

# Illiquidity and Under-Valuation of Firms

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## Abstract

We study a competitive model in which debt-financed firms may default in some states of nature. Incomplete markets prevent firms from hedging the risk of asset fire-sales when markets are illiquid. This is the only friction in the model and the only cost of default. The anticipation of such losses alone may distort firms' investment decisions. We characterize the conditions under which competitive equilibria are inefficient and the form the inefficiency takes. We also show that endogenous financial crises may arise as a result of pure sunspot events. Finally, we examine alternative interventions to restore the efficiency of equilibria.

Keywords: illiquid markets, default, incomplete markets, price distortions, inefficient investment

JEL Classification: D5, D8, G1, G33

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# 1 Introduction

Financial markets play an important role in the efficient allocation of resources. One important function of financial markets is to provide the price signals that guide investment decisions. If the market price of a firm is distorted, the firm's investment decisions will also be distorted. In this paper we present a general equilibrium model in which debt-financed firms face the risk of bankruptcy in some states of nature.<sup>1</sup> The price at which the firm's assets can be liquidated in those states is one of the determinants of the present market value of the firm. If those prices are liquidity constrained, the current market value of the firm is reduced and that in turn will affect current investment decisions.

The efficient markets hypothesis requires, *inter alia*, that markets for financial assets are liquid, both in the sense that prices are insensitive to the volume of trades and in the sense that traders are not liquidity constrained. We investigate an environment where traders may be liquidity constrained and hence asset prices may also reflect the amount of liquid assets in the buyers' possession, and not only the assets' future returns. Liquidity matters particularly in the event of default, where creditors are paid off with the proceeds from the liquidation of the borrower's assets. In the absence of a liquid market, these assets may be sold at firesale prices, causing a significant loss to the creditors. The anticipation of such a loss will in turn increase the cost of borrowing and reduce the firm's initial investment. We consider the case where, besides this possible loss, there are no other costs of default and there are no other events where liquidity considerations matter.

The impact of anticipated defaults and illiquid asset markets is intimately tied up with the incompleteness of markets. We consider an environment where there are no commitment issues; hence, if markets are complete, there is no need for default in the first place. Borrowers and lenders can achieve whatever state-contingent incomes they want by trading contingent claims. As a consequence, when markets are complete, there is never a shortage of liquidity and assets are always efficiently priced. By contrast, if the available debt instruments do not allow for state contingent payments, it is possible that in some states borrowers will have insufficient resources to pay their debts. Further, there may be no way to hedge against capital losses resulting from default, in which case investment decisions may be distorted. Thus, incomplete markets, default and liquidity are jointly responsible for the distortion of prices and investment decisions.

To illustrate these ideas, we use a three-period model in which firms owned by risk neutral entrepreneurs may undertake projects requiring an investment in the first period and producing output in the later periods. Entrepreneurs have no resources and must finance their investment by issuing debt, which is purchased by a large set of identical consumers. The only uncertainty concerns the timing of output: the project undertaken by any entrepreneur will produce output in either the second or third period, but not both. This uncertainty about the timing of production together with the unavailability of contingent debt instruments are what generate the risk of default. An entrepreneur who is unable to repay or to

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<sup>1</sup>We assume that all investment is debt financed for simplicity. Similar arguments could be made with both debt and equity, but the analysis would be much more complicated.

renegotiate his debt is forced to default, liquidate his firm's assets and give the proceeds to the creditors.

The crucial friction in our model arises from a "cash in advance" constraint. This constraint is binding only in the event of default. The bankruptcy code is assumed to require the resolution of the defaulted debt by means of an immediate payment to creditors in cash, not of an IOU for future payment. Hence, the assets of a defaulting firm (its claims to present or future production) must be sold for cash and creditors are not allowed to use anticipated receipts from the bankruptcy proceedings as collateral to buy such assets in the market. As a consequence, firms' asset prices are sometimes determined by the amount of cash in the market rather than by future earnings.

It is important to note that a firm's revenue stream is unaffected by default: if the firm is sold for less than its fundamental value, the sellers' loss is the buyers' gain. Moreover, all consumers are identical, so that default does not even have an effect on the distribution of wealth.<sup>2</sup> Hence, bankruptcy is always efficient *ex post* and, since the representative consumer takes both sides of every trade being at the same time creditor and buyer of the firms that are liquidated, his consumption is unaffected by a firm's liquidation. Nonetheless, a profit-maximizing entrepreneur, anticipating the firm's loss of market value when the firm is liquidated, will make inefficient investment decisions.

The heart of the paper is the characterization of the conditions under which competitive equilibria are efficient, that is, liquidity constraints do not bind, and of the consequences when they do bind. If the entrepreneurs who produce early default in equilibrium, there is no future output to sell and default does not generate any demand for liquidity in the asset market. We call this the case of *no asset sales*. Alternatively, if the entrepreneurs who default are late producers, their claims to future output have to be liquidated in order to pay "cash" to the creditors. If the liquidity available in the market is sufficiently high, the buyers will pay the fundamental value for the liquidated firms. This is the case of a *liquid market*. But if the amount of liquidity is too low, there is an *illiquid market* and the market-clearing price will be *liquidity-constrained*, that is, lower than the fundamental value.

In the case of no asset sales or a liquid market, the liquidation value of the firm is equal to its fundamental value and there is no distortion of investment decisions made at the first date. On the other hand, in the case of an illiquid market the firm's value is liquidity-constrained and this will lead to distortions in the decisions made at the first date. The form this distortion takes is quite intuitive. Firms adjust their investment decisions, that is the project they choose, so that less output appears when they are in default (and forced to sell it at fire sale prices) and more when they are solvent and their creditors are able to buy up the assets of bankrupt firms cheaply. In other words, they will choose more liquid projects, that produce more in the second period and less in the third period. Both tendencies reduce the roundaboutness of production and increase the liquidity of the asset market in the second period but at the same time distort investment decisions.

Even though there is no intrinsic aggregate uncertainty in the model, we show that it

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<sup>2</sup>Since there is a representative consumer, the incompleteness of asset markets imposes no effective constraint on the allocation of consumption.

is possible to have endogenous financial crises as the result of purely extrinsic uncertainty (sunspots). Suppose that, at the beginning of the second period, agents observe the realization of a sunspot variable that affects the equilibrium asset price. With some probability, the market value of late producing firms is high and equal to the fundamental value, in which case there is no default, and with some probability the asset price collapses and late producers are forced to default. Suppose the probability that the price equals the fundamental is high. Investment decisions will then give little weight to the possibility of making capital gains when the asset price falls and, hence, will give little weight to providing liquidity to the asset market in the second period. This sets up the conditions for a self-fulfilling collapse in the asset price: a fall in the asset price causes firms to default, this triggers demand for liquidity (through asset sales), but since the supply of liquidity is small, market clearing can only be restored by a large drop in the price at which assets can be sold. The probability of a collapse is small but, if it occurs, its effects are extreme.

Having shown how market failures arise, we attempt to clarify the source of the inefficiency of equilibrium. We do this by considering three alternative ways in which the efficiency of competitive equilibria can be restored. First, we show that the introduction of firm-specific contingent securities removes the possibility of default and liquidation and makes the liquidity constraint always redundant. The introduction of such securities amounts, in effect, to completing the market. Secondly, the removal of the cash-in-advance constraint present in the event of default (for instance by allowing the payment of creditors with IOUs) ensures that the market clearing price in the asset market is always equal to the fundamental value of the firm. Finally, we show how the distortion of investment decisions can be corrected by the use of Pigovian taxes, that tax the adoption of more liquid projects and allow so to correct the distortion caused by liquidity-constrained asset prices.

**Related literature.** The effect of liquidity on asset prices and its role as a source of financial crises has been studied by numerous authors. The effect of “cash in the market pricing” in banking crises was first studied by Allen and Gale (1994) and related themes have been pursued in a series of papers (see, for example, Allen and Gale, 1978, 2004a and 2004b). Diamond and Rajan (2000, 2001, 2005) also study liquidity in a banking context. By contrast, we focus on a purely market based economy in which there are no depository institutions and firms take production decisions entirely financed by the issue of debt. Shleifer and Vishny (1992) argued that the most likely buyers of the assets of a bankrupt firm would be other firms in the same industry. Since all firms would likely be affected by the same negative business cycle shocks, asset prices are likely to be low when a firm has to be liquidated. They did not study the general equilibrium effects of default or allow for other methods of financing asset sales.

