{Cost}[\mathbf{x}, G]$ be the function that returns the cost of the shortest (directed) path in the graph G between the vertices $\mathbf{x}_{\text{init}}$ and $\mathbf{x}$. Let $\texttt{Path}[\mathbf{x}, G]$ be the function that returns the path achieving that cost. The DFMT* algorithm is given in Algorithm 1. The algorithm uses two mutually exclusive sets, namely H and W. The *unexplored* set W stores all samples in the sample set V that have not yet been considered for addition to the tree of paths. The *wavefront* set H, on the other hand, tracks in sorted order (by cost from the root) only those nodes which have already been added to the tree that are near enough to tree leaves to actually form better connections. A detailed description of the algorithm would parallel the one provided in [15] and is omitted due to space limitations, we refer the interested reader to [15]. An extension of PRM*, which we denote by DPRM*, may also be defined in a straightforward manner as in [10], although we omit the full description here due to space constraints. Briefly, DPRM* searches the graph of all local collision-free connections that appear in any $\texttt{Near}$ set (as opposed to the tree subgraph constructed by DFMT*) for the least cost trajectory.

Algorithm 1 Differential Fast Marching Tree (DFMT*)

$V \leftarrow \{\mathbf{x}_{\text{init}}\} \cup \texttt{SampleFree}[N]$; $E \leftarrow \emptyset$
$W \leftarrow V \setminus \{\mathbf{x}_{\text{init}}\}$; $H \leftarrow \{\mathbf{x}_{\text{init}}\}$
$\mathbf{z} \leftarrow \mathbf{x}_{\text{init}}$
while $\mathbf{z} \notin \mathcal{M}_{\text{goal}}$ **do**
  $H_{\text{new}} \leftarrow \emptyset$
  $X_{\text{near}} = \texttt{Near}^{+}_{\tau_N}[V \setminus \{\mathbf{z}\}, \mathbf{z}, d_N] \cap W$
  for $\mathbf{x} \in X_{\text{near}}$ **do**
    $Y_{\text{near}} \leftarrow \texttt{Near}^{-}_{\tau_N}[V \setminus \{\mathbf{x}\}, \mathbf{x}, d_N] \cap H$
    $\mathbf{y}_{\min} \leftarrow \arg\min_{\mathbf{y} \in Y_{\text{near}}} \{\texttt{Cost}[\mathbf{y}, T = (V, E)] + c^*_\tau[\mathbf{y}, \mathbf{x}]\}$ //Dynamic programming recursion
    if $\texttt{CollisionFree}_\tau[\mathbf{y}_{\min}, \mathbf{x}]$ **then**
      $E \leftarrow E \cup \{(\mathbf{y}_{\min}, \mathbf{x})\}$
      $H_{\text{new}} \leftarrow H_{\text{new}} \cup \{\mathbf{x}\}$
      $W \leftarrow W \setminus \{\mathbf{x}\}$
    end if
  end for
  $H \leftarrow (H \cup H_{\text{new}}) \setminus \{\mathbf{z}\}$
  if $H = \emptyset$ **then**
    return Failure
  end if
  $\mathbf{z} \leftarrow \arg\min_{y \in H} \texttt{Cost}[\mathbf{y}, T = (V, E)]$ //Select wavefront node with lowest cost-to-come
end while
return $\texttt{Path}[\mathbf{z}, T = (V, E)]$

VI. Asymptotic Optimality of DFMT*

In this section, we prove the asymptotic optimality of DFMT*. The asymptotic optimality of DPRM* follows as a corollary. We note that in contrast to the work required to establish probabilistic exhaustivity for this class of differentially constrained systems and cost functions, the argument that DFMT* recovers paths at least as good as any waypoint-traced trajectory is essentially equivalent to the proofs presented in [10] and [15]. Thus we outline only the differences between the proof of the following theorem and the proofs of Theorems VI.1 and VI.2 presented in [10]. The following optimality result for DFMT* also provides a *convergence rate* bound. To avoid confusion we note that this bound is given in terms of sample size N; for a discussion of how sample size relates to run time for FMT*-style algorithms see [15].

