Credit Cycles Redux

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Abstract

Theoretical studies have shown that under unorthodox assumptions on preferences and production technologies, collateral constraints can act as a powerful amplification and propagation mechanism of exogenous shocks. We investigate whether or not this result holds under more standard assumptions. We find that collateral constraints typically generate small output amplification. Large amplification is obtained as a "knife-edge" type of result.
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1 Introduction

Business cycle models typically rely on large exogenous shocks to explain fluctuations in aggregate output. This approach is often criticized because shocks of the required magnitude are hard to find in the data (Summers 1986, Cochrane 1994). An alternative explanation is that the economy has some **amplification mechanism** that transforms relatively small shocks into large output fluctuations.

Kiyotaki and Moore (1997) and Kiyotaki (1998) have argued that such mechanism is a particular form of credit-market frictions. Specifically, when debts need to be fully secured by collateral, say land, and the collateral is also an input in production, then a small shock to the economy can be largely amplified. For instance, a small negative shock that reduces the net worth of credit-constrained firms forces them to curtail their investment in land. Land prices and output fall because credit-constrained firms are by nature more productive in the use of land. The fall in the value of the collateral reduces even more the debt capacity of constrained firms, causing additional falls in investment, land prices, and output. The cumulative effect could be dramatic, as they show using a carefully designed economy.

The results of Kiyotaki and Moore (1997) (KM henceforth) have launched a significant body of mainly theoretical research. Examples are Krishnamurthy (1998), Kocherlakota (2000), Caballero and Krishnamurthy (2001), and Paasche (2001). However, there has not yet been a systematic assessment of the quantitative significance of collateral constraints as an amplification mechanism of shocks. This assessment seems particularly important because theoretical models have used some extreme assumptions in order to boost the amplification. For example, KM introduce enough assumptions to induce constrained agents to fully invest all of the unexpected income; to prevent any response of the interest rate (lenders' preferences are linear); and to enhance the role of collateral in the economy (borrowers' technology is linear in land).[1] Are shocks still significantly amplified under more standard choices of preferences and technologies?

The objective of this paper is to address this question using a simple dynamic general equilibrium

[1] Both the appendix of Kiyotaki and Moore (1997) and Kiyotaki (1998) attempt to relax some of the unorthodox assumptions, but there is no assertion on whether these models can generate large **output** amplification.

model. The model is a two-agent closed economy, in the spirit of KM, but modified to introduce standard specifications of preferences and technologies. In particular, all agents in our economy have concave preferences, have access to concave production technologies, and are required to collateralize their debts. In order to generate productivity gaps between constrained and unconstrained agents, we employ the standard, but nonessential, assumption that agents differ in their discount factors. We use the model to examine the features and parameter values needed in order to achieve large output amplification.

The main finding of this paper is that collateral constraints can in fact amplify unexpected shocks to the economy, but the effect is generally small. For the standard values of a capital share of 1/3 and an elasticity of intertemporal substitution (EIS) of 1, the amplification is close to zero. Large amplification arises as a "knife-edge" type of result: on the one hand, it occurs at the right combination of a particularly small EIS (below 0.2) and a large share of capital (the collateralizable asset) in the production function. But if the EIS is too small, or the capital share is too large, then the equilibrium may not be a saddle path. Instead, the equilibrium may exhibit jagged dynamics, or be locally indeterminate.

To understand why the amplification is typically small, it is useful to break up the response of output to a shock in the following four components:[2]

$$\begin{aligned}\text{output response} \quad = \quad & (\text{productivity gap}) \times (\text{collateral share in production}) \times \\ & (\text{production share constrained agents}) \times (\text{redistribution of collateral})\,.\end{aligned}$$

This expression states that the response of output to shocks is bigger the larger the productivity gap between constrained and unconstrained agents, the larger the share of collateral in the production function, the larger the fraction of output produced by constrained agents, and the larger the redistribution of collateral from unconstrained to constrained agents originated by the shock. Notice that the amplification is caused by the redistribution of collateral from low-productive unconstrained agents to high-productive constrained agents. The expression suggests that the re-

[2]This equation will be derived later in the paper.

sponse of output is generally small. For example, if constrained agents are twice more productive (so that the productivity gap is 1/2), produce half of the total output, and the collateral share is 1/2, then constrained agents must increase their holdings of collateral by 800% just to increase output by 1%.

More specifically, there are three main reasons why output amplification is typically small. First, the concavity of the production function imposes a natural limit on the size of the first three components of the expression above. In that case, the share of collateral is below 1, and there is a trade-off between the productivity gap and the production share: a large productivity gap requires constrained agents to hold a small fraction of the collateral in the economy, which means that their share of the total production must be small. KM avoid this trade-off by assuming that the technology of constrained agents is linear in the collateral.

Second, the concavity of the preferences imposes a natural limit on the size of the fourth component. As constrained agents use the unexpected resources from a positive shock to secure more debt and demand more capital, the interest rate increases to induce unconstrained agents to provide the additional loans. This response of the interest rate limits the magnitude of the redistribution of capital and the response of output to the shock. If preferences are linear, as is the case in KM, then constrained agents can provide the additional loans without any increase in the interest rate. Thus, the asset price effect emphasized by KM is partially offset by the interest rate effect when preferences are concave. We find that for plausible values of the EIS the response of the interest rate significantly offsets the asset-price effect.

Finally, concave preferences also limit the size of the fourth component in a second way. Consumption smoothing implies that part of the unexpected resources are invested and part are consumed. In KM economy, however, constrained agents invest all the unexpected resources in capital.

A second important finding is that there exists a trade-off between amplification and persistence. Large amplification requires almost zero persistence. The reason is the following: large amplification occurs when constrained agents can significantly increase their capital holdings via additional debt. Furthermore, the larger the debt, the larger the compensation lenders require for postponing consumption due to the concavity of preferences. Thus, large amplification in one period implies large debt repayments next period which eat up borrowers' resources and capital holdings. Using the predator-prey analogy, "the fatter the prey, the sooner the killing."

Our results are robust to changes of the different assumptions of our benchmark model. We study possible responses of the non-collateralizable inputs, in particular labor, allow aggregate collateral to be accumulated, allow for probabilistic shocks and differences in capital shares and EIS across agents. Overall, our results show that for empirically plausible calibrations collateral constraints by themselves are not enough to account for the large fluctuations of output observed in the data.

Our exercise is similar in spirit to Kocherlakota (2000). He shows that the quantitative significance of the amplification effects generated by endogenous collateral constraints depends crucially on the parameters of the economy, in particular on factor shares. Our paper differs from Kocherlakota's in two ways. First, our economy is closed so that the interest rate is endogenously determined. This allows us to account for general equilibrium effects. Second, as in KM, the distribution of collateral across agents plays a crucial role in our model: this role is lost in Kocherlakota's representative-agent specification, which eliminates the leverage effect present in KM.

There is a related literature on the importance of financial factors on the investment behavior of firms which emphasizes the role agency costs (see, for example, Bernanke and Gertler, 1989; Bernanke, Gertler and Gilchrist,1999; and Calstrom and Fuerst, 1997 and 2000), and of limited enforceability (see Cooley, Quadrini and Marimon, 2001). These models do not directly incorporate collateral constraints, and consider a different mechanism from the one analyzed here. Fuerst (1995) analyzes such mechanism and also finds little propagation.

The reminder of the paper is organized as follows. Section 2 presents our benchmark model economy. In Section 3 we characterize the dynamics of the model and derive the conditions under which there is global and local determinacy of the equilibrium for the special case of CRRA utility, and Cobb-Douglas production function. We also present and discuss impulse responses to a one-time unexpected productivity shock. In Section 4 we present different extensions to the benchmark model in order to evaluate the robustness of our results. In particular, we introduce labor, allow the collateralizable asset (capital) to be reproducible, consider a stochastic environment where debt contracts are non-contingent, and allow capital shares and EIS to differ across agents. We show how our results are robust to these different extensions. Section 5 concludes.

2 Benchmark model

2.1 Economic environment

Consider an economy inhabited by two types of agents who differ in their rate of time preference. Agents may also differ in other dimensions such as the degree of risk aversion or the production technologies. There are two goods in this economy: a durable asset (capital, K), and a non-durable commodity (output, C). Agents maximize their expected lifetime utility as given by

$$E \sum_{t=0}^{\infty} \beta_i^t u_i(c_{it}) \text{ for } i = 1, 2 \tag{1}$$

where $1 > \beta_1 > \beta_2 > 0$, and c_{it} is consumption of agent i at time t. The momentary utility function, u_i, is assumed to satisfy usual properties. For the most part we use $u_i(c_{it}) = \frac{c_{it}^{1-\sigma_i}}{1-\sigma_i}$. We allow for the possibility that u differ across agent's types. There is a continuum of agents of each type with population size $m_i > 0$, $i = \{1, 2\}$. For simplicity, we normalize $m_2 = 1$ and refer to m_1 as m. Following steady state considerations, we often call agents type 1 lenders and agents type 2 borrowers. Except for the unanticipated shock, there is not uncertainty in the model.[3]

Agent i produces using a concave technology, $f_i(k_i)$, where k_i is capital and $\lim_{k \to 0} f_i'(k) = \infty$.[4] Similar to u, f may also differ across agent's types. For the most part we use $f_i(k) = k^{\alpha_i}$. Agents face a budget constraint given by

$$c_{it} + q_t(k_{it+1} - k_{it}) + a_{it} = f_i(k_{it}) + p_t a_{it+1} \tag{2}$$

[3]Section 4.3 considers a stochastic environment where debt contracts are non-contingent.

[4]One can think that the decreasing returns to scale in capital are due to implicit fixed labor. Section 4.1 introduces labor explicitly so that the production function is constant returns to scale in capital and labor.

where q is the price of capital in terms of consumption good, a_{it} are debt payments (including interests), and p_t is the price of one-period ahead bond at time t. Agents behave competitively taking prices as given.

We assume that borrowers cannot commit to repay their loans. They can escape with the production with no other penalty than losing their capital. As a result, loans need to be secured by the value of the capital, i.e.

$$a_{it+1} \leq q_{t+1} k_{it+1}. \tag{3}$$

Capital is available in a fixed aggregate amount, $\overline{K}$. This assumption can be interpreted as either investment taking a long time-to-build, or as the adjustment costs of investment being very high. As will become clear below, this assumption helps the model to generate larger output amplification. The more costly it is to accumulate capital, the larger is the response of asset prices to unexpected shocks, and the larger the redistribution of resources.[5] Finally, as in KM, we exclude the possibility of renting capital. Adding this possibility would not change the perfect-foresight equilibrium path but it would affect how the economy responds to an unanticipated shock. In particular, if agents own all the capital they employ, they will fully benefit from an unexpected increase in the value of collateral.

2.2 Competitive Equilibrium

Definition 1. A competitive equilibrium are sequences of prices $\{p_t, q_t\}_{t=0}^{\infty}$ and allocations $\{c_{it}, k_{it}, a_{it}\}_{t=0}^{\infty}$ for $i = 1, 2$, such that:

1. $\{c_{it}, k_{it}, a_{it}\}_{t=0}^{\infty}$ maximize (1) subject to (2) and (3) given $\{p_t, q_t\}_{t=0}^{\infty}$ and initial endowments k_{i0} and a_{i0} for $i = 1, 2$.

2. Capital, goods, and asset markets clear: $\sum_{i=1}^{2} m_i k_{it} = \overline{K}$, $\sum_{i=1}^{2} m_i c_{it} = \sum_{i=1}^{2} m_i f_i(k_{it})$, and $\sum_{i=1}^{2} m_i a_{it} = 0$.

This completes the description of the economy and the equilibrium concept. It is important to

[5] Section 4.2 relaxes this assumption and allows capital to be reproducible.

stress three features of the model that make it suitable for our purpose. First, the model is a slight modification of a standard representative-agent economy. If borrowing constraints are eliminated (or discount factors are identical) then the economy will collapse into a standard representative-agent economy. The model is thus designed to highlight the role of collateral constraints as the sole cause for amplification and persistence effects.

Second, we make no assumptions to keep the interest rate (the inverse of p) constant as do other papers in the literature.[6] We can thus study if changes in the interest rate dampen or enhance the asset price effect usually stressed as the key element behind the amplification. Third, the model requires only a small set of parameters on preferences and technologies: the intertemporal elasticity of substitution, factor shares, discount factors, and the relative mass of credit-constrained agents. We can use evidence about some of these parameters to impose some discipline in the analysis.

Let $s_t \equiv q_t - p_t q_{t+1}$ be the user cost (or down payment) of capital. Standard arguments can be used to show that the optimal choices of capital and bonds are characterized by

$$u_i'(c_{it})s_t = \beta_i f_i'(k_{it+1})u_i'(c_{it+1}), \tag{4}$$

$$u_i'(c_{it})p_t \geq \beta_i u_i'(c_{i+1}). \tag{5}$$

The first condition equates the marginal cost of holding capital to its marginal benefit. The second condition states that unconstrained agents equate the marginal benefit of borrowing to its marginal cost. For constrained agent, however, the marginal benefit of borrowing is larger than the marginal cost.

In the absence of credit constraints, equations (4) and (5) imply that production would be efficient, i.e., marginal products of capital would be equal across agents. In that case, the economy has no transitional dynamics and the distribution of capital is determined by the condition

$$f_1'\left(\left(\overline{K} - K^e\right)/m\right) = f_2'(K^e)$$

[6] Kiyotaki and Moore (1997) and Kiyotaki (1998) assume linear preferences or technologies, and Kocherlakota (2000) assumes a small open economy.

where K^e is the capital held by impatient agents. Furthermore, if the production functions are identical for both types of agents, then $K^e = \frac{\overline{K}}{1+m}$, so that all agents would hold the same amount of capital.

2.3 Steady State

Let variables in capital letters denote the aggregate quantities corresponding to the variables in lowercase. The following proposition summarizes the main properties of the steady state.