Liquidity also affects the firms’ investment decisions in Holmstrom and Tirole (1997, 2001), who study models where moral hazard limits the pledgeable income of firms. To ensure firms’ access to funds, the constraint that an appropriate share of the firms’ investment is in ‘liquid’, or pledgeable assets is thus imposed. In such framework, the firm’s future valuation plays then no role for its current investment decisions, the role of liquidity is also different and

the liquidity needs are exogenous, only the liquidity premium is endogenously determined. Finally, Kiyotaki and Moore (1997) study the effect of fluctuations in the value of collateral on the firm's ability to access liquidity.

The possibility of default in competitive environments is also investigated in various recent papers (see Kehoe and Levine (1993), Dubey, Geanakoplos and Shubik (2005) for the first contributions). Some important differences from our paper are the facts that default arises from a limited commitment problem (hence is also present when markets are complete) and liquidity issues play no role in the payments received by creditors.

The rest of the paper is organized as follows. The primitives of the economy considered are laid out in Section 2. The investment and portfolio choices of firms and consumers are described in Section 3, together with the decisions concerning the renegotiation of debt and default and the firms' liquidation process. Competitive equilibria are then defined and some properties of consumers' and firms' choices determined. This allows to obtain a simpler set of equilibrium conditions that is useful in the rest of the analysis. Section 4 characterizes the parameter values for which efficient equilibria exist. Since an equilibrium is shown to always exist, the complementary set of parameters can only support an inefficient equilibrium. The properties of these equilibria are analyzed in more detail in Section 5, where we show the consequences of the scarcity of liquidity. Here we also investigate the existence of inefficient sunspot equilibria. Finally, in Section 6, we show that efficiency can be restored by introducing new markets or using tax-transfer schemes. Some of the proofs are relegated to the Appendix.

## 2 The Environment

Time is divided into three dates, indexed by  $t = 0, 1, 2$ . At each date, there is a single good that can be used for consumption or investment. Investment and financing decisions are made at the first date ( $t = 0$ ); consumption and production occur at the second and third dates ( $t = 1, 2$ ).

There is a large number of identical consumers (strictly speaking, a non-atomic continuum with unit measure), each of whom has an endowment  $e = (1, 0, 0)$  consisting of one unit of the good at date 0 and nothing at dates 1 and 2. The utility of the representative consumer is denoted by  $u(c_1, c_2)$  and defined by

$$u(c_1, c_2) = u_1(c_1) + u_2(c_2),$$

for any consumption stream  $(c_1, c_2) \geq 0$ . The period utility functions  $u_1(\cdot)$  and  $u_2(\cdot)$  have the usual properties: they are continuously differentiable, increasing, and concave.

The good can be invested in risky projects at date 0 to produce outputs of the good at dates 1 and 2. The only uncertainty concerns the timing of production. Each project requires one unit of the good at date 0 and produces output at one and only one of the future dates  $t = 1, 2$ . With probability  $\alpha > 0$  the output appears at date 1 and with probability

$1 - \alpha > 0$  it appears at date 2. The probability  $\alpha$  is constant and the same for all projects. Since there is a large number of independent projects, we assume that the “law of large numbers” is satisfied, meaning that the fraction of projects producing at date 1 is precisely  $\alpha$ .

A project is described by an ordered pair  $a \equiv (a_1, a_2)$ , where  $a_1$  is the amount of the good produced at date 1 and  $a_2$  is the amount produced at date 2. The set of available projects is defined by a smooth production possibility frontier  $a_2 = \varphi(a_1)$ , that is, the project  $a = (a_1, a_2)$  is feasible if and only if

$$0 \leq a_1 \leq 1 \text{ and } 0 \leq a_2 \leq \varphi(a_1),$$

where  $\varphi(\cdot)$  satisfies the usual properties: it is continuously differentiable, decreasing and strictly concave on  $(0, 1)$ , satisfies the boundary condition  $\varphi(1) = 0$  and the Inada conditions

$$\lim_{a_1 \rightarrow 0} \varphi'(a_1) = 0 \text{ and } \lim_{a_1 \rightarrow 1} \varphi'(a_1) = -\infty.$$

Projects are operated by firms owned by entrepreneurs<sup>3</sup>. More specifically, there is assumed to be a large number of risk neutral entrepreneurs, each of whom can undertake a single project requiring the investment of one unit of the good at date 0. Entrepreneurs have no resources of their own and consumers cannot undertake investment projects themselves, so projects are undertaken by firms and financed by consumers. The number of entrepreneurs is assumed to be greater than the number of consumers, so the number of entrepreneurs willing to undertake a project is greater than the number of projects that can be financed by consumers. This “free entry” assumption ensures that firms earn zero profits in equilibrium.

Given that entrepreneurs earn zero profits in equilibrium, in characterizing Pareto-efficient allocations we restrict our attention to allocations where all the projects’ output goes to the consumers. At a symmetric, Pareto-efficient allocation all endowments are invested at date 0 in feasible projects whose output maximizes the expected utility of the representative consumer. In addition, since  $\varphi$  is strictly concave, Pareto efficiency requires that all endowments be invested in a unique type of project.

Suppose that a project  $a$  is chosen at date 0. At each date  $t = 1, 2$ , consumption equals total output. Total output at date 1 is equal to  $\alpha a_1$  since a fraction  $\alpha$  of the projects produce  $a_1$  at date 1; similarly, consumption at date 2 is equal to  $(1 - \alpha) a_2$  since a fraction  $1 - \alpha$  of projects produce  $a_2$  at date 2. Thus, the representative consumer consumes  $\alpha a_1$  at date 1 and  $(1 - \alpha) a_2$  at date 2. We say that a project  $a^*$  supports a symmetric, *Pareto-efficient allocation* if it maximizes

$$u(c_1, c_2) = u_1(\alpha a_1) + u_2((1 - \alpha) a_2).$$

among the set of feasible projects. The Inada conditions imply that the efficient project must have positive output at each date  $t = 1, 2$ , that is,  $0 < a_1^* < 1$ . Thus,  $a^*$  is Pareto-efficient if and only if it satisfies the first-order condition for an interior maximum,

$$\alpha u_1'(\alpha a_1^*) + (1 - \alpha) u_2'((1 - \alpha) a_2^*) \varphi'(a_1^*) = 0. \tag{1}$$

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<sup>3</sup>In what follows, we use the terms firm and entrepreneur interchangeably.

The efficient allocation is illustrated in Figure 1.

— Figure 1 here —

## 3 Equilibrium

### 3.1 Overview

We make the extreme assumption that short-term debt is the only financial instrument available in the economy. A bond issued at date 0 is a promise to pay one unit of the good at the beginning of date 1. Entrepreneurs issue bonds, collateralized by future output, to finance their investment in risky projects. They make their production and financing decisions to maximize their firm's profits. Consumers purchase bonds issued by entrepreneurs to finance their future consumption. They choose the type of bonds that maximizes their expected utility, given the entrepreneurs' choice of project and the market price of the bonds.

Since projects are risky and the promised return on debt is non-contingent, entrepreneurs may not have enough resources to fulfil their debt obligations at date 1. In that event they may have to default. The institution of bankruptcy requires the resolution of the defaulted debt by means of an immediate payment to creditors in cash and not in the form of claims to future payments. The entrepreneurs whose projects produce output at date 1 (*early producers*) can make an immediate payment. The others (*late producers*) have no income readily available. They can avoid default by renegotiating the debt with their creditors and rolling it over to the next period. If they fail to renegotiate the debt, however, they must declare bankruptcy and liquidate the firm's assets (i.e., its claims on future production) by selling them in the asset market. The proceeds of this sale are used to repay creditors.

To clarify the timing of these events and their consequences, we divide the second date into three sub-periods, labelled *A*, *B*, and *C*, corresponding to the three phases of the bankruptcy process, repayment/renegotiation/default, liquidation and resolution, respectively. In sub-period *A*, each entrepreneur discovers whether he is an early or late producer. If he is an early producer, he immediately pays his creditors. If he is a late producer, he either renegotiates the debt (i.e., rolls it over) or defaults. Late producers who fail to renegotiate their debt sell the firms' assets in the market that opens in sub-period *B*. The liquidated value of these firms is paid to the creditors, *up to the nominal value of their debt*, in sub-period *C*. This time line is illustrated in Figure 2.

— Figure 2 here —

The process of renegotiation and bankruptcy influences the actual payoff to bondholders and hence the value of the debt associated with different types of projects. In particular, it implies that the value of the debt at date 0 will depend on the value at date 1 of claims to date-2 output. The entrepreneurs' choice of project and the efficiency of the equilibrium allocation may also be affected.