Theorem VI.1 (DFMT* asymptotic optimality)**.** *Let* $(\Sigma, \mathcal{M}_{\textit{free}}, \mathbf{x}_{\textit{init}}, \mathcal{M}_{\textit{goal}})$ *be a trajectory planning problem satisfying the assumptions* A_Σ *and with* $\mathcal{M}_{\textit{goal}}$ ξ*-regular, such that there exists an optimal path* π^* *with weak* δ*-clearance for some* $\delta > 0$*. Let* c_N *denote the cost of the path returned by DFMT* with* N *vertices using the time interval:*

$$\tau_N = \left(\frac{2\mu[\mathcal{M}_{\textit{free}}]}{C_{\det}\zeta_n}\right)^{1/\tilde{D}} \left(\frac{1+\eta}{\tilde{D}}\right)^{1/\tilde{D}} \left(\frac{\log N}{N}\right)^{1/\tilde{D}}$$

and cost radius $d_N = f(N)\tau_N$*, where* $\eta \geq 0$ *is an implementation-specific parameter,* $\tilde{D} = (D+n)/2$*, and* f *is an implementation-specific function such that* $f(N) \to \infty$ *as* $N \to \infty$*. Then for fixed* $\varepsilon > 0$*,*

$$\mathbb{P}\left(c_N > (1+\varepsilon)c(\pi)\right) = O\left(N^{-\eta/\tilde{D}} \log^{-1/\tilde{D}} N\right).$$

Proof. As in Theorem VI.2 [10], for fixed $\varepsilon > 0$ we select an approximation to the optimal trajectory from the weak δ-clearance homotopy for π^*. The nature of this homotopy and the regularity of the goal region allow for the approximant $\pi = (\mathbf{x}, \mathbf{u}, T)$ to a) have strong δ_α-clearance for some $\delta_\alpha > 0$, b) extend into the interior of $\mathcal{M}_{\text{goal}}$ so that there exists $\gamma > 0$ such that $B[\mathbf{x}[T], \gamma] \subset \mathcal{M}_{\text{goal}}$, and c) have cost bounded as $c[\pi] \leq (1+\varepsilon/2)c[\pi^*]$.

Now we approximate the approximant: in order to apply probabilistic exhaustivity we consider N sufficiently large so that $f(N) \geq 7(1 + M_U[\pi])$ and $p_N = C_p(1 + \sqrt{M_U[\pi]})\tau_N^{-s+1} < \min\{\delta, \gamma/2\}$. Note that even though $M_U[\pi]$ is an unknown parameter (from the algorithm's perspective), it is still a constant and the above inequalities must hold in the asymptotic limit $N \to \infty$. Then we may apply Theorem IV.7 to produce, with probability at least $1 - O\left(N^{-\eta/\tilde{D}} \log^{-1/\tilde{D}} N\right)$, a sequence of waypoints $\{\mathbf{y}_k\}_{k=0}^K \subset V$ which $(\tau_N, \varepsilon/4, d_N, \min\{\delta, \gamma/2\})$-trace π. In the event that such $\{\mathbf{y}_k\}$ exist, we note that $\mathbf{y}_{k+1} \in \texttt{Near}^{+}_{\tau_N}[V \setminus \{\mathbf{y}_k\}, \mathbf{y}_k, d_N]$, $\texttt{CollisionFree}_\tau[\mathbf{y}_k, \mathbf{y}_{k+1}]$ is true for all k, and $\mathbf{y}_K \in \mathcal{M}_{\text{goal}}$.

In other words, with high probability there exist waypoints $\{\mathbf{y}_k\}$ in the sample set for which the connecting trajectory σ is obstacle-free and has cost close to the optimum. These are the elements required to prove that the "lazy" dynamic programming of the DFMT* algorithm returns a path with cost upper bounded as $c_n \leq c[\sigma] \leq (1+\varepsilon/4)c[\pi] \leq (1+\varepsilon)c[\pi^*]$; for the full details see the proof of Theorem VI.1 [10]. □

VII. Numerical Experiments

The DFMT* and DPRM* algorithms were implemented in Julia and run using a Unix operating system with a 2.0 GHz processor and 8 GB of RAM. We tested DFMT* and DPRM* on the double integrator system, a standard LQDMP formulation as studied in [9]. We also implemented variants of DFMT* and DPRM* where the local connection cost is optimized over time τ, in addition to control input $\mathbf{u}$, and the $\texttt{Near}$ sets are defined on cost radius alone. This is similar to how the $\texttt{Near}$ sets of Kinodynamic RRT*