Proposition 1. There exist a unique steady state. In steady state impatient agents are credit constrained, and their capital holdings satisfy $K_2^* < K^e$. In addition, the following equations hold:

$$p^* = \beta_1$$

$$\frac{f_2'(K_2^*)}{f_1'\left(\left(\overline{K} - K_2^*\right)/m\right)} = \frac{\beta_1}{\beta_2} > 1$$

$$s^* = \beta_1 f_1'\left(\left(\overline{K} - K_2^*\right)/m\right), \; q^* = \frac{s^*}{1-\beta_1}$$

Proof: In equilibrium agents of at least one type are not credit constrained. Therefore, equation (5) evaluated at the steady state implies that

$$p^* \geq \beta_i \text{ for } i = 1,2 \text{ and } p^* = \beta_i \text{ for at least some } i.$$

Since $\beta_1 > \beta_2$, it follows that $p^* = \beta_1$ and $p^* > \beta_2$. Thus, impatient agents are credit constrained. In addition, equation (4) evaluated at steady state implies that

$$\frac{f_2'(k_2^*)}{f_1'(k_1^*)} = \frac{f_2'(K_2^*)}{f_1'\left(\left(\overline{K} - K_2^*\right)/m\right)} = \frac{\beta_1}{\beta_2} > 1$$

Thus $K_2^* < K^e$. The remaining equations are easy to derive using (2) through (5). ■

The first equation of Proposition 1 states that the steady-state interest rate is completely determined by the discount factor of the patient agents. The second equation stresses the role played by β_2 in the model. It determines the degree of inefficiency, i.e., the gap in marginal productivities. The lower the β_2 the larger the gap in marginal productivities.

Figure 1 illustrates the determination of the steady state of the economy. The efficient allocation with no debt-enforcement friction would produce $K_2^* = K^e$. In that case, impatient agents' consumption would drift toward zero. The existence of credit constraints reduces the borrower's capital holdings to $K_2^* < K^e$ and, more importantly, induces a gap in the marginal productivities. This gap is crucial for the model to generate amplification effects. If marginal products were equal in equilibrium, then small changes in the distribution of capital would have no effect on aggregate output.

3 Dynamics

We now describe the equilibrium path of the economy when one agent is constrained. The following proposition shows that unless all agents hold the same amount of capital, at least one agent, the one with lower capital holdings, is credit constrained. Thus, there is an equivalence between being credit constrained and holding capital below the efficient level.

Proposition 2. $a_{it} = q_t k_{it}$ if and only $k_{it} < \overline{K}/(1+m)$.

Proof: For sufficiency, divide (4) by (5) to obtain

$$\frac{s_{t-1}}{p_{t-1}} \leq f'(k_{it}) \text{ with strict inequality if } a_{it} = q_t k_{it}.$$

Thus, credit constrained agents have larger marginal productivity of capital and less capital than unconstrained agents. Therefore, they have less capital than the efficient level of capital. For necessity, note that $k_{it} < \overline{K}/(1+m)$ implies $f'(k_{it}) > f'\left(\left(\overline{K} - k_{it}\right)/m\right)$ This implies that $f'\left(\left(\overline{K} - k_{it}\right)/m\right) = \frac{s_{t-1}}{p_{t-1}} < f'(k_{it})$ so that agent i has to be credit constrained. ■

For simplicity consider the case when agent 2 is constrained so that agent 1 is unconstrained. The opposite case can be derived either by a symmetry argument or using the same reasoning as for agent 2. Assume the following standard functional forms for preferences and technologies: $f(k) = k^{\alpha}$ and $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, so that $1/\sigma$ is the intertemporal elasticity of substitution. Note that we assume that α and σ are the same for both agents, so that they only differ in their discount factors. We analyze below of differences in α and σ. As shown in Appendix A, the equilibrium can be described by a system of two first-order difference equations in k_2 and z, where z is the relative consumption of type-1 agents with respect to type-2 agents, i.e., $z \equiv \frac{mc_1}{c_2}$.

Proposition 3. Given $k_{2t} < K^e$, the competitive equilibrium is completely characterized by the dynamic system

$$\frac{z_{t+1}}{z_t} = \left(\frac{\beta_1}{\beta_2}\right)^{1/\sigma} \left(\frac{k_{2t+1}}{\left(\overline{K} - k_{2t+1}\right)/m}\right)^{(1-\alpha)/\sigma} \tag{6}$$

and

$$\frac{mf\left(\left(\overline{K} - k_{2t}\right)/m\right)}{f\left(k_{2t}\right)} = z_t - \left(\frac{F(k_{2t})}{F(k_{2t+1})}\right)^{\sigma} (1 + z_{t+1})^{\sigma} (1 + z_t)^{1-\sigma} \beta_2 \alpha \left(\frac{k_{2t+1}}{k_{2t}}\right)^{\alpha} \tag{7}$$

where

$$F(k_{2t}) \equiv mf\left(\frac{K - k_{2t}}{m}\right) + f(k_{2t}).$$

Proof: See Appendix A.

3.1 Global determinacy

The qualitative properties of the equilibrium can be described using a phase diagram based on these two equations (see details in Appendix B). Figure 2.a. shows a typical phase diagram. The curve $g_z = 1$ describes the stationary path along which $z_{t+1} = z_t$. According to equation (6) such path requires $k_2 = k_2^*$. Similarly, the curve $g_{k2} = 1$ describes the locus of (k_2, z) points such that $k_{2t+1} = k_{2t}$. According to equations (6) and (7) such path is described by the following equation

$$z = \frac{mf\left(\left(\overline{K} - k_2\right)/m\right)}{f\left(k_2\right)} + \alpha\beta_2 \left(1 + \left(\frac{\beta_1}{\beta_2}\right)^{1/\sigma} \left(\frac{k_2}{\left(\overline{K} - k_2\right)/m}\right)^{\frac{1-\alpha}{\sigma}} z\right)^{\sigma} (1 + z)^{1-\sigma}$$

Denote $g(k_2)$ the solution for z from this equation. Based on these two curves, the diagram shows the case in which the competitive equilibrium is unique and described by the saddle point path E-E.[7] Appendix C shows that this is the case under the following necessary and sufficient condition.

Lemma 4. The competitive equilibrium exhibits global saddle-path stability if and only if

$$\left(\frac{(1-\alpha)\,g(k_2)}{1+g(k_2)}\frac{\overline{K}}{\overline{K}-k_2}+\alpha\right)\frac{1}{\sigma}>\alpha\frac{f(k_2)}{F(k_2)}\left(1-\frac{f'\left(\frac{\overline{K}-k_2}{m}\right)}{f'(k_2)}\right)\text{ for all }k_2\in(0,k^e]. \tag{8}$$

Moreover, a sufficient condition for global saddle-path stability is

$$\frac{1}{\sigma}>\frac{\alpha\,(1-\tau(k_2))}{\alpha\,(1-\tau(k_2))+m\,(\tau(k_2))^{\frac{\alpha}{\alpha-1}}+\tau(k_2)}\text{ for all }k_2\in(0,k^e] \tag{9}$$

where $0<\tau(k_2)\equiv\frac{f'((\overline{K}-k_2)/m)}{f'(k_2)}0<1$ is the ratio of productivities.[8]

Proof: See Appendix C.

Equation (8) has no closed-form solution, although it could be evaluated numerically. This equation suggests that there is a region of indeterminacy that could potentially be important particularly if σ is sufficiently large. To evaluate this possibility we perform a numerical exercise using the sufficient condition (9), which is simpler than (8) because $g(k_2)$ does not need to be computed. For each m, we maximize the right hand side of (9) over $0<\alpha<1$ and $0<\tau<1$. The result is a lower bound for $\frac{1}{\sigma}$, so that any EIS above this bound guarantees global determinacy. Figure 2.b. shows this lower bound as function of the share of constrained agents $\left(\frac{1}{1+m}\right)$. The bound increases with the fraction of credit constrained agents. Notice that even if half of the household-firms are constrained, any EIS above 0.1 guarantees global determinacy.

The main message of Figure 2.b. is that as long as the EIS is not extremely low (say below 0.2), and the mass of credit constrained agents extremely large (say above 70%), then global determinacy

[7] Standard arguments can be used to ruled out other paths as equilibrium ones.

[8] Recall that we are considering the case in which type-2 agents are constrained, and that according to Proposition 2, $0<k_2<k_1$.

is guaranteed. The empirical evidence suggest that the EIS is well above 0.3 (see Vissing-Jorgensen, 2002), and that the fraction of constrained agents is well below 50%.[9]

In what follows, we assume that condition (8) holds and study its implications around the steady state on the model's parameters. We assume this for two reasons. First, it is standard in the literature to focus attention on fluctuations originated by fundamental shocks rather than on endogenous fluctuations. For example, analogous restrictions are used by Kiyotaki and Moore (1997) and Kiyotaki(1999). Second, and more importantly, global determinacy is lost only for extreme and implausible values of $1/\sigma$ and m.

3.2 Local Dynamics

We now focus our analysis on the equilibrium properties around the steady state. For this purpose, we evaluate condition (8) for global determinacy at the steady state and derive a restriction on the underlying parameters of the economy. This yields the following restriction:

$$\frac{1+x^{\alpha}+\frac{(1-\alpha)}{\alpha}\frac{1+x}{x}\left(\alpha\beta_2+x^{\alpha}\right)}{1-\frac{\beta_2}{\beta_1}}>\sigma \tag{10}$$

where $x=m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1-\alpha}}$. As argued, this restriction is weak. In the frictionless economy $1-\frac{\beta_2}{\beta_1}=0$, and the previous condition is always satisfied. Among the economies that satisfy this condition are economies with log utility function ($\sigma=1$); representative-agent economies ($\beta_1=\beta_2$); economies with a large mass of unconstrained agents ($m\rightarrow\infty$); and AK or AL type economies (i.e. $\alpha\simeq 1$ or $\alpha\simeq 0$). In contrast, provided that some inefficiency exist so that $1-\frac{\beta_2}{\beta_1}>0$, this condition is violated by economies with large σ (i.e., close to zero EIS), low values of m, and certain values of α. In these cases, the equilibrium is not locally determined.

In addition to requiring local determinacy, we preclude the possibility of jagged dynamics (that occurs when the only stable root is negative), which we do not consider to be an empirically plausible description of business cycles. A necessary and sufficient condition to ensure a unique

[9] See Fazzari et al (1988), Hoshi et al (1991), Whited (1992), Gilchrist et al (1995), and Erickson et al (2000).

positive stable root is[10]

$$\sigma < 1 + \frac{1 + m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1-\alpha}}}{\alpha\left(\beta_1 - \beta_2\right)} \tag{11}$$

Proposition 5. The steady state exhibits saddle-path stability with a positive stable root if and only if (10) and (11) hold.

Proof: See Appendix D.

3.3 A one-time unexpected shock

We now study the response of the model economy to a one-time unexpected shock by using numerical simulations. For that purpose, we log-linearize equations (6) and (7) around the steady state. We assume that the economy is at the steady state at time zero, when an unexpected one-time increase in productivity of 1% for all agents occurs. If there were no collateral constraints, the economy would be back to its steady state immediately after the shock (i.e., at time one), and so amplification would be zero. In contrast, when collateral constraints are present, the shock provides more resources to both constrained and unconstrained agents. Since the shock is temporary, both types of agents save part of the extra resources in order to smooth consumption. The difference between the two types is that the unconstrained agents are indifferent between buying capital or bonds while the constrained, who are borrowers, will smooth the shock by buying capital. In fact, constrained agents are at a corner solution so that the only way to borrow more is to buy more capital. Since borrowers' marginal product of capital is higher, aggregate output increases following the productivity shock. Thus, the fundamental channel behind amplification is the redistribution of capital toward agents with high productivity.

The two main variables of interest are amplification and persistence. We define amplification as the elasticity of output in period one with respect to a productivity shock in period zero, ϵ_{YZ}.[11]

[10] We could have alternatively assumed that condition (9) holds. This condition ensures both global determinancy and monotonic saddle-path stability around the steady state. We did not to impose this condition at the local level because it is only a sufficient but not necessary condition, and it excludes cases of large amplification, as we show below.

[11] The elasticity of output in period zero with respect to a productivity shock in period zero is always 1 in this model.

Persistence is measured by the stable root of the log-linearized system.[12]

3.3.1 Amplification

Output in period one can only vary if K_2 moves from its steady state value at time 1. We can then write ϵ_{YZ} as the product of two components: the elasticity of output at time one with respect to K_2 at period 1, ϵ_{YK_2} , times the elasticity of K_2 at period one with respect to z_0, ϵ_{K_2Z}

$$\begin{aligned} \epsilon_{YZ} &= \epsilon_{YK_2}\epsilon_{K_2Z} = \left(f_2' - f_1'\right)\frac{X}{Y}\epsilon_{K_2Z} \\ &= \underbrace{\frac{(f_2' - f_1')}{f_2'}}_{\text{productivity gap}} \cdot \underbrace{\alpha}_{\text{collateral share}} \cdot \underbrace{\frac{Y_2}{Y}}_{\text{output share}} \cdot \underbrace{\epsilon_{K_2Z}}_{\text{redistribution of capital}} \end{aligned} \tag{12}$$

This equation suggests that ϵ_{YK_2} is typically a small number. For example, if constrained agents were twice more productive, produce half of the total output, and have a capital share of $1/2$, then $\epsilon_{YK_2} = \frac{1}{8}$. One can in principle try to increase ϵ_{YK_2} by inducing a larger productivity gap and a larger output share, given certain plausible value for the capital share. There is a limit, however, to how much can be accomplished this way due to the trade-off between the productivity gap and the output share. Under standard concave technologies, a large productivity gap requires borrowers to hold little capital. But if borrowers hold little capital, then $\frac{Y_2}{Y}$ is small. Thus, for the model to produce large amplification ϵ_{K_2Z} must be significantly large to compensate for the small value of ϵ_{YK_2}. This means that significant output amplification requires a very large redistribution of capital toward constrained agents. However, just a large redistribution of capital toward constrained agents is not sufficient to guarantee significant output amplification.

KM show that in their model ϵ_{KZ} is significantly large, in the order of $\frac{1}{1-\beta_1}$. However, they do not discuss at all the size of ϵ_{YZ} or ϵ_{YK}. Their claims about the power of the propagation mechanism refer only to the redistributive properties of the model, but not to its ability to generate

[12] The stable root determines how fast the variables in the system return to the steady state. We use this definition of persistence because it is the one adopted by KM. However, there are other possible definitions. For example, Cogley and Nason (1995) suggest to use the autocorrelation of ouput as a better test for business cycles models.

large output response. It turns out, however, that under the assumptions used in KM, ϵ_{YK_2} can be made arbitrarily close to 1. For instance, constrained agents in their model use a linear technology which avoids the trade-off between the productivity gap and the output share.[13]

Figure 3 illustrates both the magnitude of output amplification and the size of capital redistribution in our model for different pairs $(\alpha, 1/\sigma)$, given $\beta_1 = 0.99$, $\beta_2 = 0.9 \cdot \beta_1$, and $m = 0.5$. As shown below, our main results are not sensitive to the particular choice of parameters. There are at least four important observations from this figure: (i) Output amplification is "small" (below one) for most parameter configurations. (ii) There are configurations of parameters that produce significant amplification (larger that one). They require a low EIS and large capital share. (iii) The transition between the area of low to high amplification is sharp: amplification is generally small, but it quickly changes to be very large for certain configurations of parameters. (iv) Although capital redistribution is also "small" for a large set of parameters, it responds more than output and can be quite sizeable particularly for low EIS and large α.