At date 1, consumers have to decide whether to use any of the income they receive from early producers to purchase the assets of the liquidated firms in sub-period  $B$  and, in so doing, transfer this income to the final period. At date 2, the bonds issued at date 1 pay off and there is no further trade.

Markets are competitive and prices set at a level such that markets clear in equilibrium. In particular, at date 0, the supply of bonds issued by entrepreneurs equals the demand by consumers. Similarly at date 1 the supply of bonds by defaulting entrepreneurs is equal to the consumers' demand.

In the remainder of this section we provide a more precise statement of the equilibrium conditions at the same time as deriving some basic equilibrium properties. By the end of the section we will have derived the reduced-form set of equilibrium equations that we analyze in the sections that follow. In Section 3.2, we provide a precise account of the renegotiation game between firms and their creditors that determines whether the debt can be paid off or renegotiated and rolled over, or the firm is forced to default. We show that the renegotiation game results in default if and only if the present value of the firm's revenue stream is less than the face value of its debt. In Section 3.3, we summarize the creditors' payoffs in each of the situations that can arise at date 1. Having characterized the outcome at date 1, taking as given the entrepreneurs' decisions at date 0, in Section 3.4 we proceed to analyze the entrepreneur's problem, which is to raise finance and choose a production plan that maximizes the value of his firm. The value of the firm depends on the consumers' marginal valuation for consumption at each future date, on the market price of bonds at date 1, and on the possibility of default. Once the entrepreneur has made his financial and production decisions at date 0, his future actions are all determined. It remains to characterize the behavior of consumers, which we do in Section 3.5. At the first date, consumers inelastically supply their funds to the firms that offer the best returns. At date 1, they make the optimal consumption and savings decision, taking the bond price and the firms' payouts and defaults as given. The last step in our characterization of equilibrium is the statement of the market-clearing conditions, in Section 3.6.

## 3.2 Renegotiation and default

Consider an entrepreneur who invested 1 unit of the good at date 0, issued debt with a face value of  $d_0 > 0$  and chose the project  $a = (a_1, a_2)$ . At the beginning of date 1, the entrepreneur learns whether he is an early producer who receives output  $a_1$  at date 1 or a late producer who receives output  $a_2$  at date 2.

**Sub-period A: repayment/renegotiation/default.** Payments on debt obligations are due in this first sub-period. There are two cases to be analyzed, depending on whether the firm's output appears in the present or in the future.

**Early producers:** An early producer receives a revenue of  $a_1$  at the beginning of date 1. If the face value of the short-term debt is less than or equal to his revenue ( $d_0 \leq a_1$ ), the entrepreneur is solvent and immediately pays the amount  $d_0$  to his creditors. On the other hand, if the face value of the debt is greater than his revenue ( $a_1 < d_0$ ), the entrepreneur is

insolvent. In this case he defaults and pays as much as he can (i.e.,  $a_1$ ) to the bond holders. Since the project's future output is zero no further payment can be made.

Thus, an early producer, whether he is solvent or not, makes a payment  $\min\{a_1, d_0\}$  to the bond holders in sub-period  $A$ .

**Late producers:** A late producer has no current output, but expects to receive  $a_2$  in the next period. To avoid default, he must renegotiate or “roll over” the debt  $d_0$ . The renegotiation procedure is structured as follows. The entrepreneur makes a “take it or leave it” offer to the bond holders, offering to exchange new short-term debt with a face value of  $d_1$  for the old debt  $d_0$  issued at date 0. Once the entrepreneur has made an offer, the creditors simultaneously accept or reject it. The renegotiation succeeds if a majority of the creditors accept and the entrepreneur can afford to pay off the bond holders who reject the offer. Otherwise it fails and the entrepreneur is forced to default and to liquidate his firm's assets, giving the proceeds to his creditors. All this has to be done before the end of the current period (date 1).

**Sub-period B: liquidation.** In this sub-period the market for the assets of defaulting late producers opens. The only asset these entrepreneurs possess is their claim to the project's future output. Since there is no uncertainty about the amount of future output, this claim can be realized by issuing riskless debt, fully collateralized by the future output. The debt trades at a uniform price  $q_1$  regardless of the firm's project since there is no default risk.

**Sub-period C: resolution.** At the end of date 1, bankrupt late producers settle their debts by distributing the liquidated value of their projects,  $q_1 a_2$ , pro rata among their creditors.

Because of the timing of default, liquidation, and resolution, there is a marked asymmetry between early and late producers in default. If an early producer defaults in sub-period  $A$ , he immediately hands over his output  $a_1$  in partial payment of his debt. Defaulting late producers are in a different situation. Because they have no current revenue, they must liquidate their assets in sub-period  $B$  before making any payment to their creditors. So these are forced to wait until sub-period  $C$  for payment. The delay is important because income received from liquidation in sub-period  $C$  cannot be used to purchase bonds in sub-period  $B$ . This can affect the equilibrium price of bonds  $q_1$  which in turn will affect the amount,  $\min\{q_1 a_2, d_0\}$ , that the creditors eventually receive.

### The renegotiation game

Now that we have described the sequence of events at date 1, we can analyze the outcome of the renegotiation process between a late producer and the bond holders who financed his project. Suppose that, for the entrepreneur in question, the chosen project is  $a = (a_1, a_2)$  and the face value of debt is  $d_0$ .

The renegotiation game consists of two stages:

- The entrepreneur makes a “take it or leave it” offer  $d_1 \leq a_2$  to the bond holders.
- The bond holders simultaneously accept or reject the firm's offer.

Two conditions must be satisfied in order for the renegotiation to succeed.

- (i) First, a majority of the bond holders must accept the offer.
- (ii) Secondly, the rest of the bond holders must be paid off in full. Hence, if a fraction  $\gamma > 0.5$  of bond holders accept they must be paid  $d_1$  at date 2, while the remaining fraction  $1 - \gamma$  must be paid  $d_0$  at date 1 in sub-period  $A$ . This is feasible if the budget constraint

$$\gamma q_1 d_1 + (1 - \gamma) d_0 \leq q_1 a_2$$

is satisfied.

If either condition is not satisfied, the renegotiation fails and the entrepreneur is forced to default, liquidate the project, and distribute the proceeds to the bond holders at the end of the period.

If a bond holder accepts the offer and renegotiation succeeds, he receives  $d_1$  at date 2. If he rejects the offer and renegotiation still succeeds, he must be paid  $d_0$  immediately, i.e., in sub-period  $A$ . If renegotiation fails, the bond holder receives  $\min\{q_1 a_2, d_0\}$  at the end of date 1, regardless of whether he accepts or rejects. Let  $(c_1, c_2)$  be the consumption profile of the representative consumer. In equilibrium each consumer holds a negligible amount of the debt issued by any firm, to fully diversify firm specific risk, so his payoffs for accepting and rejecting the renegotiation offer are described in the following table:

	Success	Failure
Accept	$u'_2(c_2) d_1$	$u'_1(c_1) \min\{q_1 a_2, d_0\}$
Reject	$\max\left\{u'_1(c_1), \frac{u'_2(c_2)}{q_1}\right\} d_0$	$u'_1(c_1) \min\{q_1 a_2, d_0\}$

If the consumer rejects a successful offer, he can choose to consume his payment  $d_0$  at date 1 or he can invest it in bonds and consume  $d_0/q_1$  at date 2. Thus, his payoff is the maximum of  $u'_1(d_0) d_0$  and  $u'_2(c_2) d_0/q_1$ . This gives us the entry in the lower left hand cell. The others are self-explanatory.

The subgame given by the second stage of the renegotiation game has some of the features of a coordination game, so it is not surprising that there may be multiple equilibria. In particular, if a majority of bond holders rejects the entrepreneur's offer, renegotiation fails and the individual bond holder receives the same payoff whether he accepts or rejects the offer. Thus, there is always an equilibrium of the subgame in which all bond holders reject the offer, renegotiation fails, and the project is liquidated prematurely.

There is also a pure-strategy equilibrium of this subgame in which renegotiation succeeds if and only if

$$u'_2(c_2) d_1 \geq \max\left\{u'_1(c_1), \frac{u'_2(c_2)}{q_1}\right\} d_0, \quad (2)$$

that is, the payoff from accepting the entrepreneur's offer, conditional on success, is at least as great as the payoff from rejecting it. In what follows, we will consider the case where

renegotiation fails only if it is unavoidable. That is, bond holders are assumed to accept the offer if acceptance is optimal *when everyone else accepts*. This minimizes the incidence of default and restricts default to those cases where it is *essential* (Allen and Gale, 1998).