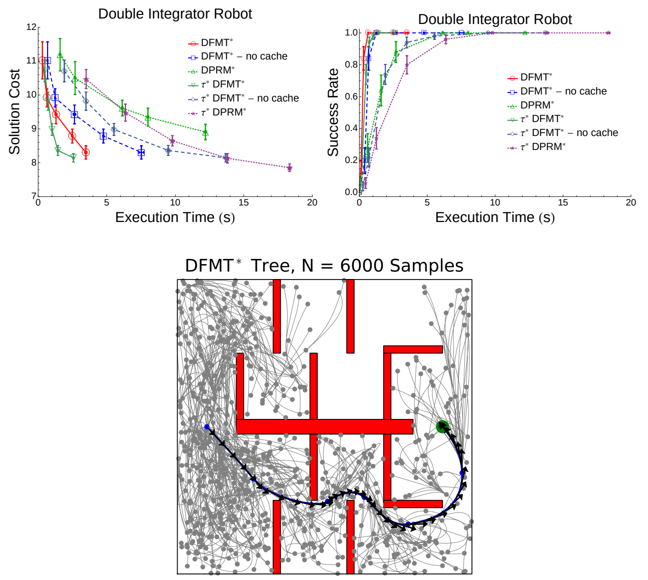

Fig. 1. Top: Simulation results for the double integrator system with a maze obstacle set. The error bars in each axis represent plus and minus one standard error of the mean for a fixed sample size n. Bottom: Example DFMT* tree for $n = 6000$. The position of the the feasible trajectory returned is highlighted in blue with velocity denoted by arrows.

studied in [9] are defined. We call these variants τ^* DFMT* and τ^* DPRM*. The simulation results are summarized in Figure 1. A maze was used for $\mathcal{M}_{\text{obs}}$, and our algorithm implementations were run 50 times each on sample sizes up to $N = 12000$ for fixed τ and $N = 6000$ for the τ^* variants. We plot results for both versions of DFMT* run with and without a cache of near neighbor sets and local connection costs; as discussed in [10] this information, which does not depend on the problem-specific obstacle configuration, may be precomputed for batch-processing algorithms such as DFMT* and DPRM*– the price to pay is a moderate increase in memory requirements. We see that the extra time for optimizing over local connection duration τ is significant (DFMT* – no cache vs. τ^* DFMT* – no cache), but may be mitigated by precomputation (DFMT* vs. τ^* DFMT*).

VIII. Discussion and Conclusions

In this paper we have provided a thorough and rigorous theoretical framework to assess optimality guarantees of sampling-based algorithms for linear affine systems with a mixed time/energy cost function. In particular, we leveraged the study of small-cost perturbations to show that optimum-approximating waypoints may be found among randomly sampled state sets with high probability. We applied this analysis to design and theoretically validate an asymptotically optimal algorithm, DFMT*, for the LQDMP problem.

We believe this analysis framework may be extended to study "batch-processing" motion planners of similar design, that is sampling-based algorithms for which the random state sample is selected at initialization (as opposed to during operation, the case for "anytime" algorithms). For example, our aim in future work is to extend our analysis to the τ^* variant of DFMT* in which the time-optimal, as opposed to the fixed-time, two-point boundary value problem forms the basis for the `Near` set.

Although this work is limited to linear affine drift systems, it not only provides a good model for many real systems, but is a crucial first step towards modelling nonlinear systems as well. Indeed, since DFMT* can be applied to a nonlinear system by linearizing the dynamics, an important next step will be to assess the theoretical guarantees of DFMT* applied to such a linearized approximation. A similar linearization approach has been experimentally validated by Kinodynamic RRT* in [9], and we are optimistic that the theoretical analysis may be accomplished, given the similarities in analysis already evident between the linear affine drift systems studied in this paper and the (possibly non-linear) control-affine driftless systems studied in [10]. In particular, the parallel notions of controllable/bracket-generating systems and controllability index/Hausdorff dimension give hope that a unifying theory for non-linear systems with drift may be achieved. There are a number of additional directions open for further research. In particular, we plan to deploy DFMT* on robotic platforms, specifically helicopters and floating platforms emulating the dynamics of spacecraft. Also, it is of interest to study a bidirectional version of DFMT*. Finally, it is of interest to devise strategies whereby the tuning parameters are "self regulating," with the objective of making the algorithm "parameter-free."