An additional important observation is obtained by looking more closely into the area of largest output amplification, around the hill in Figure 3. Figure 4.a. shows a top perspective of this area. The white hump-shaped area corresponds to $(\alpha, 1/\sigma)$ parameters for which there is either jagged-stable dynamics, local indeterminacy or instability. Notice how the largest amplification, which corresponds to the darkest shade, is right at the border of the hump. Thus, the configurations of parameters that produce the largest amplification are at the edge of the space of monotonic saddle-path stability.

Figure 4.b. illustrates the types of dynamic behavior generated by the parameters on the hump-shaped area. First, the top-left part of the hump corresponds to the area in which the only stable root is negative. Second, notice that most of the hump corresponds to unstable roots, i.e., the region where the steady state equilibrium is a source. Finally, there are two stretches that correspond to multiple equilibria cases, i.e. two positive stable roots, and two complex roots. We do not analyze dynamics for the parameters on the hump-shaped area for at least two reasons. First, on this region

[13] In addition to the linear technology, a low saving rate is required to generate large amplification in KM.

the EIS is implausibly low and α implausible high (see discussion below in this section). Second, this hump-shaped area can be easily eliminated by allowing agents to differ not only in β but also in α and σ (see section 4.4).

The results shown in Figures 3 and 4.a. cast doubts on the ability of collateral constraints to produce significant amplification for two main reasons. First, large amplification is not a robust result of the model. The model produces large amplification only as a "knife-edge" type of result. It requires a very particular combination of parameters at the edge of the parameter space of local determinacy. In other words, a small change in parameters, for instance a small increase in the EIS can reduce the amplification dramatically.

Second, the parameters required to generate large amplification are not empirically plausible. On the one hand, the share of collateral in the production function is probably lower than 1/3, which is approximately the capital share of output in the U.S. But the results in Figure 3 (and Figure 6 below) indicate that the capital share must be at least 0.5 in order to obtain some significant amplification. In addition, the EIS in the U.S. is probably well above 0.3, as recently documented by Vissing-Jorgensen (2002). However, the results in Figure 3 (and Figure 6 below) indicate that large amplification requires the EIS to be well below 0.2.

We now comment on the impact of α on amplification. Figure 4 shows that amplification is non-monotonic in α: it is first increasing and then decreasing. This can be seen by taking a fixed σ and moving across the α axis. At first, as α increases, amplification is higher just because the elasticity of output with respect to capital increases (see equation (12)). However, as α gets closer to 1 the amplification decreases. In order to get some intuition for this result, note first that if $\alpha = 1$, then the impatient agent would disappear from the economy. In that case the marginal productivity is constant and the steady-state users cost of capital equals β_1. Since $\beta_1 > \beta_2$, then the users cost is higher than the marginal product for impatient agents, and therefore they would not own any capital. Similarly, as α approaches 1, the share of capital owned by impatient agents decreases so that their role in the aggregate is much lower.

Figure 4 also shows that the lower the EIS, the larger the amplification. To understand this, notice first that in this economy, constrained agents smooth consumption by buying capital: they are at a corner solution so that the only way to borrow more is to buy more capital. Since borrowers' marginal product of capital is higher, aggregate output increases following the productivity shock. Now, with a lower EIS the smoothing motive becomes stronger, and constrained agents spend an even larger fraction of the unexpected resources buying capital. Thus, a lower EIS implies an even larger redistribution of capital toward the more productive agents, and a larger amplification.

Mass of unconstrained and productivity gap Up to now we have illustrated the magnitude of amplification for pairs $(\alpha, 1/\sigma)$ but for a given mass of unconstrained agents, m, and productivity ratio, β_2/β_1. How does amplification depend on m and β_2/β_1? Figure 5 illustrates this relationship for given values of α, σ and β_1. Notice that amplification is non-monotonic in β_2/β_1: it first increases and then decreases. For low β_2 the productivity gap is large but borrowers own very little capital in the economy and their effect on aggregate variables is small. Therefore, amplification effects are low. As β_2 increases, borrowers own a larger fraction of capital in the economy, and so amplification effects become more important. However, as β_2 gets closer to β_1 then the productivity differentials start to vanish, so that the amplification is small. The impact of m on the amplification is mixed but overall a small m seems to help amplification. However, m cannot be arbitrarily small because the credit market could become unstable.

It may seem important at this point to come up with some empirically plausible values for β_2/β_1 and m. However, it is hard to find convincing information about this parameters. Fortunately, we do not really need to know much about these parameters for our purposes. We can choose β_2/β_1 and m to maximize the amplification (ϵ_{YZ}) for each pair $(\alpha, 1/\sigma)$, given a plausible value for β_1. This procedure provides an upper bound for ϵ_{YZ}. If the upper bound is small, then we must conclude the model cannot generate much amplification. Figure 6 depicts the outcome of this exercise given $\beta_1 = 0.99$. It confirms that for empirically plausible values of α and σ the amplification is almost nil. Large amplification requires a very large α and a very low EIS.

3.3.2 Persistence, prices, and other variables

It turns out that persistence in the model is also generally small. Further, the region of parameters for which amplification is largest corresponds to close-to-zero persistence. This is so because the largest amplification is achieved with a substantial redistribution of capital toward borrowers, which implies a large increase in the interest rate, that in turn makes this amplification short lived.

Figure 7 presents the impulse responses of the borrowers' output Y, capital stock K_2, bond prices p, capital prices q, the users cost of capital s, and the split of Y into C_1 and C_2. For the purpose of illustration, we choose a set of parameters to enhance amplification, i.e. large α and σ. In particular, $\beta_1 = 0.99$, $\beta_2 = 0.9\beta_1$, $\alpha = 0.8$, $\sigma = 15$ and $m = 0.3$. First notice that at the time of the shock $t = 0$, Y increases by 1%, which is the magnitude of the shock, while next period $t = 1$, output reaches a maximum amplification of about 1.2%. This amplification is obtained because borrowers increase their capital holdings K_2 by about 30%. Part of this increase is explained by increase in the value of the collateral q, which increases around 30% the period after the shock. This large price increase could have produce a much larger redistribution of capital but the large increase of the interest rate, of around 20%, partially offsets the price effect.

In the period of the shock, borrowers are both consuming more and buying more capital. In fact, C_2 increases around 1%, almost the full increase in Y. Instead, lenders increase consumption very little in the period of the shock, but they wait until next period to enjoy the higher bond returns. In effect, C_1 barely increases at $t = 0$, but it is around 0.6% higher than the steady state in $t = 1$. In summary, as in KM, most of the action in this model occurs in the period of the shock and is associated to a large redistribution of capital from lenders to borrowers. This redistribution is so large that prices react substantially.

4 Extensions to the benchmark model

In this section we check the robustness of the results obtained in the benchmark economy by relaxing, one at a time, some of its assumptions.

4.1 Labor

In the benchmark model, inputs other than the collateralizable asset are kept constant. This explains the decreasing returns in the capital input. One may think that if not only capital, but also other inputs flow towards the more productive, constrained "firms", then output amplification may be larger than the one found in the benchmark economy. To address this concern this section explicitly introduces labor as another input in production. We consider two cases: in the first one, there is a labor market to which households supply labor inelastically. We show that the equalization of the marginal products of labor across firms either eliminates any amplification or leaves it the same as in the benchmark model. If labor is elastically supplied then the model performs worse than the benchmark model.

Since labor markets do not help to increase the amplification, we consider a second case with no labor market but only household work. We allow labor to be elastically supplied. We find that some small additional amplification is possible if labor supply is very elastic and wealth effects are small, but even in the extreme cases, the total amplification is still very small. Thus, our main results are robust to the introduction of labor, or other non-collateralizable factors into the model.

4.1.1 Market for labor

As in the benchmark model, the two types of households differ in their discount factor $\beta_1 > \beta_2$, and own capital k_1 and k_2, but now households can supply labor into a labor market. Each household inelastically supplies one unit of labor so that total labor supply in the economy is $\overline{H} = (1+m)$. Let w_t be the wage, l_{it} labor demand, and $f_i(k_{it}, l_{it})$ a CRS production function. The budget constraint for household i now reads

$$c_{it} + q_t(k_{it+1} - k_{it}) + a_{it} + w_t l_{it} = f_i(k_{it}, l_{it}) + p_t a_{it+1} + w_t$$

where w_t on the right-hand side is the household's labor income, and $w_t l_{it}$ its wage bill.

Introducing labor only adds the following static optimality condition to the benchmark model:

$f_i^l(k_{it}, l_{it}) = w_t$. It means that the marginal products of labor are equal across firms with positive capital,

$$f_1^l(k_{1t}, l_{1t}) = f_2^l(k_{2t}, l_{2t}) = w_t.$$

Other equations of the model are similar to the benchmark's. Consider the steady state of this economy. There is now the possibility of a corner a solution in which only one type of households employs all the capital and labor in the economy. In this case aggregate production equals $m_i f_i(\frac{\overline{K}}{m_i}, \frac{\overline{H}}{m_i})$ for some i. Clearly shocks cannot be amplified since aggregate capital and labor are constant. Consider next the case of an interior steady state. The distribution of capital in this case satisfies

$$\beta_1 f_1^k(k_1^*, l_1^*) = \beta_2 f_2^k(k_2^*, l_2^*).$$

Note that around the steady state

$$dY = f_1^k(k_1^*, l_1^*)dk_1 + f_2^k(k_2^*, l_2^*)dk_2 + f_1^l(k_1^*, l_1^*)dl_1 + f_2^l(k_2^*, l_2^*)dl_2 \tag{13}$$

and since $dk_1 = -dk_2$, $dl_1 = -dl_2$, and $f_1^l(k_1^*, l_1^*) = f_2^l(k_2^*, l_2^*)$,

$$dY = \left[f_2^k(k_2^*, l_2^*) - f_1^k(k_1^*, l_1^*)\right] dk_2$$

which can be reduced to

$$\frac{dY}{Y} = \frac{\beta_1 - \beta_2}{\beta_2} \frac{f_2^k(k_2^*, l_2^*)k_2}{Y_2} \frac{Y_2}{Y} \frac{dk_2}{k_2} = \frac{\beta_1 - \beta_2}{\beta_2} \alpha_2 \frac{Y_2}{Y} \frac{dk_2}{k_2}.$$

This equation is identical to that of the benchmark model without labor, i.e. equation (12). Thus, if total labor fixed but mobile across firms no additional amplification effects are obtained. The reason is simple: as long as labor is mobile across sectors, its marginal productivity will be equalized, and any redistribution of labor across sectors will have no additional impact in aggregate output.

If aggregate labor is not fixed but optimally supplied, then the model will perform worse than the benchmark. A one time positive aggregate shock is equivalent to a positive wealth effect. If leisure is a normal good, then households will like to work less, which reduces output. Households would be willing to supply more labor if the wage increases. However, wages can only increase if aggregate labor decreases because aggregate capital is fixed.

4.1.2 Household labor

Suppose instead that labor is not mobile across firms, but that leisure enters in the utility function, i.e., there is no market for labor, but households can decide to substitute labor for leisure. Labor choice can only help the amplification if labor is "procyclical". This means that income effects must be weak. We thus consider the following preferences, which eliminate any income effect on labor supply,

$$u_i(c_i, l_i) = \frac{1}{1-\sigma_i}\left[c_i - \frac{1}{\gamma_i} l_i^{\gamma_i}\right]^{1-\sigma_i}.$$

In addition, we assume Cobb-Douglas technologies. This combination of technologies and preferences produces

$$l_i^{\gamma_i - 1} = \text{wage} = (1-\alpha_i)\, k_i^{\alpha_i} l_i^{-\alpha_i}$$

where $\frac{1}{\gamma_i - 1}$ is the elasticity of labor supply, and so the production function for household i can be simply written as

$$f_i(k_i, l_i) = (1-\alpha_i)^{\frac{1}{\alpha_i + \gamma_i - 1}}\, k_i^{\frac{\alpha_i \gamma_i}{\alpha_i + \gamma_i - 1}}.$$

As before, to evaluate the amplification of aggregate output Y, note that around the steady state

$$dY = f_1^k(k_1^*, l_1^*)dk_1 + f_2^k(k_2^*, l_2^*)dk_2 + f_1^l(k_1^*, l_1^*)dl_1 + f_2^l(k_2^*, l_2^*)dl_2$$

and since in this model $dk_1 = -dk_2$ and

$$dl_i = \frac{\alpha_i}{\alpha_i + \gamma_i - 1}\frac{l_i}{k_i}dk_i$$

then

$$dY = \left[f_2^k(k_2^*, l_2^*) - f_1^k(k_1^*, l_1^*)\right] dk_2 + \frac{(1-\alpha_1)}{\alpha_1 + \gamma_1 - 1} f_1^k(k_1^*, l_1^*) dk_1 + \frac{(1-\alpha_2)}{\alpha_2 + \gamma_2 - 1} f_2^k(k_2^*, l_2^*) dk_2.$$

Assume for simplicity that $\alpha_2 = \alpha_1$ and $\gamma_1 = \gamma_2$. Then

$$\frac{dY}{Y} = \frac{\gamma}{\alpha + \gamma - 1}\left[\frac{\beta_1 - \beta_2}{\beta_2} \alpha \frac{Y_2}{Y} \frac{dk_2}{k_2}\right].$$

Thus, output response to an exogenous shock in this model corresponds to the constant $\frac{\gamma}{\alpha+\gamma-1}$ times the same four components of the benchmark model as in equation (12). When the elasticity of labor supply is zero, i.e. $\gamma_i \to \infty$, the constant approaches 1, and we have the exact same formula as in the benchmark model with no labor. At the other extreme, a perfectly elastic labor supply, i.e. when $\gamma_i \to 1$, produces an upper bound for output amplification given by

$$\left(\frac{dY}{Y}\right)_{\max} = \frac{1}{\alpha}\left[\frac{\beta_1 - \beta_2}{\beta_2} \alpha \frac{y_2}{y} \frac{dk_2}{k_2}\right] \tag{14}$$

This equation states that for a given redistribution of capital, dk_2/k_2, a model with perfectly elastic labor supply and zero wealth effects on labor produces $\frac{1}{\alpha}$ times the amplification of the benchmark model.[14] Thus, this model can only increase the amplification significantly if α is small, or the labor share is large. But, if α is small then the amplification of the benchmark model is almost nil as shown before. If for instance $\alpha = 1/3$, then this model would generate up to three times more amplification than the benchmark model. Recall though that when $\alpha = 1/3$ the amplification generated by the benchmark was very small. Similarly, when α approaches 1, the

[14] Another popular utility function used in the business cycles literature is of the form $lnc - Al$ (see Hansen, 1985). It can be shown that the output amplification generated by this function is

$$\frac{dY}{Y} = \frac{1}{\alpha}\left[\frac{\beta_1 - \beta_2}{\beta_2} \alpha \frac{y_2}{y} \frac{dk_2}{k_2}\right] - \epsilon$$

where

$$\epsilon = \frac{1-\alpha}{\alpha}\frac{1}{\alpha}\left(f_k^1(k_1^*, l_1^*) k_1 \frac{dc_1}{c_1} + f_k^2(k_2^*, l_2^*) k_2 \frac{dc_2}{c_2}\right).$$

Thus, this utility function does not attain the upper bound for output amplification because even though labor supply is infinitely elastic, income effects on labor are present.

amplification would be the same as in the benchmark, which could be large only if the elasticity of intertemporal substitution in consumption is close to zero.