Letting  $M(c) = \frac{u'_2(c_2)}{u'_1(c_1)}$  denote the intertemporal marginal rate of substitution, condition (2) can be equivalently written as:

$$d_0 \leq \min \{M(c), q_1\} d_1.$$

In analyzing the renegotiation game, it is convenient to anticipate a property of the equilibria of the economy that we establish later. For the moment, we treat this property as an auxiliary assumption:

$$q_1 \leq M(c). \quad (3)$$

Condition (3) implies that, in equilibrium, consumers might want to purchase more riskless debt at date 1 than they are able to. With this temporary assumption we can prove the following result.

**Proposition 1** *If:*

(a)

$$d_0 \leq q_1 a_2,$$

*there exists a subgame perfect equilibrium of the renegotiation game in which the entrepreneur offers  $d_1 = d_0/q_1$  and the creditors all accept. If*

(b)

$$d_0 > q_1 a_2,$$

*there is no equilibrium in which renegotiation succeeds: in every subgame perfect equilibrium of the renegotiation game the entrepreneur is forced to default and liquidate the project.*

**Proof.** (a) The proof is constructive. Suppose that the creditors' strategy is to accept offers  $d_1 \geq d_1^* \equiv d_0/q_1$  and reject offers  $d_1 < d_1^*$  and the entrepreneur's strategy is to offer  $d_1 = d_1^*$ . We claim that these strategies constitute a subgame perfect equilibrium.

We begin by showing that the strategy of an individual creditor is a best response to the strategies of the other creditors and the entrepreneur. When the entrepreneur offers  $d_1 \geq d_1^*$ , all the other creditors accept the offer and renegotiation succeeds even if the creditor under consideration rejects. Hence, the creditor receives  $d_1 \geq d_1^*$  at date 2 if he accepts the offer and  $d_0/q_1 = d_1^*$  at date 2 if he rejects it. So it is (weakly) optimal to accept the offer. If the entrepreneur offers  $d_1 < d_1^*$ , all the other creditors reject the offer, so the renegotiation fails and the bond holder under consideration receives the same payoff whether he accepts or rejects. So rejecting the offer is (weakly) optimal in this case.

It remains to show that the entrepreneur's strategy is a best response to the creditors' strategy. If the entrepreneur offers  $d_1 = d_1^*$  his offer is accepted, he pays out  $d_1^* = d_0/q_1$  at date 2, and his firm's profit is  $a_2 - d_0/q_1 \geq 0$ . Any offer  $d_1 \in (d_1^*, a_2]$  will also be accepted, but clearly yields lower profits. On the other hand, if he offers  $d_1 < d_1^*$ , the offer will be

rejected, he is forced to default and ends up paying out  $\min\{q_1 a_2, d_0\}$  at the end of date 1. By assumption,  $d_0 \leq q_1 a_2$  so  $\min\{q_1 a_2, d_0\} = d_0$  and the payment under default leaves the entrepreneur a non-negative profit  $q_1 a_2 - d_0$  at date 1. Since the present value at date 1 of the expression we found for the profits when  $d_1 = d_1^*$  is also  $q_1 a_2 - d_0$ , the entrepreneur does not gain by offering  $d_1 < d_1^*$  either. This completes the proof that the strategies constitute a subgame perfect equilibrium.

(b) The proof is by contradiction. Suppose there exists a subgame perfect equilibrium in which the renegotiation succeeds. Then it must be optimal for creditors to accept an offer  $d_1 \leq a_2$ . But this cannot be, since by rejecting the offer (when everyone else accepts) a creditor obtains  $d_0/q_1 > a_2 \geq d_1$  at date 2. ■

### 3.3 Payments to bond holders

In the preceding analysis we have seen that a bond issued at date 0 yields different payments, depending on whether the entrepreneur turns out to be an early or late producer and whether early or late producers default. These payments are displayed in the table below.

	Payment if $d_0 \leq q_1 a_2$ (late producers solvent)	Payment if $d_0 > q_1 a_2$ (late producers default)
Early producer	$\min\{a_1, d_0\}$ at date 1	$\min\{a_1, d_0\}$ at date 1
Late producer	$d_0/q_1$ at date 2	$q_1 a_2$ at (the end of) date 1

If the entrepreneur is a late producer and  $d_0 > q_1 a_2$ , renegotiation always fails, as shown in Proposition 1. Hence the entrepreneur defaults and pays creditors an amount  $\min\{q_1 a_2, d_0\} = q_1 a_2$ . The other cases are self-explanatory.

A bond issued at date 0 is identified by its face value  $d_0$  and the project  $a$  it finances. The bond market at date 0 is competitive. For any feasible project  $a$ , let  $V(a, d_0)$  denote the market value of debt with face value  $d_0$  issued to finance project  $a$ . Given the presence of a representative consumer, in equilibrium  $V(a, d_0)$  equals the ratio of the consumer's marginal utility of the payoff from a unit investment in a bond of type  $(a, d_0)$  to his marginal utility of income at date 0.

### 3.4 Production and financing decisions

Now we can describe the entrepreneur's production and financing decisions. Taking the price function  $V(\cdot)$  as given, the entrepreneur's decision problem consists of choosing an admissible project  $a$  and face value of the debt  $d_0$  to maximize his firm's profits<sup>4</sup>:

$$\begin{aligned} \max_{a, d_0} \quad & \max\{V(a, d_0) - 1, 0\} \\ \text{s.t.} \quad & 0 \leq a_1 \leq 1, \quad 0 \leq a_2 \leq \varphi(a_1) \end{aligned} \quad (4)$$

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<sup>4</sup>The entrepreneur, the sole owner of the firm, is assumed to be risk neutral. His welfare is clearly maximized by the profit-maximizing choice of  $(a, d_0)$ .

Equivalently, we can interpret this problem as maximizing the value of the debt issued. Since entrepreneurs have no resources of their own and there is limited liability, the firm's revenue can never be negative. The specification of the objective function in (4) reflects the fact that, if the value of the debt issued is lower than the cost of the initial investment, that is,  $V(a, d_0) < 1$ , it will be impossible for the entrepreneur to undertake a project at all and he will be forced to remain inactive.

As we argued in Section 2, free entry by entrepreneurs ensures that firms earn zero profits in equilibrium. This fact is helpful in studying the solutions to the entrepreneurs' problem (4). The zero-profit condition implies that, for all  $(a, d_0)$ ,

$$V(a, d_0) \leq 1 \tag{5}$$

and, for all projects whose initial investment can be financed, that is, ordered pairs  $(a, d_0)$  satisfying  $V(a, d_0) = 1$ , the face value of the debt  $d_0$  must satisfy

$$d_0 \geq \max \{a_1, q_1 a_2\}. \tag{6}$$

Condition (6) says that the entrepreneur has no revenue left after paying bond holders, whether he is an early producer or a late producer. When he is an early producer,  $d_0 \geq a_1$  implies that the face value of the debt is at least as great as his firm's revenue. When he is a late producer,  $d_0 \geq q_1 a_2$  ensures that either he defaults and pays out  $\min \{d_0, q_1 a_2\} = q_1 a_2$  at (the end of) date 1 or (when  $d_0 = q_1 a_2$ ) he renegotiates the debt and pays out  $d_0/q_1 = a_2$  at date 2. In either case, his firm realizes zero profit.

We show, in addition, that

**Lemma 1** *The value of the firm's debt, and hence its profits, are always maximized by setting*

$$d_0 = \max \{a_1, q_1 a_2\}. \tag{7}$$

**Proof.** We show that, if  $d_0 > \max \{a_1, q_1 a_2\}$ , a reduction in  $d_0$  has either no effect or increases  $V(a, d_0)$ . We consider two cases in turn. If

$$a_1 > q_1 a_2$$

the payments to bond holders, and hence the bond's value, are the same whether  $d_0 = a_1$  or  $d_0 > a_1$ , because default by early producers makes no difference to the outcome and late producers must default in any case. If

$$a_1 \leq q_1 a_2,$$

late producers do not default if  $d_0 = q_1 a_2$  whereas they must default if  $d_0 > q_1 a_2$ . Hence, the payment to bond holders is  $a_2$  at date 2 in the first case and  $q_1 a_2$  at (the end of) date 1 in the second. Under (3),  $u'_2(c_2) a_2 \geq u'_1(c_1) q_1 a_2$ , and the inequality is strict if  $q_1 < M(c)$ .