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APPENDIX

Lemma IV.4. *Let $\pi = (\mathbf{x}, \mathbf{u}, T)$ be a piecewise-optimal trajectory and consider a fixed partitioning time $\tau < T$. Denote $\mathbf{x}[k\tau] = \mathbf{x}[k\tau]$ for each $k \in \{0, 1, \dots, K = \lfloor T/\tau \rfloor\}$. Suppose that $\{\mathbf{y}_k\}_{k=1}^K \subset \mathcal{M}$ satisfy:*

(a) $\mathbf{y}_k \in \Delta[\mathbf{x}[k\tau], \tau, \sqrt{\tau}]$ for all $k \in \{0, 1, \dots, K = \lfloor T/\tau \rfloor\}$,

(b) more than a $(1-\alpha)$ fraction of the $\mathbf{y}_k$ satisfy $\mathbf{y}_k \in \Delta[\mathbf{x}[k\tau], \tau, \beta\sqrt{\tau}]$,

for fixed parameters $\alpha, \beta \in (0, 1)$. Then $c_\tau^*[\mathbf{y}_{k-1}, \mathbf{y}_k] \leq 7(1 + M_U)\tau$ for all k. Additionally, the trajectory $\sigma = \pi_\tau^*[\mathbf{y}_0, \dots, \mathbf{y}_K] = (\mathbf{y}, \mathbf{v}, K\tau)$ satisfies

$$c[\sigma] \leq c[\pi]\left(1 + 6\left((1 + M_U)\alpha + \beta\right)\right).$$

Finally, the maximum distance deviation σ takes from π is $O(\tau^{-s})$, that is:

$$\max_{t \in [0, K\tau]} \left(\min_{t' \in [0,T]} \|\mathbf{y}[t] - \mathbf{x}[t']\| \right) = O((1+\sqrt{M_U})\tau^{-s+1/2}).$$

Proof. In order to bound the cost of σ we compare it to the trajectory:

$$\tilde{\pi} = \pi_\tau^*[\mathbf{x}[0], \mathbf{x}[\tau], \dots, \mathbf{x}[K\tau]],$$

which in turn must have cost at most $c(\pi)$. Indeed, any difference between $c[\tilde{\pi}]$ and $c[\pi]$ results from the fact that $\tilde{\pi}$ may "shortcut" the pivot nodes $\mathbf{x}_j$ of π. We consider each node $\mathbf{y}_k$ to be a perturbation of $\mathbf{x}[k\tau]$ by the vector $\boldsymbol{\delta}_k = \mathbf{y}_k - \mathbf{x}[k\tau]$. Denote $d_k = \max\left\{ \|\boldsymbol{\delta}_k\|_{G[\tau]^{-1}}, \|\exp[A\tau]\boldsymbol{\delta}_k\|_{G[\tau]^{-1}} \right\}$. We have by assumption that $d_k \leq \sqrt{\tau}$ for all k and $d_k \leq \beta\sqrt{\tau}$ for at least $(1-\alpha)K$ values of k. We apply Lemma IV.1, in the special case of Remark IV.2, to bound the cost of σ:

$$\begin{aligned} c[\sigma] &= \sum_{k=1}^K c_\tau^*[\mathbf{y}_{k-1}, \mathbf{y}_k] \\ &\leq \sum_{k=1}^K c_\tau^*[\mathbf{x}[(k-1)\tau], \mathbf{x}[k\tau]]\left(1 + 3(d_{k-1} + d_k)/\sqrt{\tau}\right), \end{aligned}$$

and therefore:

$$\frac{c(\sigma)}{c(\tilde{\pi})} \leq 1 + \frac{3\sum_{k=1}^K c_\tau^*[\mathbf{x}[(k-1)\tau], \mathbf{x}[k\tau]](d_{k-1}/\sqrt{\tau} + d_k/\sqrt{\tau})}{\sum_{k=1}^K c_\tau^*[\mathbf{x}[(k-1)\tau], \mathbf{x}[k\tau]]}. \tag{14}$$

Now, note that $\tau \leq c_\tau^*[\mathbf{x}[(k-1)\tau], \mathbf{x}[k\tau]] \leq \tau(1 + M_U)$ for each k. The lower bound is clear; the upper bound follows from the fact that $c_\tau^*[\mathbf{x}[(k-1)\tau], \mathbf{x}[k\tau]]$ is at most the cost of the portion of π between $\mathbf{x}[(k-1)\tau]$ and $\mathbf{x}[k\tau]$. We note here that this fact, in combination with the fact that $d_k \leq \sqrt{\tau}$ for all k, implies our first desired result: $c_\tau^*[\mathbf{y}_{k-1}, \mathbf{y}_k] \leq 7(1 + M_U)\tau$. The ratio in equation (14) is greatest in the case that $c_\tau^*[\mathbf{x}[(k-1)\tau], \mathbf{x}[k\tau]]$ achieves this upper bound and lower bound precisely when $d_k = \sqrt{\tau}$ and $d_k = \beta\sqrt{\tau}$ respectively. That is,

$$\begin{aligned} c(\sigma)/c(\tilde{\pi}) &\leq 1 + \frac{3\left(2\tau(1+M_U)\alpha K + 2\tau(1-\alpha)K\beta\right)}{\tau(1+M_U)\alpha K + \tau(1-\alpha)K} \\ &\leq 1 + 6\left(\frac{\tau(1+M_U)\alpha K}{\tau K} + \frac{\tau(1-\alpha)K\beta}{\tau(1-\alpha)K}\right) \\ &= 1 + 6\left((1+M_U)\alpha + \beta\right), \end{aligned}$$

and the second desired result follows from the fact that $c(\tilde{\pi}) \leq c(\pi)$.