In conclusion, even considering the best case scenario with perfectly elastic labor supply and no wealth effect on labor supply, the amplification is still small. If in addition, one takes into account that a very large elasticity of labor supply is not empirically plausible and that wealth effects are important, then it is safe to conclude that introducing endogenous labor supply does not change the conclusions on output amplification obtained for the benchmark model.[15]

4.2 Reproducible capital

One of the assumptions of our benchmark model is that capital $\overline{K}$ is non reproducible. As mentioned before, this assumption helps the model generate larger amplification. The intuition is simple: if the quantities of the collateralizable asset cannot adjust, then all the effect of a positive unexpected productivity shock is reflected in the asset prices. In the benchmark model, a large change in asset prices goes along with a large redistribution of capital toward constrained agents, and the corresponding increase in aggregate output.

In contrast, if capital is reproducible, then the effect of the shock on the price of capital will be smaller. However, one could argue that in this case the additional investment can increase output. Thus, it is a quantitative question whether or not the increase in output due to additional investment is as large as the one achieved with fixed capital supply and higher capital prices. In this section we address this question by allowing capital to be reproducible. We find that capital accumulation significantly reduces the amplification.

Suppose agent i can allocate his capital k_i to produce either consumption or investment goods.[16]

[15] Empirical studies document that in the U.S. the elasticity of labor supply for men is close to zero (Pencavel, 1987). Although the one for women is higher, the real business cycle literature has used an elasticity of about 1.7 (Greenwood, J. et al, 1988).

[16] For the purpose of comparison, we choose to allow each agent to produce investment goods in order to avoid creating a rental market for capital. This market is absent in KM as well as in our benchmark economy.

As before, let q be the price of capital in terms of consumption goods. Thus, agent i maximizes

$$F(k_i) \equiv \max_{0 \leq k_{ic} \leq k_i} k_{ic}^{\alpha} + q\,(k_i - k_{ic})^{\alpha}$$

where k_{ic} is the capital allocated to the production of consumption goods. Notice that both production functions exhibit decreasing returns[17] and, for simplicity, we have assumed that the functions are identical. This assumption simplifies the solution without limiting our ability to evaluate how output amplification is affected when the cost of producing capital is low, i.e. when α is large. Recall that, in contrast, in the benchmark economy the cost of producing capital could be thought of as being very high, since the supply of capital was fixed at $\overline{K}$. Further, recall that when α is large, but not too close to 1, output amplification is large.

In any interior solution, the marginal products must be equal, i.e. $k_{ic}^{\alpha-1} = q\,(k_i - k_{ic})^{\alpha-1}$, which implies that total production by individual i can be written as

$$F(k_i) = [\theta^{\alpha} + q\,(1-\theta)^{\alpha}]\, k_i^{\alpha} \equiv A(q) k_i^{\alpha}$$

where

$$\theta(q) = \frac{1}{1 + q^{\frac{1}{1-\alpha}}}.$$

This expression simplifies the problem substantially, as the budget constraint for agent i can be simply written as

$$c_{it} + q_t(k_{it+1} - k_{it}) + a_{it} = A_t k_{it}^{\alpha} + p_t a_{it+1} - \delta q_t k_{it}$$

where δ is the depreciation rate.

Optimality conditions are the same as in the benchmark model, except that the first order condition for k_{it+1} now reads

$$u_i'(c_{it}) s_t = \beta_i u_i'(c_{it+1}) \left[A(q_{t+1}) f'(k_{it+1}) - \delta q_{t+1} \right].$$

[17] This feature is similar to the common assumption of convex adjustment costs for investment.

As in the benchmark economy, impatient agents are credit constraint in the steady state, and thus $p^* = \beta_1$. Since by definition $s^* = q^*(1 - \beta_1)$, using the optimality condition for capital accumulation we have

$$q^*(1 - \beta_1 + \delta\beta_i) = \beta_i A^* f'(k_i^*)$$

which implies

$$\frac{f'(k_2^*)}{f'(k_1^*)} = \frac{\beta_1}{\beta_2}\frac{1 - \beta_1 + \delta\beta_2}{1 - \beta_1 + \delta\beta_1} > 1$$

so that when capital is reproducible, it is still the case that borrowers have a higher marginal product of capital than lenders.

Since no analytical results can be found, we solve the model numerically. Figure 8 illustrates impulse responses from a one-time 1% productivity shock. For purpose of comparison, this figure uses the same parameters as Figure 7, and $\delta = 0.1$. The figure confirms that once capital is reproducible, the powerful effect of asset prices on agents' net worth is lost. In fact, the maximum increase in q is only about 1%, and it occurs in the period of the shock. This increase in q triggers an increase in investment, which allows output to increase by more than the shock. The maximum increase in output, which occurs in the period of the shock, is only about 1.4%. Notice that out of this 1.4%, 1% is due to the exogenous shock, so that only 0.4% can be considered amplification generated by the model. This is small compared to the 1.2% obtained in Figure 7, when capital was not reproducible. In the period after the shock, there is still redistribution of capital toward borrowers, but it is very small compared to the benchmark economy. Finally, notice that there is almost no persistence: the effects of the shock almost disappear after two periods.

4.3 A stochastic model

In our benchmark analysis we considered a dynamic economy that, at the aggregate level, was deterministic, and then evaluated the effects of a small, one-time unanticipated shock. The idea of this analysis was that only when shocks are small and uncorrelated, one can truly evaluate how much more amplification and persistence can be achieved from the endogenous propagation mechanism of the model. However, it remains the question of what would happen in a fully stochastic model in which shocks were not a zero-probability event, and could be rationally anticipated as in the standard real business cycles models.

One issue that emerges when shocks can be rationally anticipated is whether or not contingent debt contracts can be written. If this was the case, then the amplification and persistence obtained in our benchmark model will be lessened. This is so because contingent contracts would avoid the large redistribution of capital that lies at the heart of output amplification. Thus, the amplification effects found in the deterministic economy can potentially be preserved only when contingent debt contracts are not possible. In this section we solve a stochastic version of our benchmark economy with non-contingent debt contracts. We then analyze impulse responses from shocks, and compute some standard business cycle statistics by generating a series of shock realizations.

One of the issues in writing a stochastic version of the benchmark economy is how to specify the collateral constraint. First, notice that in our deterministic benchmark model we followed the assumption of KM that the shock occurs after the agents have decided to stay or leave. This implies is that it is too late for the agent to repudiate his debt contract after the shock, and thus borrowers will always pay back the amount they agreed to in the previous period. We maintain this assumption in the stochastic economy. This implies that ex-post, agents pay what they agreed to regardless of how the collateral constraint is specified.

Since it is beyond the scope of this paper to solve for the optimal collateral constraint in a stochastic environment, here we consider the case in which credit is limited by the expected value of the collateral. We are aware that this is not the optimal credit limit because agents are risk averse, but taking the expected value of the collateral is a reasonable counterpart of the deterministic case.[18]

[18]Since agents are risk averse, we would expect the optimal limit on credit to be below the expected value of collateral. Ultimately, what this limit affects is the amount of redistribution of collateral that occurs following the shock. In particular, if the limit on credit is very tight, a positive productivity shock would imply more redistribution of capital toward borrowers than if the credit limit is looser.

We assume that the productivity shock follows an AR(1) process given by

$$z_t = (1 - \rho_z)z^* + \rho_z z_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon)$. We choose σ_ε small enough in order to ensure that the collateral constraint for borrowers binds, as it does in the steady state. Figure 9 illustrates impulse responses from a 1% productivity shock when $\rho_z = 0.9$ and $\sigma_\varepsilon = 1$. In this simulation we used the same parameters as in Figure 7. It can easily be seen that qualitatively, the two figures are very similar. Quantitatively, notice that the maximum deviation obtained for aggregate output in Figure 9 is about 2.8%, versus 1.2% obtained in Figure 7. Notice that the 2.8% obtained in Figure 9 is not purely endogenous: we can decompose it into an exogenous part of 0.9%, which is simply the autocorrelation of the shock, and the remaining 1.9% which can be attributed to the endogenous propagation mechanism of the model. Thus, when shocks are rationally anticipated and debt contracts are non-contingent, we still obtain some amplification, but again, it is small as in the benchmark model.

Up to now we have measured amplification as the elasticity of output in period $t = 1$ with respect to a one-time shock in period $t = 0$. Using the stochastic model, we can now compute population moments and use as a measure of amplification the ratio of standard deviation of output to shock. We computed this ratio for the cases in which $\rho_z = 0.9$ and $\rho_z = 0$. It turns out that if $\rho_z = 0.9$, so that $\sigma_z = 2.294$, the standard deviation of output is $\sigma_Y = 3.365$. In contrast, if $\rho_z = 0$, case in which $\sigma_z = 1$, the standard deviation of output is $\sigma_Y = 1.532$. Thus, when the shock is autocorrelated, amplification is $\sigma_Y/\sigma_z = 1.467$ versus 1.532 when it is not. Once again, these numbers indicate small amplification, even when we have chosen parameter values that would help the model achieve larger amplification.

4.4 Asymmetric Case

Up to this point we have discussed simulations in which agents only differ in their discount factors. One of the conclusions from these simulations is that large amplification can be obtained with a low EIS, and a large, but not too-close-to-one capital share. The evidence on the value of σ is controversial, and some of it indicates that at least for a set of agents in the economy, the EIS is very close to zero (Guvenen, 2002). Thus, an interesting exercise would be one in which we allow agents to differ in σ. In particular, in order to "help the model" generate large amplification, we would like borrowers to have a low EIS.

Another interesting simulation is to allow for different α's across borrowers and lenders. If we want to "help the model" generate large amplification, then we can let borrowers have a high α, and lenders a low α, so that the aggregate capital share is consistent with the empirical evidence. Figure 10 shows the amplification achieved when agents differ in β, σ and α. In particular, $\alpha_1 = 0.3$, $\sigma_1 = 0.1$, $m = 0.5$, $\beta_1 = 0.99$, and $\beta_2 = 0.9\beta_1$. This figure confirms our previous finding that large amplification is a knife-edge type of result: it requires a very large α_2 and a very low EIS for the credit-constrained agent. Small variations on these parameters imply a sharp reduction in amplification. Finally, notice that all parameter combinations in Figure 10 guarantee local determinacy with non-jagged dynamics, i.e. the hump-shaped area of Figure 4.a. has disappeared.

5 Concluding comments

The purpose of this paper is to evaluate the role of collateral constraints as an amplification mechanism of exogenous shocks to the economy. In particular, we analyze several stylized models that incorporate the main mechanism proposed by KM. According to this mechanism, what causes amplification is the fact that a group of agents in the economy are credit-constrained and have a higher marginal product of capital. Thus, adverse shocks to the net worth of constrained agents negatively affect investment in collateral, output and asset prices. The fall in the value of the collateral worsens the downturn because it further limits the ability of constrained agents to borrow.

We analyze how amplification changes for different parameters when we allow for standard utility and production functions. Our approach is to "help the model" generate amplification by analyzing equilibrium paths along which a group of agents is always against the constraint. The idea is that if even under these "favorable" conditions the model does not generate amplification, then it would be difficult for more general, less-stylized models with collateral constraints to do so.

As the simulations indicate, large amplification can be obtained only under the “right” combination of a low EIS, and a large but not too-close-to-one capital share. Large amplification is a “knife-edge” result because it is not robust to small changes in parameters. In addition, for typical parameter values used in the business cycle literature the amplification is close to zero. These results are robust to the inclusion of labor, aggregate investment, probabilistic shocks and differences in EIS and collateral shares across agents.

Our findings would still hold if agents were heterogenous in other dimensions. Here we introduce heterogeneity in the discount factors, but this is nonessential. Any heterogeneity that induces differences in productivity across agents would produce similar results. This is so because the fundamental channel to produce amplification is the redistribution of a productive asset from lower to higher-productivity agents. In general, when technology exhibits marginal decreasing returns in the productive asset, the largest output amplification would be attained when this asset is transferred to agents who hold a very small fraction of it. However, by the same token, since high-productivity agents hold a very small fraction of the productive asset, their impact on aggregate production is small. All in all, our results show that collateral constraints by themselves are not enough to account for the large fluctuations of output observed in the data.