Thus, the value of the debt is at least as high in the case where  $d_0 = q_1 a_2$  as it is in the case where  $d_0 > q_1 a_2$  and is strictly higher if  $q_1 < M(c)$ . ■

In the sequel we restrict our attention to the case where, for any project the entrepreneur considers undertaking in equilibrium, the face value of the debt issued satisfies (7). On this basis, the specification of the payoffs for bondholders obtained in Section 3.3 can be further simplified:

(i) if  $a_1 \leq q_1 a_2$ , we have:

- $d_0 = q_1 a_2$
- early producers default and pay  $a_1$  at date 1
- late producers are solvent and pay  $a_2$  at date 2.

(ii) if  $a_1 > q_1 a_2$ :

- $d_0 = a_1$ ,
- early producers are solvent and pay  $a_1$  at date 1
- late producers default and pay  $q_1 a_2$  at (the end of) date 1

Note that the possibility of default introduces a discontinuity in payoffs and hence a non-convexity into the entrepreneur's decision problem. For this reason, we divide the analysis of the entrepreneur's decision into two parts, depending on whether the late producer is solvent or in default. Each case corresponds to a convex sub-problem.

**(i) Late producers solvent** Consider first projects such that  $q_1 a_2 \geq a_1$ . In this case, as we argued above, creditors receive  $a_1$  at date 1 with probability  $\alpha$  and  $a_2$  at date 2 with probability  $1 - \alpha$ . They can use the payment received at date 1 for immediate consumption or to purchase bonds for future consumption, whichever gives them the greater utility. Because of our auxiliary assumption,  $q_1 \leq M(c)$ , it is always weakly optimal at the margin to save the payment until date 2. Hence, the market value of the debt issued to finance these projects, with face value  $d_0 = q_1 a_2$ , is

$$V(a, q_1 a_2; q_1, c, \lambda) = \frac{1}{\lambda} \left[ \frac{u'_2(c_2)}{q_1} \alpha a_1 + u'_2(c_2) (1 - \alpha) a_2 \right], \quad (8)$$

where  $\lambda > 0$  denotes the marginal utility of consumption at date 0.<sup>5</sup>

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<sup>5</sup>This expression and the next one ensure, as we will see, that for every type of bond  $(a, d_0)$ ,  $V(a, d_0)$  is a market-clearing price. Alternatively, we can interpret  $V(a, d_0)$  as an entrepreneur's rational conjecture about how much money he can raise by issuing short-term debt  $(a, d_0)$ .

**(ii) Late producers in default** For projects such that  $q_1 a_2 < a_1$ , the entrepreneur defaults when he is a late producer. Bond holders again receive  $a_1$  at date 1 with probability  $\alpha$ , but now they get a payment  $q_1 a_2$  at the end of date 1 with probability  $1 - \alpha$ . In the first event, we can again suppose without loss of generality that the payment received at date 1 is saved until date 2. Then the market value of the debt issued at date 0 with face value  $d_0 = a_1$  to finance project  $a$  is

$$V(a, a_1; q_1, c, \lambda) = \frac{1}{\lambda} \left[ \frac{u'_2(c_2)}{q_1} \alpha a_1 + u'_1(c_1) (1 - \alpha) q_1 a_2 \right]. \quad (9)$$

We can formalize the properties of the entrepreneurs' decision in the following claim.

**Claim 1** *In a competitive equilibrium where (some) entrepreneurs are active, each entrepreneur chooses an admissible project  $\bar{a}$  which solves his decision problem (4), where  $V(a, d_0)$  is given by (8) for projects such that  $q_1 a_2 \geq a_1$  and by (9) for projects such that  $q_1 a_2 < a_1$ . In addition,*

$$V(\bar{a}, \max\{\bar{a}_1, q_1 \bar{a}_2\}) = 1 \text{ and } V(a, \max\{a_1, q_1 a_2\}) \leq 1$$

for any other feasible project  $a$ .

Given the non-convexity of the firms' choice problem, it is possible that an array of projects is chosen in equilibrium. To keep the notation simple, however, we stated both the above claim and the following characterization of equilibrium for the *symmetric* case in which all entrepreneurs choose the same project  $a$ .<sup>6</sup>

### 3.5 The consumer's decision

At date 0, consumers supply their endowments in exchange for bonds. The price function  $V(\cdot)$  specified in (8) and (9) ensures that, provided  $c$  is the representative consumer's optimal consumption plan and  $\lambda$  his marginal utility of income at date 0, he is willing to finance any project  $a$  the entrepreneurs may choose. More precisely, for any  $(a, d_0)$ , consumers are willing to purchase bonds with value  $V(a, d_0)$  at date 0 from any entrepreneur who chooses a project  $a$  and issues bonds with face value  $d_0$ .

The consumer's problem at date 2 is trivial, because there is no further trade and the consumer simply consumes all his income. It remains to analyze his choice problem at date 1, when the consumer has to decide how much of his current income to use for immediate consumption and how much to save in the form of short-term debt. In particular, we need to verify that condition (3), which we have used as an auxiliary assumption, actually holds in equilibrium.

The consumer's decision problem at date 1 differs according to whether late producers are solvent or in default. As in the previous section, we consider each case in turn.

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<sup>6</sup>The general case, with an array of projects chosen in equilibrium, is described in the appendix.

**(i) Late producers solvent:**  $d_0 = q_1 a_2 \geq a_1$  In this case, as we saw, the early producers pay out  $a_1$  in sub-period  $A$  of date 1, and the late producers roll over their debt and pay out  $a_2$  at date 2. Each consumer receives a deterministic payment<sup>7</sup> equal to  $\alpha a_1$  in sub-period  $A$ . This income can be used to purchase  $b_1$  units of bonds in sub-period  $B$ . Since the consumer receives no further payment in sub-period  $C$ , the remaining income,  $\alpha a_1 - q_1 b_1$ , constitutes the maximal amount he can spend on consumption at date 1. At date 2, the consumer's income will be  $(1 - \alpha) a_2 + b_1$  and will be entirely devoted to consumption.

Thus, the consumer's problem at date 1 is to choose a consumption plan  $c = (c_1, c_2)$  and bond holding  $b_1$  to solve

$$\begin{aligned} \max_{(c_1, c_2) \geq 0, b_1} \quad & u_1(c_1) + u_2(c_2) \\ \text{s.t.} \quad & q_1 b_1 \leq \alpha a_1 \\ & c_1 = \alpha a_1 - q_1 b_1 \\ & c_2 = (1 - \alpha) a_2 + b_1. \end{aligned} \tag{10}$$

It is clear that in this case the liquidity constraint,  $q_1 b_1 \leq \alpha a_1$ , requiring that the expenditure on bonds does not exceed the consumer's available income, is implied by the date-1 budget constraint,  $c_1 = \alpha a_1 - q_1 b_1$ , and the condition  $c_1 \geq 0$ . Then the necessary and sufficient conditions for  $(c_1, c_2)$  and  $b_1$  to be a solution of the consumer's decision problem are the two budget constraints, i.e., the second and third constraints in (10), and the first-order condition

$$q_1 = M(c). \tag{11}$$

**(ii) Late producers in default:**  $d_0 = a_1 > q_1 a_2$  The only difference for the consumer with respect to the previous case is that he now receives no payment at date 2, but instead receives an amount  $(1 - \alpha) q_1 a_2$  in sub-period  $C$  of date 1. Hence the income available to buy bonds when the bond market opens is still equal to  $\alpha a_1$  while the income which can be used for consumption is now  $\alpha a_1 + (1 - \alpha) q_1 a_2 - q_1 b_1$  at date 1 and  $b_1$  at date 2. Hence, the consumer's problem at date 1 is to choose a consumption plan  $c$  and bond holding  $b_1$  to solve

$$\begin{aligned} \max_{(c_1, c_2) \geq 0, b_1} \quad & u_1(c_1) + u_2(c_2) \\ \text{s.t.} \quad & q_1 b_1 \leq \alpha a_1 \\ & c_1 = \alpha a_1 - q_1 b_1 + q_1 (1 - \alpha) a_2 \\ & c_2 = b_1. \end{aligned} \tag{12}$$

In this case, the liquidity constraint is no longer redundant. The necessary and sufficient conditions for  $(c_1, c_2)$  and  $b_1$  to be an optimum are the three constraints in (12) and the first-order condition

$$q_1 \leq M(c), \tag{13}$$

where the inequality (13) is strict only when the liquidity constraint is binding, i.e.,  $q_1 b_1 = \alpha a_1$ . When  $q_1 < M(c)$ , the consumer would like to save more, but is unable to use the

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<sup>7</sup>This follows from the fact that, as already said, in equilibrium each consumer holds a negligible amount of the debt issued by any entrepreneur and all projects are independent, so the law of large numbers applies.

payment  $q_1(1 - \alpha)a_2$  he anticipates receiving in sub-period  $C$  as collateral in order to borrow the cash needed in sub-period  $B$  to purchase additional short-term debt.