To bound the distance that σ deviates from π we consider the distance from each point $\mathbf{y}[t]$ of σ to the starting point $\mathbf{x}[\lfloor t/\tau \rfloor \tau]$ of its corresponding trajectory segment in π, noting that:

$$\max_{t \in [0, K\tau]} \left(\min_{t' \in [0,T]} \|\mathbf{y}[t] - \mathbf{x}[t']\| \right) \leq \max_{t \in [0, \tau K]} \|\mathbf{y}[t] - \mathbf{x}[\lfloor t/\tau \rfloor \tau]\|.$$

Let $t \in [0, \tau K]$ and denote $k = \lfloor t/\tau \rfloor$. Then:

$$\|\mathbf{y}[t] - \mathbf{x}[k\tau]\| \leq \|\mathbf{y}[t] - \tilde{\mathbf{x}}[t]\| + \|\tilde{\mathbf{x}}[t] - \mathbf{x}[k\tau]\|. \tag{15}$$

From Lemma IV.1 we know that $\|\mathbf{y}[t] - \tilde{\mathbf{x}}[t]\| = O(\tau^{-s+1/2}\sqrt{\tau}) = O(\tau^{-s+1})$, which depends only on the fact that the perturbation sizes $d_k \leq \sqrt{\tau}$ are small, i.e. the bound is uniform over all t. For the second term in the inequality, we note from equation (7) the explicit formula for the portion of π between times $k\tau$ and $(k+1)\tau$:

$$\tilde{\mathbf{x}}[t] = \bar{\mathbf{x}}'[t] + G[t'] \exp[A^T(\tau - t')] G[\tau]^{-1} (\mathbf{x}[(k+1)\tau] - \bar{\mathbf{x}}'[\tau]),$$

where $t' = t - k\tau$ and $\bar{\mathbf{x}}'[t]$ is the drift of the state $\mathbf{x}[k\tau]$ over time t. Now,

$$\begin{aligned} \|L[\tau]^{-1}(\mathbf{x}[(k+1)\tau] - \bar{\mathbf{x}}'[\tau])\| &= \|\mathbf{x}[(k+1)\tau] - \bar{\mathbf{x}}'[\tau]\|_{G[\tau]^{-1}} \\ &\leq \sqrt{M_U \tau}, \end{aligned}$$

and $\|\bar{\mathbf{x}}'[t] - \mathbf{x}[k\tau]\| = O(\tau)$, so thus:

$$\begin{aligned} \|\tilde{\mathbf{x}}[t] - \mathbf{x}[k\tau]\| &\leq O(\tau) + \|G[\tau]\| \cdot e^{\|A\|\tau} \cdot \|L[\tau]^{-1}\| \cdot \sqrt{M_U \tau} \\ &= O(\sqrt{M_U}\tau^{-s+1}), \end{aligned}$$

where the asymptotic bound depends only on the dynamics of the system, and is again uniform over all t. Then plugging both of these bounds into equation (15) we have that $\|\mathbf{y}[t] - \mathbf{x}[k\tau]\| = O((1 + \sqrt{M_U})\tau^{-s+1})$ and therefore:

$$\max_{t \in [0, K\tau]} \left(\min_{t' \in [0,T]} \|\mathbf{y}[t] - \mathbf{x}[t']\| \right) = O((1 + \sqrt{M_U})\tau^{-s+1}).$$

□

Remark VIII.1. *We note that if the time interval $[k\tau, (k+1)\tau]$ contains no pivot nodes of π, the comparison $\|\tilde{\mathbf{x}}[t] - \mathbf{x}[k\tau]\|$ may be replaced with $\|\tilde{\mathbf{x}}[t] - \mathbf{x}[t]\| = 0$ in equation* (15)*, giving a distance deviation bound $O(\tau^{-s+1})$ for that interval. That is, we may guarantee that the deviation of σ from π depends only on the perturbation size, except near pivot nodes. If we know that the maximum control effort in the neighborhood of pivot nodes is small, then we may improve the deviation bound in those regions as well.*