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A Proof of Proposition 3

We derive a system of two difference equations in two variables: k_2 and the ratio of aggregate consumptions $z \equiv \frac{mc_1}{c_2}$. To write the system in a compact way, let $g_{zt} \equiv \frac{z_{t+1}}{z_t}$ and $g_{kt} = \frac{k_{2t+1}}{k_{2t}}$. Using the optimality condition (4) for each agent assuming isoelastic utility, and $f(k) = k^\alpha$, one can obtain

$$\frac{z_{t+1}}{z_t} = g_z(k_{2t+1}) := \left(\frac{\beta_1}{\beta_2}\right)^{1/\sigma} \left(\frac{k_{2t+1}}{\left(\overline{K} - k_{2t+1}\right)/m}\right)^{(1-\alpha)/\sigma}. \tag{15}$$

Next, aggregating the budget constraints for each agent when borrowers are constrained, and using equation (4) for borrowers it can be shown that

$$\frac{mf\left(\left(\overline{K} - k_{2t}\right)/m\right)}{f\left(k_{2t}\right)} = z_t - \left(\frac{F(k_{2t})}{F(g_{kt}k_{2t})}\right)^\sigma \left(1 + g_z\left(g_{kt}k_{2t}\right)z_t\right)^\sigma \left(1 + z_t\right)^{1-\sigma} \beta_2 \alpha g_{kt}^\alpha \tag{16}$$

where

$$F(k_{2t}) \equiv mf\left(\frac{K - k_{2t}}{m}\right) + f(k_{2t}).$$

B Phase diagram

In this appendix we use a phase diagram to analyze the global dynamics of the benchmark model. Since we are analyzing a system of difference equations, the economy will not move continuously along the trajectories, but rather jump from point to point on the trajectory. For the case of discrete time one has to additionally check that the steady state is locally a saddle path to preclude unstable equilibria that may arise along the path that converges to the steady state. Conditions for local saddle path stability are derived below.

B.1 Stationary curves

There are two stationary curves in the figures: $g_z \equiv \frac{z_{t+1}}{z_t} = 1$ and $g_k = \frac{k_{2t+1}}{k_{2t}} = 1$. From now on we omit time subscripts whenever it is not confusing. Consider first the locus of points (z, k_2) such that $g_z = 1$. According to (15) these points are given by the steady state level of k_2

$$1 = \frac{\beta_1 f_1'\left(\left(\overline{K} - k_2\right)/m\right)}{\beta_2 f_2'(k_2)} \tag{17}$$

so that $g_z = 1$ is a vertical straight line at k_2^*.

Next, consider the locus of points (z, k_2) such that $g_k = 1$. According to (16) this locus is given by

$$\underbrace{\frac{mf\left(\left(\overline{K} - k_2\right)/m\right)}{f\left(k_2\right)}}_{LHS(k_2)} = \underbrace{z - \alpha\beta_2\left(1 + g_z(k_2)z\right)^\sigma \left(1 + z\right)^{1-\sigma}}_{RHS(k_2,z)} \tag{18}$$

where we denote the mapping from k_2 to z implicitly defined by this equation as $z = h(k_2)$. The following lemma summarizes some important properties of this map:

Lemma 1A $h(0) = +\infty$ and $h(k_2) > 0$ for all $k_2 \in [0, K^e]$

Proof It follows from the following observations: (i) $LHS(0) = +\infty$; (ii) $RHS(0,z) = z - \alpha\beta_2 (1+z)^{1-\sigma}$ so that

$$\frac{\partial RHS(0,z)}{\partial z} = 1 + \frac{\alpha\beta_2}{(1+z)^\sigma}(\sigma - 1) > 1 - \frac{\alpha\beta_2}{(1+z)^\sigma} > 0.$$

(iii) $\underset{z\to\infty}{lim} RHS(0,z) = \underset{z\to\infty}{lim} z(1 - \alpha\beta_2 \left(\frac{1+z}{z}\right)^{1-\sigma} z^{-\sigma}) = \underset{z\to\infty}{lim} z(1 - \alpha\beta_2 z^{-\sigma}) = +\infty$; (iv) $\underset{z\to 0}{lim} RHS(k_2, z) = \underset{z\to 0}{lim} z - \alpha\beta_2 \left(\frac{1+g_z(k_2)z}{1+z}\right)^\sigma (1+z) = -\alpha\beta_2$. This means that $z = 0$ is not a solution for any $k_2 \in [0, k^e)$ since $LHS(k_2)$ is always positive. (v) $LHS(k^e) = m$; (vi) $RHS(k^e, z) = z - \alpha\beta_2 \left(\frac{1+(\beta_1/\beta_2)^{1/\sigma}z}{1+z}\right)^\sigma (1+z)$. ■

Next, in order to determine the slope of $g_k = 1$, use the implicit function theorem to obtain

$$h'(k_2) = \frac{\partial z}{\partial k_2} = \frac{\partial LSH/\partial k_2 - \partial RHS/\partial k_2}{\partial RHS/\partial z}$$

First, it can be shown that

$$\frac{\partial RHS}{\partial z} = 1 + \alpha\beta_2 \left(\frac{1+g_z z}{1+z}\right)^\sigma \left(\frac{\sigma(1-g_z)}{1+g_z z} - 1\right) \tag{19}$$

and to determine the sign of this derivative, we consider the following cases: (1) if $g_z = 1$, then

$$\frac{\partial RHS}{\partial z} = 1 - \alpha\beta_2 > 0;$$

(2) if $0 < g_z < 1$, then

$$\frac{\partial RHS}{\partial z} > 1 - \alpha\beta_2 > 0;$$

(3) if $g_z > 1$, then $\frac{\partial RHS}{\partial z} > 0$ requires

$$1 > \alpha\beta_2 \left(\frac{1+g_z z}{1+z}\right)^\sigma \left(\frac{\sigma(g_z - 1)}{1+g_z z} + 1\right)$$

where the right hand side of this expression is strictly increasing in g_z if $\sigma \geq 1$. In that case there exist a level $g_z(k_2;\sigma) > 1$ such that $\frac{\partial RHS}{\partial z} = 0$. Denote $\widetilde{K}$ the level of k_2 such that $\frac{\partial RHS}{\partial z} = 0$. Thus, we assume that $k_{20} < \widetilde{K}$ is such that $g_z < g_z(\sigma, z)$ so that $\frac{\partial RHS}{\partial z} > 0$.

To determine the sign of $h'(k_2)$, we still need to examine $\frac{\partial LHS}{\partial k_2}$ and $\frac{\partial RHS}{\partial k_2}$. By simple inspection $\frac{\partial LHS}{\partial k_2} < 0$. In addition, since from 15 $g_z'(k_2) > 0$, then $\frac{\partial RHS}{\partial k_2} < 0$. Thus, the sign of the numerator of $h'(k_2)$ is not clear. It can be shown that

$$\partial LSH/\partial k_2 = -\alpha LHS \left(\left(\overline{K} - k_2\right)^{-1} + k_2^{-1}\right) < 0$$

and

$$\partial RHS/\partial k_2 = (RHS - z)(1 + g_z(k_2)z)^{-1}(1-\alpha)\left(k_2^{-1} + \left(\overline{K} - k_2\right)^{-1}\right) g_z(k_2)z < 0$$

and after some algebra we find

$$\partial LSH/\partial k_2 - \partial RHS/\partial k_2 \sim \beta_2 \frac{g_z(k_2)z + \alpha}{z}\left(\frac{1 + g_z(k_2)z}{1+z}\right)^{\sigma-1} - 1.$$

To determine the sign of $h'(k_2)$, let us first consider the sign of the expression above in the steady state, i.e. when $g_z(k_2) = 1$, where it reads

$$\beta_2 \frac{z+\alpha}{z} - 1$$

which is positive as long as

$$\frac{\alpha\beta_2}{1-\beta_2} > z = \frac{m\left(\beta_1/\beta_2\right)^{\frac{\alpha}{1-\alpha}} + \alpha\beta_2}{1-\alpha\beta_2}$$

or as long as

$$\frac{(1-\alpha)\,\alpha\,(\beta_2)^2}{1-\beta_2}\left(\frac{\beta_2}{\beta_1}\right)^{\frac{\alpha}{1-\alpha}} > m.$$

In conclusion, if m is low enough, then $\partial LSH/\partial k_2 - \partial RHS/\partial k_2 > 0$ in the steady state, and so the slope of the implicit function $h(k_2)$ is positive when it crosses the steady state. In other words, the slope of the curve $g_k = 1$ is positive when it crosses the curve $g_z = 1$. Similarly, if m is high enough so that the condition above is violated, then the slope of the curve $g_k = 1$ is negative when it crosses the curve $g_z = 1$.

Now, what happens with the slope of $g_k = 1$ out of the steady state? We know from (15) that $g_z(k_2) \gtreqless 1$ for $k_2 \gtreqless k_2^*$. Then, $\beta_2 \frac{g_z(k_2)z+\alpha}{z}\left(\frac{1+g_z(k_2)z}{1+z}\right)^{\sigma-1} - 1 \gtreqless \beta_2 \frac{z+\alpha}{z}\left(\frac{1+z}{1+z}\right)^{\sigma-1} - 1$ for $k_2 \gtreqless k_2^*$. This holds for both $\sigma > 1$ and $\sigma < 1$. What this implies is that if the slope of the curve $g_k = 1$ is positive when it crosses the curve $g_z = 1$, then this slope will be increasing after the steady state, i.e. for $k_2 > k_2^*$. On the other hand, if the slope of the curve $g_k = 1$ is negative when it crosses the curve $g_z = 1$, then this slope will become flatter for $k_2 > k_2^*$. Finally, recall that $h(0) = +\infty$, so that as k_2 moves away from 0 toward k_2^*, the curve $g_k = 1$ has a negative slope.

As can be seen in Figure 2.a., function $g_k = 1$ changes at $k_2 = K^e = \frac{\overline{K}}{1+m}$, which is the efficient capital allocation, i.e. the one that would take place if there were no collateral constraints. As shown in Proposition 2 in the paper, as long as $k_2 < K^e$, impatient agents will be credit constrained, and when $k_2 > K^e$ then patient agents will become credit constrained. Up to now we have characterized $g_{k_2} = 1$, but $g_{k_1} = 1$ can be drawn using analogous derivations. Figure 2.a. portrays the case of a decreasing curve $g_{k_2} = 1$, i.e. the case of high enough m. The slope of this curve is not critical for the results.

B.2 Arrows

We now want to determine how z changes when k_2 is below and above $g_z = 1$, and how k_2 changes when z is below and above $g_k = 1$. From equation (15) we have

$$g_z'(k_2) = g_z(k_2)\frac{1-\alpha}{\sigma}\left(k_2^{-1} + \left(\overline{K} - k_2\right)^{-1}\right) > 0$$

Thus, if k_2 is below $g_z = 1$, then z is decreasing; and z increases when k_2 is above $g_z = 1$. Next, to determine how k_2 changes when z is below and above $g_k = 1$, rewrite (7) as

$$\underbrace{\frac{mf\left(\left(\overline{K} - k_2\right)/m\right)}{f(k_2)}}_{LSH} = \underbrace{z - \left(\frac{F(k_2)}{F(g_k k_2)}\right)^{\sigma} (1 + g_z(g_k k_2)z)^{\sigma}\, \beta_2 \alpha g_k^{\alpha}\, (1+z)^{1-\sigma}}_{RHS} \tag{20}$$

so that

$$\frac{\partial g_k}{dz}|_{k_2} = -\frac{\frac{\partial RHS}{\partial z}}{\frac{\partial RHS}{\partial g_k}}.$$

From equation (19) we see that near the curve $g_k = 1$, it is the case that $\frac{\partial RHS}{\partial z} > 0$. On the other hand, $\frac{\partial RHS}{\partial g_k} \gtrless 0$. If $\frac{\partial RHS}{\partial g_k} < 0$ then $\frac{\partial g_k}{dz}|_{k_2} > 0$, which implies that if z is below $g_k = 1$, then k_2 is decreasing, and k_2 increases when z is above the curve $g_k = 1$. In this case, the system exhibits global saddle-path stability, as portrayed in Figure 2.a. Appendix C formally derives the conditions under which $\frac{\partial RHS}{\partial g_k} < 0$ and so $\frac{\partial g_k}{dz}|_{k_2} > 0$. Finally, notice that all diverging paths in Figure 2.a. can be ruled out because they do not satisfy transversality conditions. Consider the cases to the right of $g_z = 1$ and below $g_{k2} = 1$. One divergent path implies $k_2 \to 0$ and z jumping to $+\infty$ (since $c_2 = 0$ once $k_2 = 0$) which is not optimal neither for agent 1 nor 2. A second case implies $k_2 \to \widetilde{k}_2 > 0$ and $z \to 0$. This is not optimal for agent 1 since k_1 remains bounded away from zero but $c_1 \to 0$. Symmetric arguments apply for the other divergent paths in the graph.

C Proof of Lemma 4

From (20) it can be shown that around the curve $g_k = 1$,

$$\frac{\partial RHS}{\partial g_k} = -(z - RHS)\left[(1-\alpha)\frac{C_1}{C}\frac{\overline{K}}{K_1} + \alpha - \sigma\frac{\left(f'(k_2) - f'(k_1)\right)K_2}{F(K_2)}\right] \tag{21}$$

which implies that a necessary and sufficient condition for $\frac{\partial RHS}{\partial g_k} < 0$ or, as discussed in Appendix B, a necessary and sufficient condition for global saddle-path stability is

$$\left((1-\alpha)\frac{C_1}{C} + \alpha\frac{K_1}{\overline{K}}\right)\frac{1}{\sigma} > \alpha\frac{K_1}{\overline{K}}\frac{Y_2}{Y}\left(\frac{f'(k_2) - f'(k_1)}{f'(k_2)}\right).$$

A sufficient condition for this to be true is

$$\left((1-\alpha)\frac{Y_1}{Y} + \alpha\frac{K_1}{\overline{K}}\right)\frac{1}{\sigma} > \alpha\frac{K_1}{\overline{K}}\frac{Y_2}{Y}\left(\frac{f'(k_2) - f'(k_1)}{f'(k_2)}\right).$$

Define $\tau = \frac{f'(k_1)}{f'(k_2)} = \left(\frac{k_2}{k_1}\right)^{1-\alpha}$, so that this sufficient condition can be written as

$$\left((1-\alpha)\frac{m\tau^{\frac{\alpha}{\alpha-1}}}{1+m\tau^{\frac{\alpha}{\alpha-1}}} + \alpha\frac{1}{1+\frac{1}{m}\tau^{\frac{1}{1-\alpha}}}\right)\frac{1}{\sigma} > \alpha\frac{1}{1+\frac{1}{m}\tau^{\frac{1}{1-\alpha}}}\frac{1}{1+m\tau^{\frac{\alpha}{\alpha-1}}}(1-\tau)$$

or, after some simplifications,

$$\frac{\alpha(1-\tau) + m\tau^{\frac{\alpha}{\alpha-1}} + \tau}{\alpha(1-\tau)} > \sigma.$$