Note that the first-order conditions (11) and (13) imply that our earlier auxiliary assumption (3) will indeed be satisfied in equilibrium.

### 3.6 Market clearing

Now we are ready to put together the different elements of the model to define an equilibrium. An equilibrium consists of a project  $\bar{a}$  chosen by entrepreneurs, a consumption plan  $\bar{c}$  chosen by consumers (with the implied value  $\bar{\lambda}$  of the marginal utility of date-0 income), and prices  $V(\cdot)$  and  $\bar{q}_1$  for the bonds issued, respectively, at dates 0 and 1 such that markets clear.

Market clearing at date 0 requires the entrepreneurs' supply of bonds to equal consumers' demand. As anticipated in the previous section, the specification of the bond-price function  $V(\cdot)$  in (8) and (9), with  $(\bar{q}_1, \bar{c}, \bar{\lambda})$  as above, together with Claim 1, ensure market clearing holds if

$$V(\bar{a}, \max\{\bar{a}_1, \bar{q}_1\bar{a}_2\}) = 1.$$

The market value of the debt issued allows entrepreneurs to raise just enough funds to finance the projects they have chosen.

Since markets do not re-open at date 2, the only other market-clearing condition concerns the bond market at date 1. The specification of the market-clearing condition again depends on whether late producers are solvent or in default when they choose project  $\bar{a}$  and the bond price is  $\bar{q}_1$ .

If late producers are solvent, they will roll over their debt, offering short-term debt with a face value of  $d_1 = \bar{a}_2$  to the bond holders. So, when the bond market opens in sub-period  $B$ , they have no need to issue new debt and there is no supply of bonds in the market. In equilibrium, the bond price must be such that the consumers' demand for bonds equals zero. In other words,

$$(\bar{c}_1, \bar{c}_2) = (\alpha\bar{a}_1, (1 - \alpha)\bar{a}_2) \text{ and } \bar{b}_1 = 0. \quad (14)$$

must be a solution of problem (10). This is the case if and only if  $\bar{q}_1 = M(\bar{c})$ , so that the consumers' first-order condition (11) is satisfied.

The more interesting case is the one in which late producers are forced to default, package their claims to future output as collateralized debt, and supply bonds with face value  $d_1 = \bar{a}_2$  when the market opens in sub-period  $B$ . In equilibrium, consumers must now demand a positive amount of bonds, that is,

$$(\bar{c}_1, \bar{c}_2) = (\alpha\bar{a}_1, (1 - \alpha)\bar{a}_2) \text{ and } \bar{b}_1 = (1 - \alpha)\bar{a}_2 \quad (15)$$

must solve (12). This happens if the consumers' first-order condition (13) holds. The first-order condition takes two possible forms, according to whether the liquidity constraint  $\bar{q}_1\bar{b}_1 \leq \alpha\bar{a}_1$  holds as an equality or as an inequality. In the first case, the liquidity constraint is binding and we have  $\bar{q}_1\bar{b}_1 = \alpha\bar{a}_1$  and  $\bar{q}_1 \leq M(\bar{c})$ ; in the second case, we have  $\bar{q}_1 = M(\bar{c})$  and

$\bar{q}_1 \bar{b}_1 < \alpha \bar{a}_1$ . These two conditions are equivalent to:<sup>8</sup>

$$\bar{q}_1 = \min \left\{ \frac{\alpha \bar{a}_1}{(1 - \alpha) \bar{a}_2}, M(\bar{c}) \right\}. \quad (16)$$

We can now state the definition of a symmetric competitive equilibrium (with nonzero output), that is, an equilibrium in which (some) entrepreneurs are active and choose the same value of  $(a, d_0)$ .

**Definition 1** A (*symmetric*) *competitive equilibrium* consists of a project  $\bar{a}$ , a corresponding consumption stream  $(\bar{c}_1, \bar{c}_2) = (\alpha \bar{a}_1, (1 - \alpha) \bar{a}_2)$  and a date-1 price of the bond  $\bar{q}_1$  such that (a)  $\bar{a}$  solves the entrepreneur's problem (4) when  $V(\cdot)$  is given by (8) and (9); (b) the bond market clears at date 0,

$$V(\bar{a}, \max\{\bar{a}_1, \bar{q}_1 \bar{a}_2\}) = 1;$$

and (c) the bond market clears at date 1,

$$\bar{q}_1 = \begin{cases} M(\bar{c}) & \text{if } \bar{a}_1 \leq \bar{q}_1 \bar{a}_2 \\ \min \left\{ \frac{\alpha \bar{a}_1}{(1 - \alpha) \bar{a}_2}, M(\bar{c}) \right\} & \text{otherwise.} \end{cases}$$

Given the non-convexities in the entrepreneur's decision problem, a symmetric equilibrium may not always exist. So we can only prove the existence of an equilibrium in general if we allow for the possibility that an array of projects is chosen. In that case, we say the equilibrium is *mixed*.

**Proposition 2** Under the stated assumptions on consumers' preferences and the technology, a (possibly mixed) competitive equilibrium always exists.

## 4 When are equilibria efficient?

Now that we have derived a reduced form characterization of equilibrium, we are ready to analyze its efficiency. In this section we determine the conditions under which the unique (symmetric) efficient allocation can be supported as an equilibrium. We will find that this happens under two quite different circumstances. The first one is when an equilibrium exists where early producers default and late producers roll over their debt to the third and final

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<sup>8</sup>This can be seen by noticing that the first condition can be restated as

$$\bar{q}_1 = M(\bar{c}) < \frac{\alpha \bar{a}_1}{(1 - \alpha) \bar{a}_2}.$$

and the second one as

$$\bar{q}_1 = \frac{\alpha \bar{a}_1}{(1 - \alpha) \bar{a}_2} \leq M(\bar{c}).$$

period. In such equilibrium, there is no trade in the asset markets in the middle period and the equilibrium supply of liquidity is then irrelevant. The second one is when we have an equilibrium where early producers are solvent and late producers default, but the supply of liquidity is sufficiently high that assets trade at their fundamental value. These two cases exhaust the possibilities for supporting an efficient allocation as a competitive equilibrium. In the remainder of this section, we use the reduced-form characterization of equilibrium in Definition 1 to identify the parameter values for which these cases obtain.

From Definition 1, we can partition the set of symmetric competitive equilibria according to whether  $\bar{q}_1 = M(\bar{c})$  or  $\bar{q}_1 < M(\bar{c})$ . In an equilibrium where  $\bar{q}_1 = \frac{\alpha \bar{a}_1}{(1-\alpha)\bar{a}_2} < M(\bar{c})$ , late producers default ( $\bar{q}_1 \bar{a}_2 < \bar{a}_1$ ) and the condition for an (interior) solution of the entrepreneur's problem is:

$$\frac{\alpha}{(1-\alpha)M(\bar{c})} = \frac{1}{-\varphi'(\bar{a}_1)} \left( \frac{\bar{a}_2}{\bar{a}_1} \right)^2. \quad (17)$$

Condition (17) is different from the efficiency condition (1) and, in fact, we will show that the equilibrium is always inefficient in this case. In all other equilibria we have  $\bar{q}_1 = M(\bar{c})$  and an (interior) solution of the firm's problem  $\bar{a}$  satisfies

$$\frac{\alpha}{(1-\alpha)M(\bar{c})} = -\varphi'(\bar{a}_1). \quad (18)$$

In other words, the consumers' MRS equals the project's MRT, the same as in the efficiency condition (1). Thus, a symmetric competitive equilibrium is Pareto-efficient if and only if  $q_1 = M(\bar{c})$ .

As shown in Section 2, the economy we have described admits a unique, symmetric, Pareto-efficient allocation, supported by the project  $a^*$  that satisfies equation (1) and  $a_2^* = \varphi(a_1^*)$ . The next result identifies conditions on preferences and technology under which a Pareto-efficient competitive equilibrium exists.

**Proposition 3** *Let  $a^*$  be a feasible project supporting an efficient allocation. This allocation can be decentralized as a competitive equilibrium if and only if either (i)*

$$\frac{u'_2((1-\alpha)a_2^*)}{u'_1(\alpha a_1^*)} \geq \frac{a_1^*}{a_2^*}$$

*holds, or (ii)*

$$\frac{\alpha a_1^*}{(1-\alpha)a_2^*} \geq \frac{u'_2((1-\alpha)a_2^*)}{u'_1(\alpha a_1^*)} < \frac{a_1^*}{a_2^*}.$$

Whenever (i) or (ii) holds, we show that an efficient equilibrium exists where  $\bar{q}_1 = M(c^*)$ ,<sup>9</sup> and late producers are solvent (in case (i)) or default (in case (ii)). When  $q_1 = M(c^*)$ ,  $a^*$

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<sup>9</sup> $c^* = (\alpha a_1^*, (1-\alpha)a_2^*)$ .

solves the firm's choice problem if it satisfies (18), which coincides with (1). On the other hand, when neither (i) nor (ii) hold, that is, when

$$\frac{\alpha a_1^*}{(1-\alpha) a_2^*} < \frac{u_2'((1-\alpha) a_2^*)}{u_1'(\alpha a_1^*)} < \frac{a_1^*}{a_2^*}, \quad (19)$$

an equilibrium supporting  $a^*$ , if it exists, must satisfy  $\bar{q}_1 < M(c^*)$ . As we have seen above, in such an equilibrium,  $a^*$  is a solution of the entrepreneur's problem only if it satisfies (17), which we will show is impossible. Hence we conclude that no Pareto-efficient equilibrium exists in that case. Note that (19) can only hold if  $\alpha < 1/2$ .

**Proof.** Set

$$\bar{q}_1 = M(c^*) = \frac{u_2'((1-\alpha) a_2^*)}{u_1'(\alpha a_1^*)}.$$

In case (i) we have  $\bar{q}_1 a_2^* = M(c^*) a_2^* \geq a_1^*$ , so if entrepreneurs choose  $a^*$  late producers are solvent and the bond market clears with zero trade. In case (ii)  $M(c^*) a_2^* < a_1^*$ , so if  $a^*$  is chosen, late producers default and the bond market again clears, since

$$\bar{q}_1 = M(c^*) \leq \frac{\alpha a_1^*}{(1-\alpha) a_2^*},$$

To conclude that there is a competitive equilibrium in which project  $a^*$  is chosen, it is only left to prove that  $a^*$  solves the entrepreneurs' choice problem at  $\bar{q}_1$ . Note that at  $\bar{q}_1 = M(c^*)$ , both for  $a$  such that  $\bar{q}_1 a_2 < a_1$  (late producers default) and for  $a$  such that  $\bar{q}_1 a_2 \geq a_1$  (early producers default), we have

$$V(a, \max\{a_1, \bar{q}_1 a_2\}; \bar{q}_1, c^*, \lambda) = \frac{1}{\lambda} [u_1'(\alpha a_1^*) \alpha a_1 + u_2'((1-\alpha) a_2^*) (1-\alpha) a_2].$$

The condition for a maximum of this expression is the same as the first-order condition for efficiency, given by (1) and clearly satisfied by  $a^*$ . Hence,  $a^*$  always maximizes the firm's profits at  $\bar{q}_1 = M(c^*)$  and  $(a^*, \bar{q}_1)$  is an equilibrium.

Next we show that the efficient allocation cannot be decentralized as a competitive equilibrium when neither (i) nor (ii) holds, that is under (19). The proof is by contradiction. Suppose  $(a^*, \bar{q}_1)$  is an equilibrium. Then market-clearing requires

$$\bar{q}_1 = \min \left\{ \frac{u_2'((1-\alpha) a_2^*)}{u_1'(\alpha a_1^*)}, \frac{\alpha a_1^*}{(1-\alpha) a_2^*} \right\} = \frac{\alpha a_1^*}{(1-\alpha) a_2^*}$$

In addition,  $a^*$  must be the entrepreneurs' optimal choice at  $\bar{q}_1 = \frac{\alpha a_1^*}{(1-\alpha) a_2^*}$ . Since  $a_1^* > \bar{q}_1 a_2^*$ , late producers default at  $a^*$  and  $a^*$  must then be an interior local maximum of problem (4) when  $V(\cdot)$  is given by (9), that is, (17) must hold. From (17), under (19) we get

$$\frac{a_2^*}{a_1^*} \frac{1}{(-\varphi'(a_1^*))} < 1.$$

Using the first-order condition for efficiency, (1), which must hold at  $a^*$ , this inequality can be rewritten as

$$\frac{a_2^* (1 - \alpha) M(c^*)}{a_1^* \alpha} < 1,$$

a contradiction to (19). This completes the proof of the theorem. ■

## 5 Scarce liquidity and inefficiency

Proposition 3 identifies the parameter values for which it is possible to decentralize the efficient allocation. Since a competitive equilibrium always exists, by Proposition 2, it follows that an inefficient equilibrium exists whenever the necessary and sufficient conditions of Proposition 3 are not satisfied. That is, when the parameters of the economy are such that (19) holds, an *inefficient* equilibrium must exist. What can we say about its properties? In this section we explore a special case of the economy to get a better insight into the features of an inefficient equilibrium and, in particular, the form that the distortion of investment decisions takes.

But first we state a general property of inefficient equilibria. As we saw in the proof of Proposition 3, when  $\bar{q}_1 = M(\bar{c})$ , the expressions for  $V(\cdot)$  in (8) and in (9) coincide and the solution of the entrepreneurs' problem (4) is always given by an efficient project. Hence,

**Claim 2** *At any inefficient equilibrium,  $\bar{q}_1 < M(\bar{c})$  and at least a positive fraction of entrepreneurs chooses a production plan in which late producers default.*

In the rest of this section, we restrict attention to economies where consumers have a linear utility function:

$$u(c_1) + u_2(c_2) = c_1 + \beta c_2. \quad (20)$$

The marginal rate of substitution  $M(c)$  is then constant and equal to the subjective discount factor  $\beta > 0$ . As shown in the lemma below, the conditions under which efficient allocations can be decentralized reduce in this case to conditions on  $\beta$ . Equilibria can then be conveniently classified according to the value of  $\beta$ .

Notice first that the first-order condition for efficiency, (1), simplifies to

$$-\varphi'(a_1^*) (1 - \alpha) = \alpha/\beta, \quad (21)$$

and has a unique solution  $a_1^*(\beta)$ , where  $a_1^*(\cdot)$  is a continuous function of  $\beta$ . It can immediately be verified that  $a_1^*(\beta)$  is monotonically decreasing in  $\beta$  and that  $a_1^* \rightarrow 0$  as  $\beta \rightarrow \infty$  and  $a_1^* \rightarrow 1$  as  $\beta \rightarrow 0$ . Then  $a_1^*(\beta)/\varphi(a_1^*(\beta))$  is also monotonically decreasing in  $\beta$  and tends to  $\infty$  as  $\beta$  tends to 0. The following lemma is the analogue of Proposition 3 for preferences satisfying (20).

**Lemma 2** *Suppose consumers' preferences are given by the utility function in (20). Then there exist positive numbers  $\beta^*$  and  $\beta^{**}$  such that:*

$$(i) \quad \beta \geq \beta^{**} \iff \beta \geq \frac{a_1^*(\beta)}{a_2^*(\beta)};$$

and, in this case, there is an efficient equilibrium where  $\bar{q}_1 = \beta$  and late producers are solvent;

$$(ii) \quad \beta \leq \beta^* \iff \beta \leq \frac{\alpha a_1^*(\beta)}{(1-\alpha)a_2^*(\beta)};$$

and, in this case, for all  $\beta \leq \min\{\beta^*, \beta^{**}\}$  there is an efficient equilibrium in which  $\bar{q}_1 = \beta$  and late producers default;

$$(iii) \quad \beta^* < \beta^{**} \iff \alpha < 1/2.$$

By Proposition 3 it also follows that for any  $\beta$  in the region  $(\beta^*, \beta^{**})$  — non-empty if  $\alpha < 1/2$  — any equilibrium is inefficient. Given our interest in inefficient equilibria, we will focus here on values of  $\beta < \beta^{**}$ . To characterize the properties of equilibria in this region, we investigate first how the entrepreneurs' optimal project varies with  $q_1$ , for  $q_1 \leq \beta$ . In the proof of the next proposition, we show that, if the optimal project is such that late producers default, it is an interior maximum of (9), while if it is such that late producers are solvent it is a corner solution (i.e., the solution occurs at the value  $\hat{a}_1$  such that  $\hat{a}_1 = q_1\varphi(\hat{a}_1)$ ). We can then compare the maximal value of (9) and of (8) to determine which one solves the entrepreneurs' choice problem (4) for each  $q_1 \leq \beta$  and, finally, verify at which prices the bond market-clearing condition is satisfied. In the next result we impose an additional property on the technology  $\varphi$  requiring that it be *symmetric*, that is:  $a_1 = \varphi(a_2)$  iff  $a_2 = \varphi(a_1)$ .