D Proof of Proposition 5

Log-linearizing equations (15) and (16) around the steady state gives the following solution in matricial form:

$$\underbrace{\begin{bmatrix} -\frac{1-\alpha}{\sigma}\frac{C_1}{C}\frac{\overline{K}}{K_1} & 1 \\ \frac{\alpha}{\sigma}F - F'k_2 & F \end{bmatrix}}_{A}\begin{bmatrix} \widehat{k}_{2t+1} \\ \widehat{1+z}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{1}{\sigma\beta_2}\frac{Y_1}{Y_2}C_2\frac{\overline{K}}{K_1} - \left(F'k_2 - \frac{\alpha}{\sigma}F\right) & F\left(\frac{1}{\sigma\alpha\beta_2} - \frac{1-\sigma}{\sigma}\right) \end{bmatrix}}_{B}\begin{bmatrix} \widehat{k}_{2t} \\ \widehat{1+z}_{t} \end{bmatrix}$$

where $\widehat{x} = d\log x$. The basic dynamic system can be solved as $\begin{bmatrix} \widehat{k}_{2t+1} \\ \widehat{1+z}_{t+1} \end{bmatrix} = A^{-1}B\begin{bmatrix} \widehat{k}_{2t} \\ \widehat{1+z}_{t} \end{bmatrix}$. The characteristic equation associated to $A^{-1}B$ is given by roots of the following polynomial

$$\pi(\omega_x) \equiv \theta_1\omega^2 + \theta_2\omega + \theta_3$$

where

$$\theta_1 = F'k_2 - \frac{\alpha}{\sigma}F - \frac{1-\alpha}{\sigma}C_1\frac{\overline{K}}{K_1}$$

$$\theta_2 = -2\left(F'k_2 - \frac{\alpha}{\sigma}F\right) + \frac{1}{\sigma\beta_2}\frac{Y_1}{Y_2}C_2\frac{\overline{K}}{K_1} - \frac{1-\alpha}{\sigma}C_1\frac{\overline{K}}{K_1}\left(\frac{1-\sigma}{\sigma} - \frac{1}{\alpha\sigma\beta_2}\right)$$

$$\theta_3 = \left(F'k_2 - \frac{\alpha}{\sigma}F\right) - \frac{1}{\sigma\beta_2}\frac{Y_1}{Y_2}C_2\frac{\overline{K}}{K_1}.$$

Condition (21) evaluated at steady state implies that $\theta_1 < 0$ To see this rewrite this condition around the steady state as

$$\frac{\partial RHS}{\partial g_k} = \frac{\beta_2\alpha(1+z)\sigma}{F(k_2)}\left[\left(f'(k_2) - f'(k_1)\right)k_2 - \frac{\alpha}{\sigma}F(k_2) - \frac{(1-\alpha)}{\sigma}C_1\frac{\overline{K}}{K_1}\right] = \frac{\beta_2\alpha(1+z)\sigma}{F(k_2)}\theta_1$$

In addition,

$$\pi(1) = -\frac{1-\alpha}{\sigma}C_1\frac{\overline{K}}{K_1} - \frac{1-\alpha}{\sigma}C_1\frac{\overline{K}}{K_1}\left(\frac{1-\sigma}{\sigma} - \frac{1}{\alpha\sigma\beta_2}\right) = -\frac{1-\alpha}{\sigma^2}C_1\frac{\overline{K}}{K_1}\left(1 - \frac{1}{\alpha\beta_2}\right) > 0$$

Therefore $\theta_1 < 0$ and $\pi(1) = \theta_1 + \theta_2 + \theta_3 > 0$ Thus, $\pi(\omega)$ is positive at $\omega = 1$ and eventually diverges to $-\infty$ as $\omega \rightarrow \infty$. Unless $\pi(0) < 0$, there is no positive root less than one. Thus, $\theta_3 < 0$ is required. Using the definition of θ_3, the latter condition implies

$$\sigma < \frac{1}{(\beta_1 - \beta_2)} \frac{\beta_1 C_1^* + \beta_2 C_2^*}{s^* K_2^*}$$

or after substituting the steady state values and rearranging gives

$$\sigma < 1 + \frac{1 + m\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1-\alpha}}}{\alpha\left(\beta_1 - \beta_2\right)}.$$

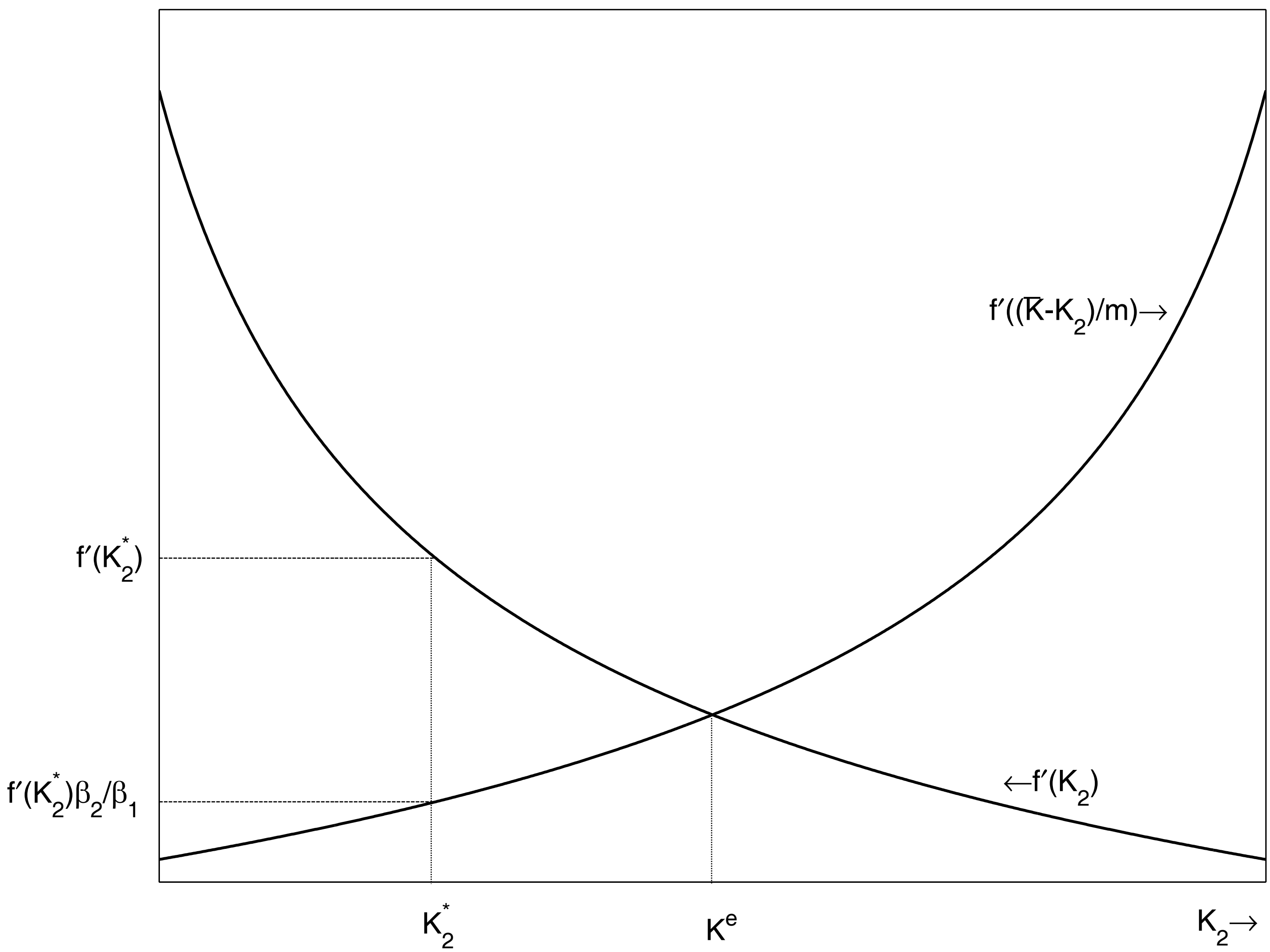


Figure 1: Steady State Distribution of Collateral

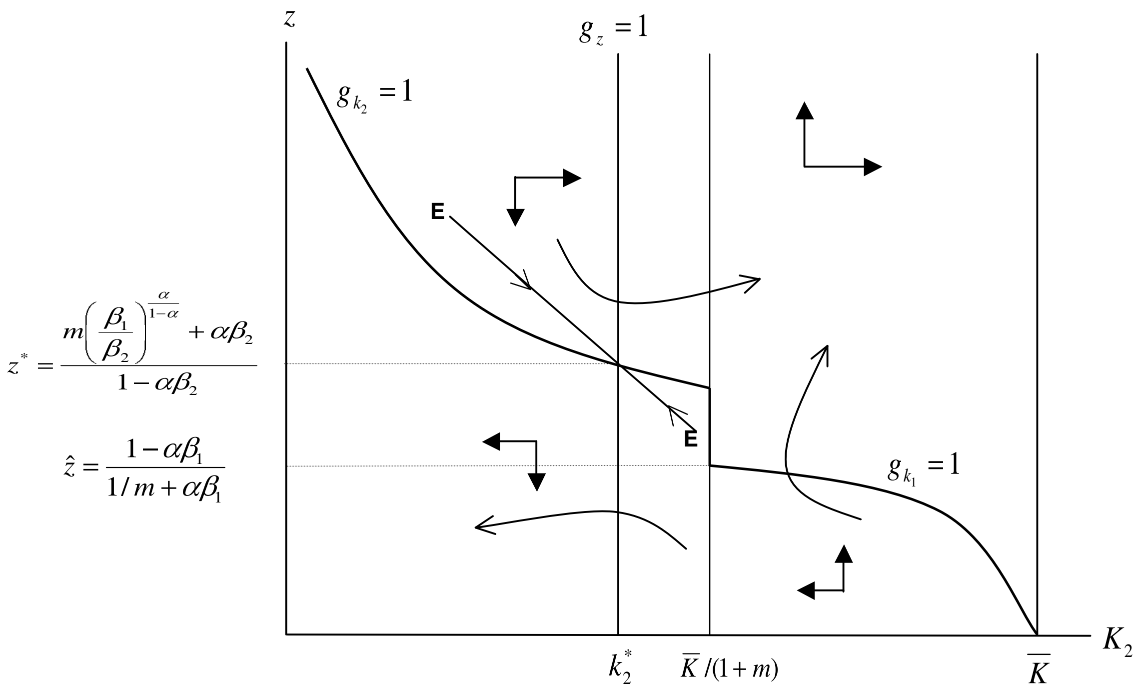


Figure 2a: Phase Diagram for Standard EIS

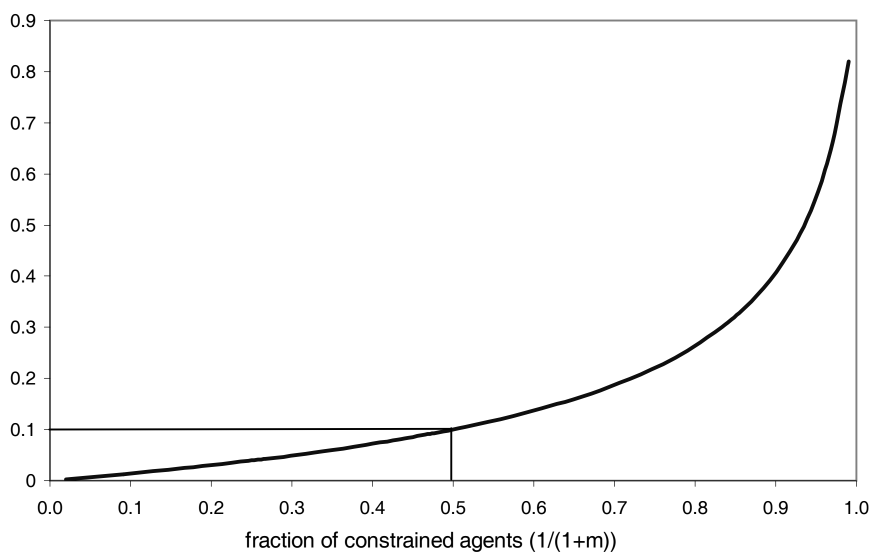


Figure 2b: Lower Bound for the EIS to Guarantee Global Determinacy

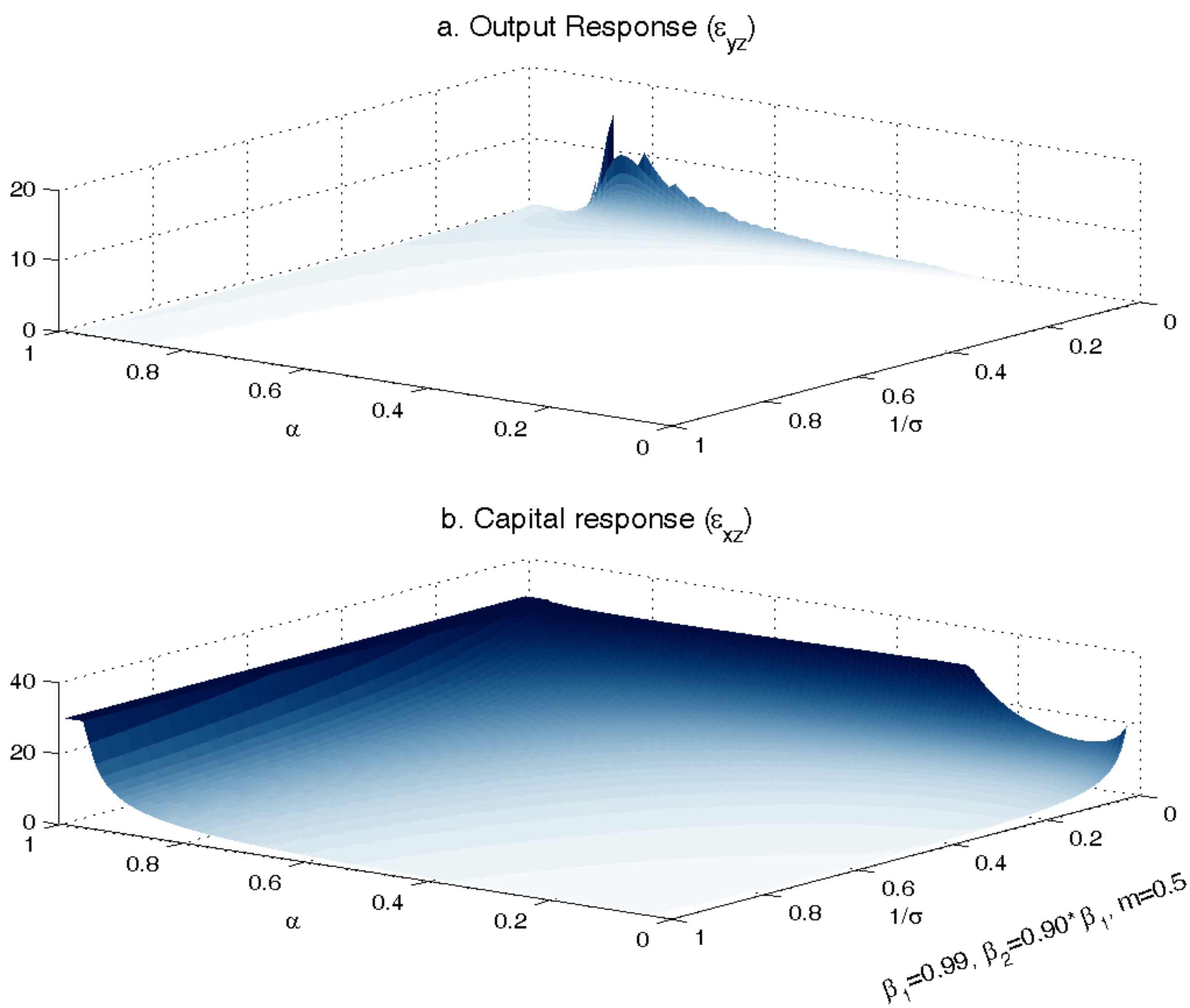


Figure 3: Response of Ouput and Collateral to an Unanticipated Shock

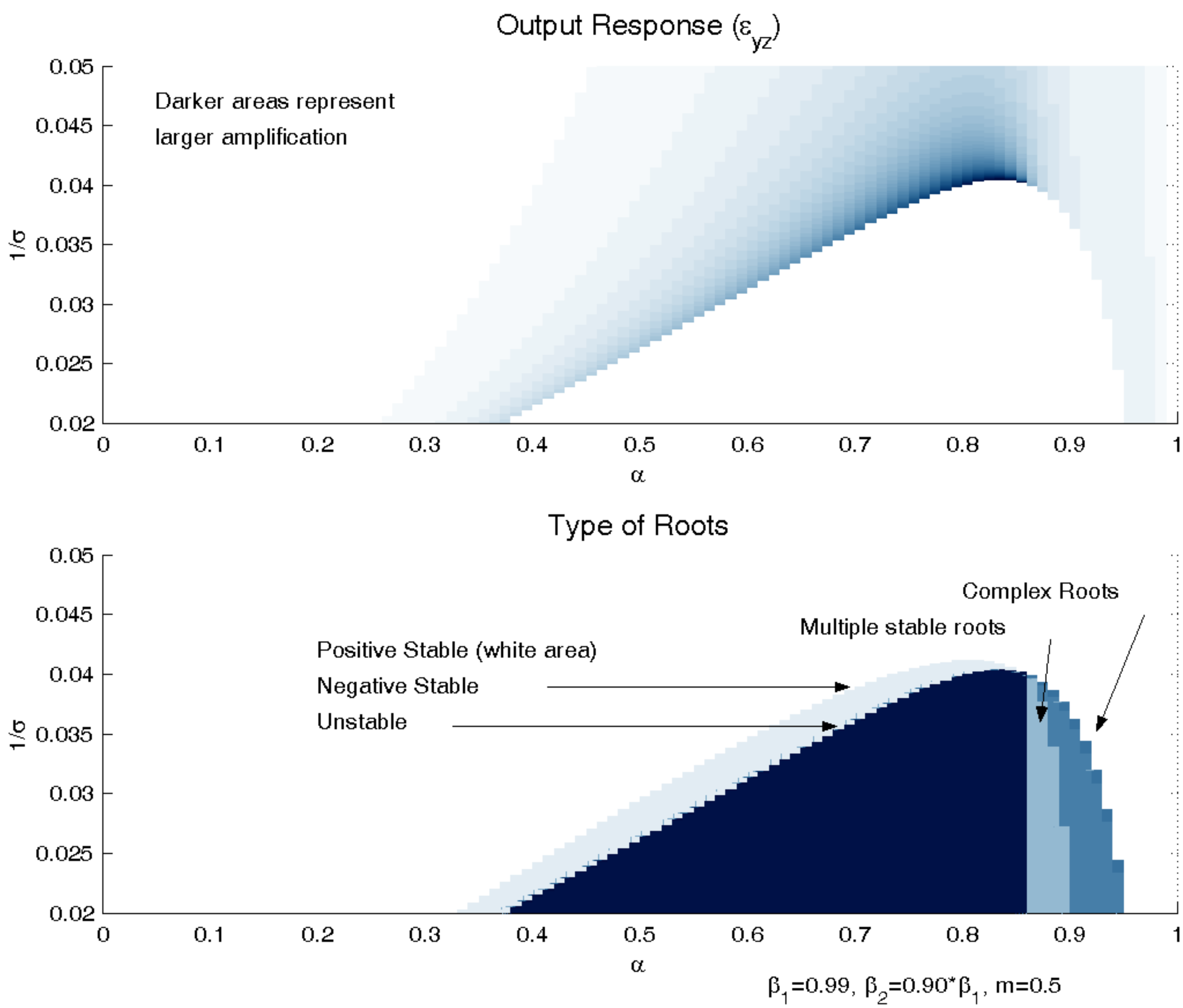


Figure 4: Output Response and Type of Roots

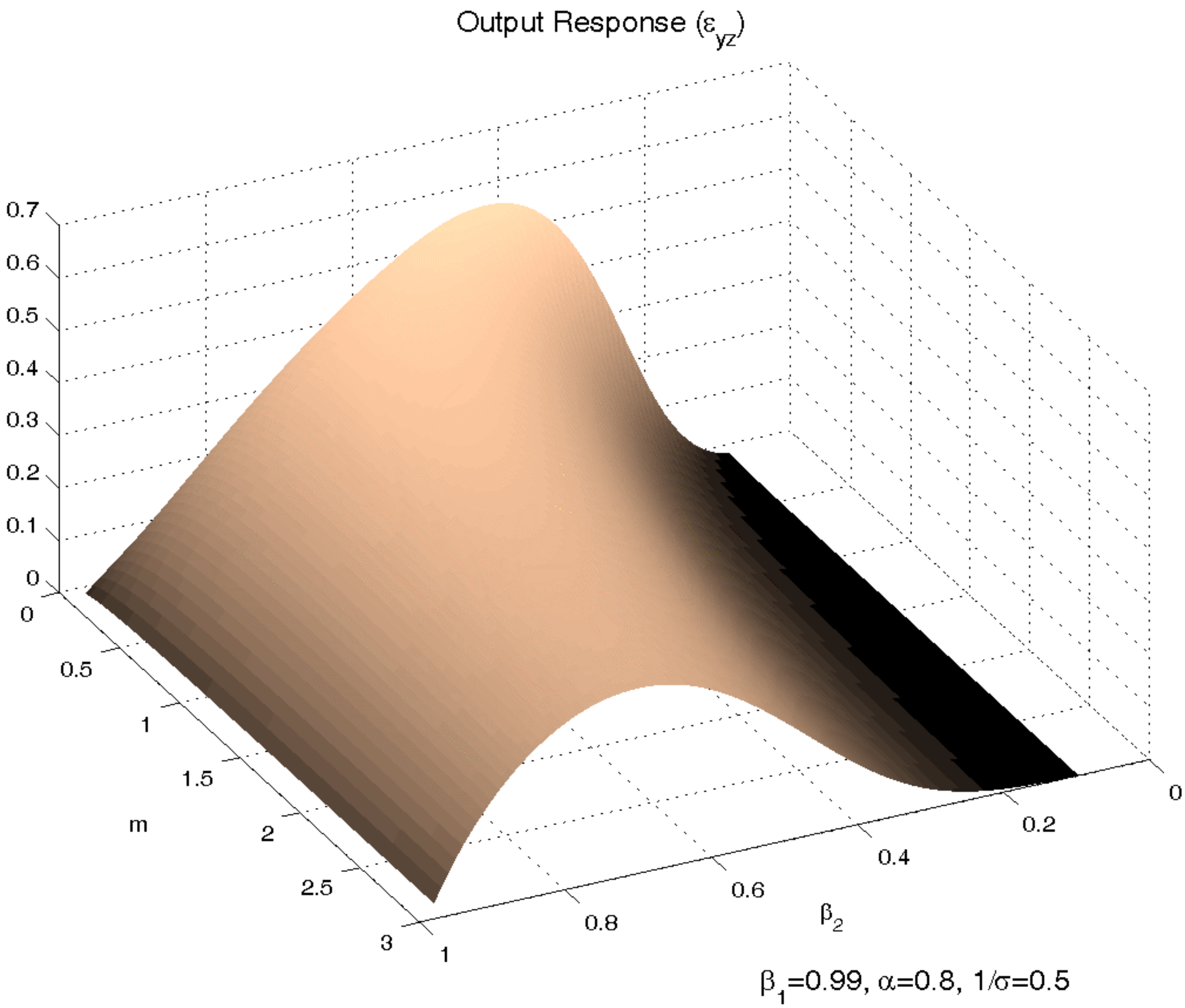


Figure 5: Output response as a function of m and the productivity ratio

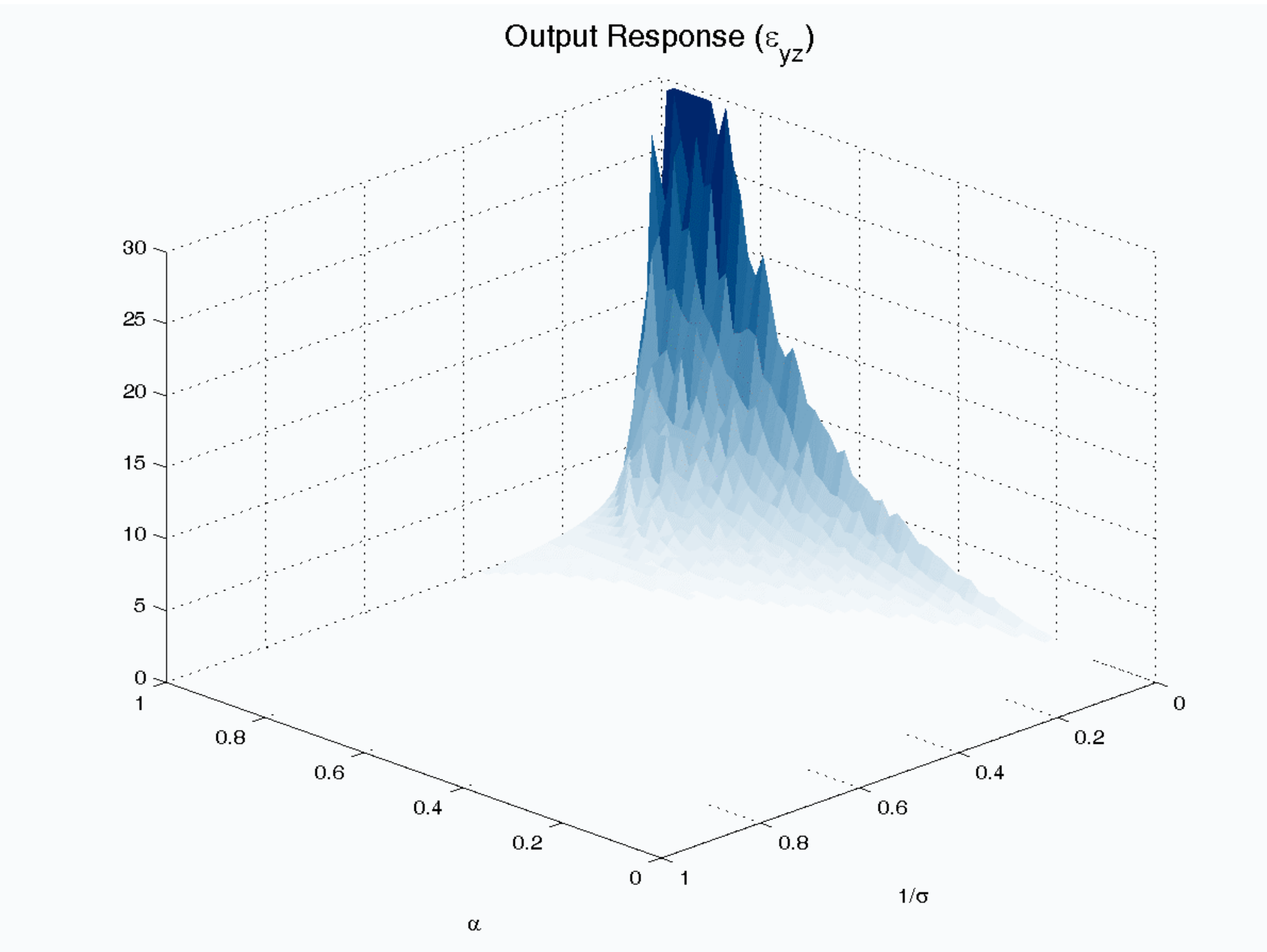


Figure 6: Output response for optimal choices of m and productivity differences

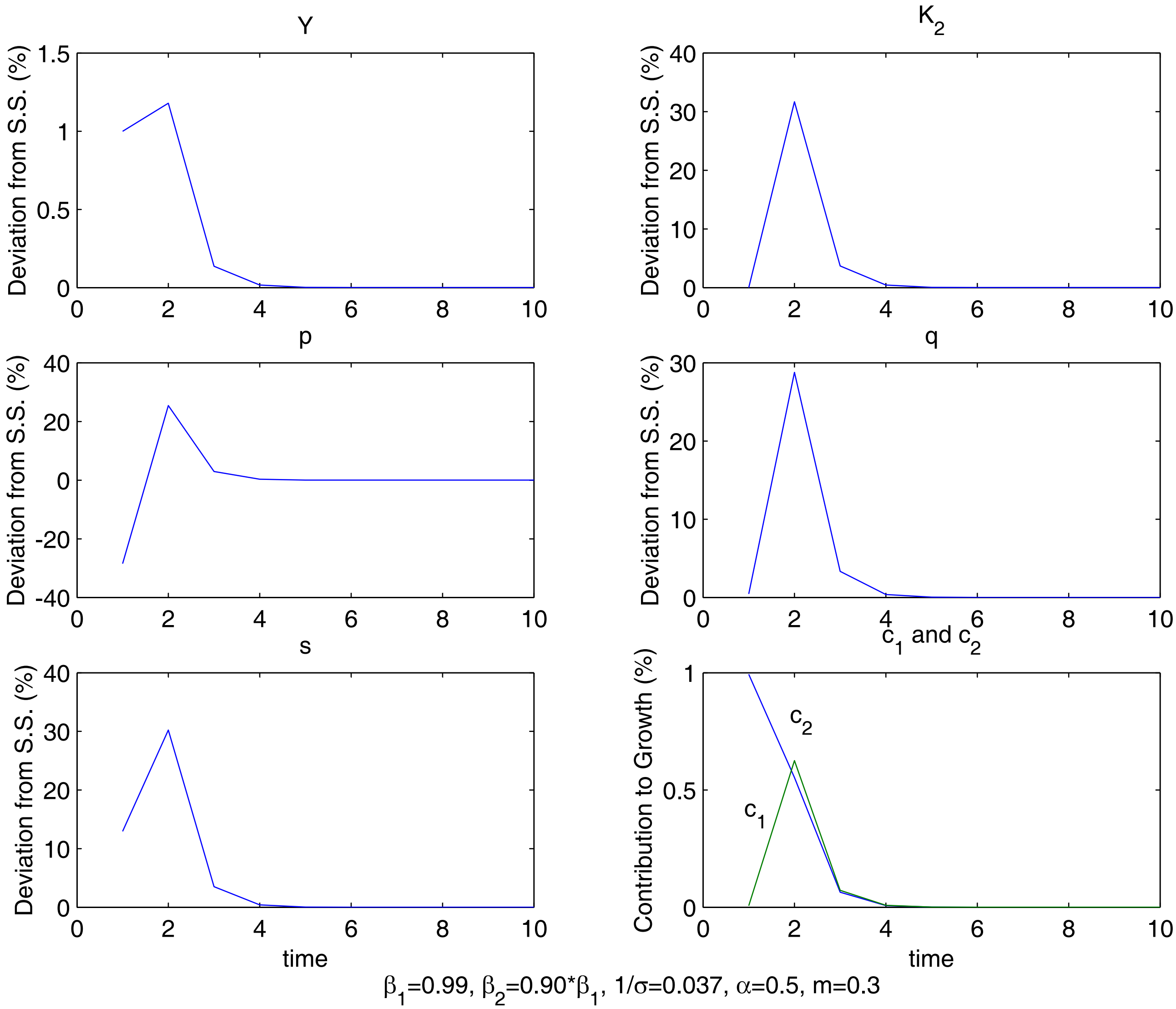
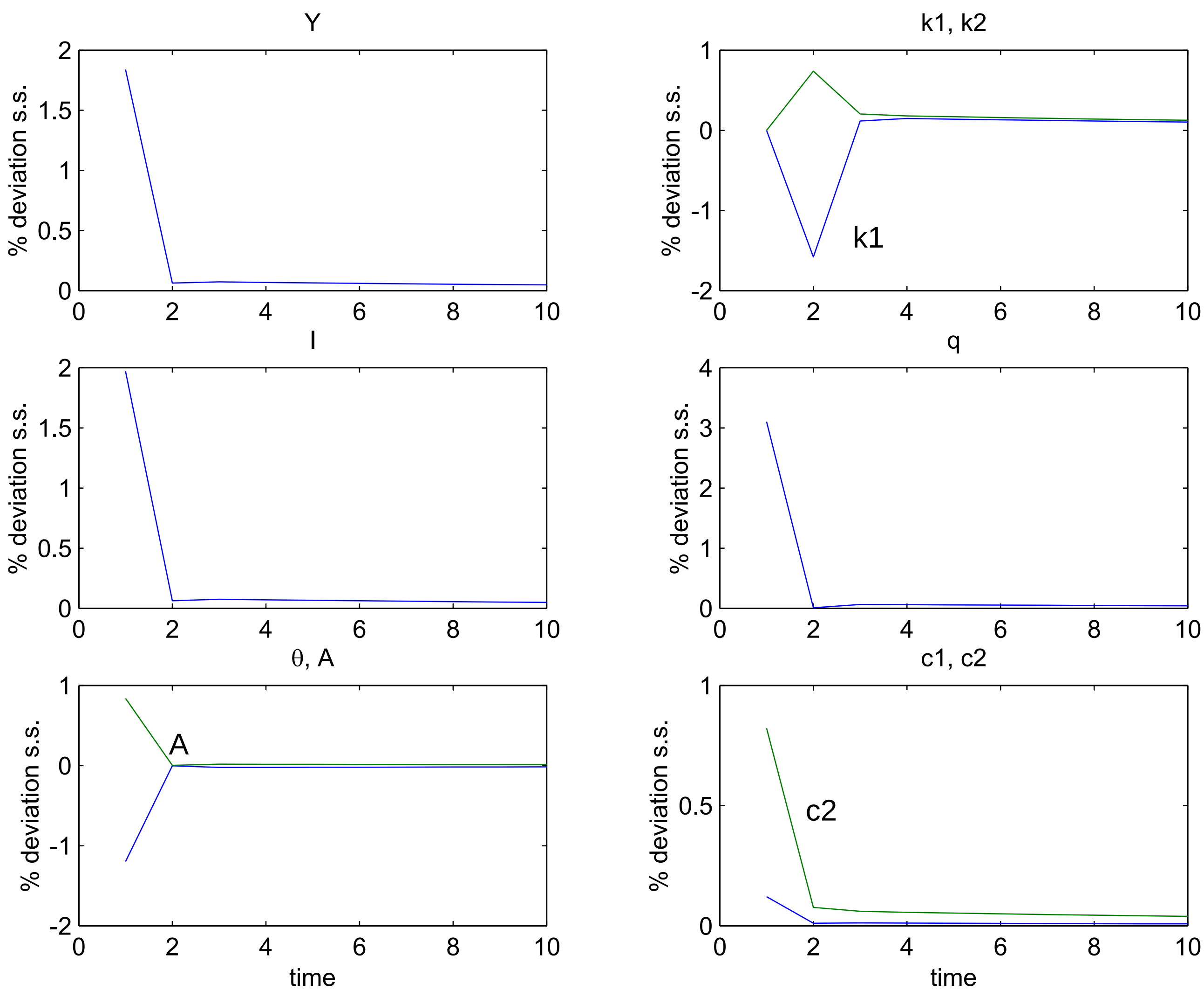


Figure 7: Impulse response functions to a 1% productivity shock



β_1=0.99, β_2=0.90*β_1, 1/σ=0.037, α=0.5, m=0.3

Figure 8: Impulse response functions to a 1% shock when capital is reproducible

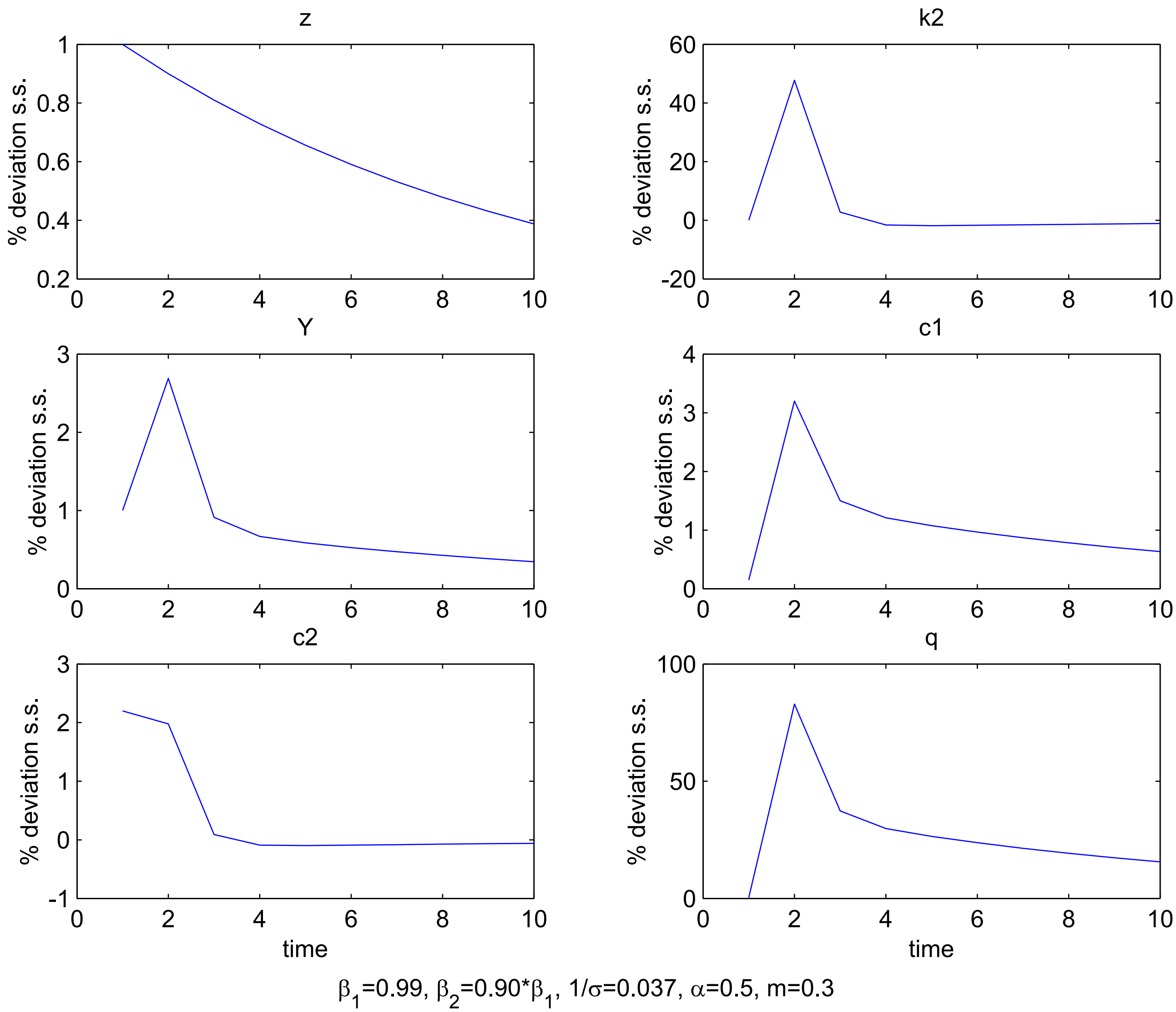


Figure 9: Impulse response functions to a 1% shock for stochastic model

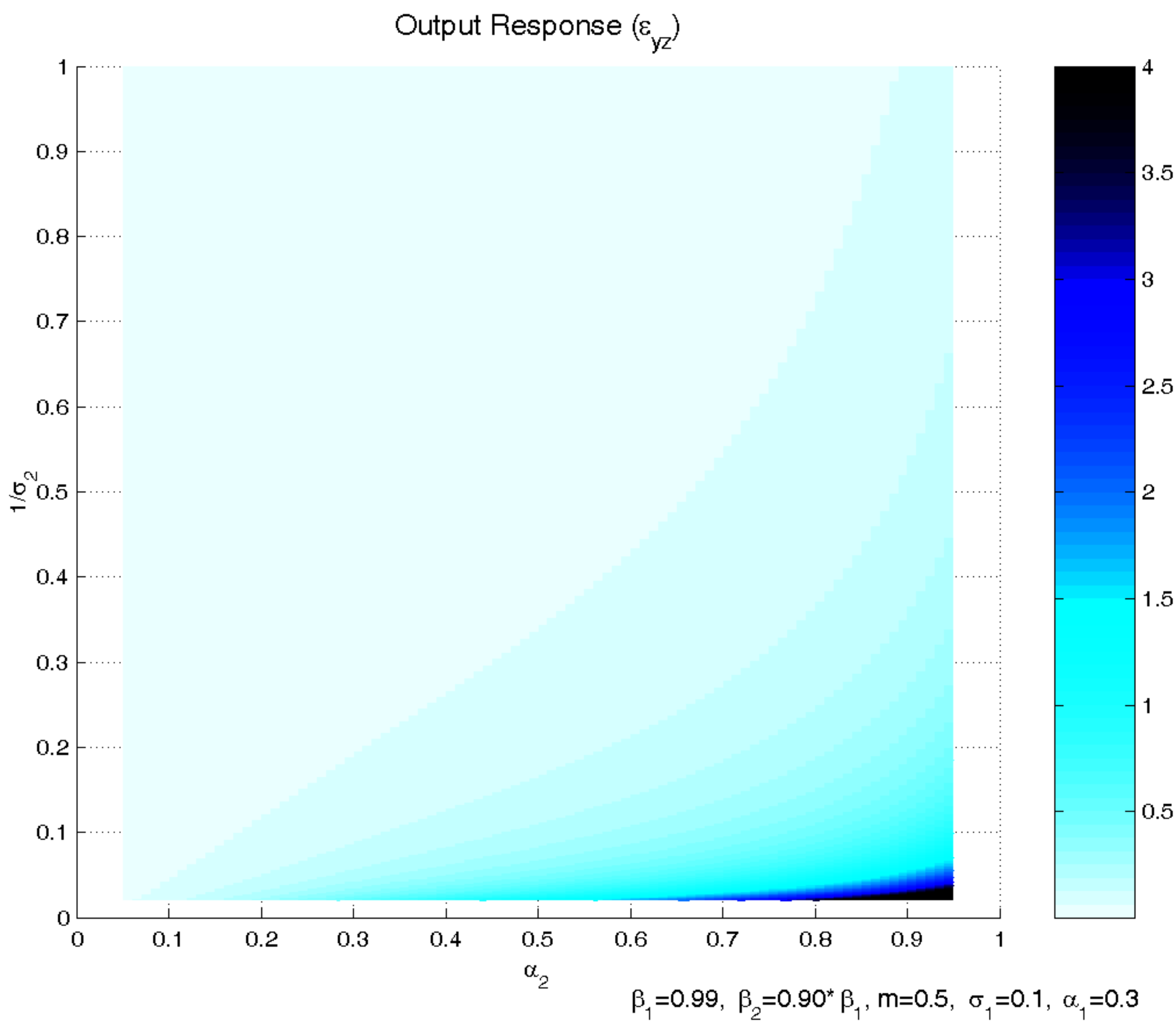


Figure 10: Output response under heterogenous preferences and technologies