**Proposition 4** *Suppose the consumers' utility function is as in (20),  $\varphi$  is symmetric and  $\beta < \beta^{**}$ . If [i]  $\beta > \beta^*$ , all equilibria are inefficient and the equilibrium set is described by one of the following three cases : (a) there is a unique, symmetric equilibrium in which late producers default; (b) there is a unique mixed equilibrium in which entrepreneurs choose between two production plans, at one of which late producers are solvent and at the other late producers default; (c) there are three equilibria, one symmetric equilibrium as in a) and two mixed equilibria as in b). On the other hand, if [ii]  $\beta < \beta^*$  a unique, efficient equilibrium exists.*

In all the inefficient symmetric equilibria characterized in the above proposition, the project chosen in equilibrium is such that  $\bar{a}_1 > a_1^*(\beta)$ , thus the distortion takes the form of a project with a higher payoff at date 1 than the efficient project  $a^*$ . This can be viewed as a response to the shortage of liquidity at  $q_1 = \beta$  (as we see from (19), the supply of liquidity by consumers, equal to  $\alpha a_1^*/\beta$ , is strictly lower than firms' demand,  $(1-\alpha)a_2^*$ ); this drives down  $q_1$  and induces entrepreneurs to choose more liquid projects, that is projects with higher payoffs at date 1 to profit from the higher rate of return available at that date. In mixed equilibria we still have  $\bar{q}_1 < \beta$ , and a positive fraction of entrepreneurs choosing a project such that  $a_1' > a_1^*(\beta)$ , but the rest of them choose a project  $a_1'' < a_1^*(\beta)$  where late producers are solvent, which reduces the demand for liquidity at date 1. Notice that when multiple equilibria exist, as in case (i.c), they are always Pareto-ranked.

How do the properties of the equilibria vary with  $\beta$  in the inefficient region  $(\beta^*, \beta^{**})$ ? If  $\beta$  increases, at the efficient project  $a_1^*$  decreases while, as shown in the proof of the proposition above, at a symmetric equilibrium  $\bar{a}_1$  increases; hence the inefficiency gap increases.

**Sunspot Equilibria.** We show next that an additional type of inefficient equilibria, where the date-1 bond price fluctuates randomly in response to the realization of a sunspot event, might also exist. The argument is constructive and general. However, both to illustrate it more simply and to allow a closer comparison with the properties of the equilibria characterized in the first part of this section, the argument will be presented for the case where the consumers' utility function is linear, as in (20).

**Proposition 5** *Suppose the consumers' utility function is given by (20) and  $\beta > \beta^{**}$ ,  $\alpha < 1/2$ . For  $\varepsilon > 0$  sufficiently small, a sunspot equilibrium exists in which entrepreneurs choose a project  $\bar{a}$  and the bond price takes the values  $\tilde{q}_1 = \beta$  and*

$$\tilde{q}_1 = \frac{\alpha \bar{a}_1}{(1 - \alpha) \bar{a}_2}, \quad (22)$$

with probabilities  $1 - \varepsilon$  and  $\varepsilon$  respectively.

**Proof.** We will show that, for  $\varepsilon$  sufficiently small, the entrepreneurs' optimal choice is obtained as a solution of the following programme<sup>10</sup>:

$$\max_{a_1} \left\{ (1 - \varepsilon) [\alpha a_1 + \beta (1 - \alpha) \varphi(a_1)] + \varepsilon \left[ \beta \frac{\alpha a_1}{\tilde{q}_1} + \tilde{q}_1 (1 - \alpha) \varphi(a_1) \right] \right\} \quad (23)$$

subject to the constraints  $a_1 \leq \beta a_2$  and  $a_1 > \tilde{q}_1 a_2$ . That is, the entrepreneurs' optimal choice is given by a project such that late producers are solvent when the date 1 bond price is  $\tilde{q}_1 = \beta$  and default when it is  $\tilde{q}_1$ .

Note that the maximum of  $\alpha a_1 + \beta (1 - \alpha) \varphi(a_1)$  is attained at the efficient project  $a^*$ . By Lemma 2,  $\beta > \beta^{**}$  implies that  $\beta > a_1^*/a_2^*$ , while  $\alpha < 1/2$  implies that  $a_1^*/a_2^* > \alpha a_1^*/[(1 - \alpha) a_2^*]$  so that  $a^*$  satisfies the constraints of problem (23) (when  $\tilde{q}_1$  is set equal to  $\alpha a_1^*/[(1 - \alpha) a_2^*]$ ) and is a solution of such problem when  $\varepsilon = 0$ . Since, again by Lemma 2,  $\alpha < 1/2$  implies  $\beta^* < \beta^{**}$  and hence for  $\beta > \beta^{**}$  we also have  $\beta > \alpha a_1^*/[(1 - \alpha) a_2^*]$ , the bonds market clearing condition (c) of Definition 1 is satisfied both when the price  $q_1$  is equal to  $\beta$  and when it equals  $\alpha a_1^*/[(1 - \alpha) a_2^*]$ . By continuity, when  $\varepsilon$  is sufficiently small the solution  $\bar{a}$  of problem (23) above (at the corresponding value of  $\tilde{q}_1$  given by (22)) will be close to  $a^*$  and the same properties hold.

To be able to say that  $\bar{a}$  is also a solution of the entrepreneur's problem (4) it remains to be shown that the entrepreneur's profits are also higher at  $\bar{a}$  than at any other project such that late producers default both when  $q_1$  equals  $\beta$  and when it equals  $\tilde{q}_1$ , that is, at the maximum of

$$(1 - \varepsilon) \{ \alpha a_1 + \beta (1 - \alpha) \varphi(a_1) \} + \varepsilon \left\{ \beta \frac{\alpha a_1}{\tilde{q}_1} + \tilde{q}_1 (1 - \alpha) \varphi(a_1) \right\} \quad (24)$$

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<sup>10</sup>It is immediate to verify that the terms multiplying  $(1 - \varepsilon)$  and  $\varepsilon$ , respectively, coincide with the terms in (8) and (9) when the utility function is given by (20).

subject to  $a_1 > \beta a_2$ ,<sup>11</sup> as well as at any project such that late producers never default, i.e. at the maximum of

$$(1 - \varepsilon) \{ \alpha a_1 + \beta (1 - \alpha) \varphi(a_1) \} + \varepsilon \left\{ \beta \frac{\alpha a_1}{\tilde{q}_1} + \beta (1 - \alpha) \varphi(a_1) \right\} \quad (25)$$

subject to the constraint  $a_1 \leq \tilde{q}_1 a_2$ .<sup>12</sup> For  $\varepsilon$  sufficiently small, (24) will be maximized at  $a$  such that  $a_1 = \beta a_2$  and (25) at  $a$  such that  $a_1 = \tilde{q}_1 a_2$ , and the firm's profits will clearly be less than at  $\bar{a}$ . ■

Combining the result in this proposition with the one in Lemma 2, when  $\beta > \beta^{**}$  and  $\alpha < 1/2$  at least two competitive equilibria exist, an efficient one where  $\bar{q}_1 = \beta$  and an inefficient sunspot equilibrium where the bond price takes the values  $\beta$  and  $\tilde{q}_1 < \beta$ , with probabilities  $1 - \varepsilon$  and  $\varepsilon$ . In the latter, the self-fulfilling belief that the bond price will collapse with positive probability at date 1 induces late producers to default when that happens. The anticipation of this event leads entrepreneurs to choose a project with an inefficiently high level of output at date 1,  $\bar{a}_1 > a_1^*$ , to profit from the low bond price at that date. For  $\varepsilon$  small, the distortion in the investment decision will also be relatively small, but the collapse in the bond price, when it occurs, will be quite significant.

## 6 Restoring efficiency

We have focused on the existence and characterization of inefficient equilibria because of the insights they offer into the role of liquidity in market failures. To understand why the market fails to allocate resources efficiently and how the market failure might be alleviated or avoided altogether, it is helpful to ask what changes in our set-up would ensure that competitive equilibria are always efficient. In this section, we consider three remedies that illustrate the role of new markets and of government intervention in achieving an efficient allocation of resources.

The decentralization of the efficient allocation is problematic only when late producers default and in particular when condition (19) holds, so we focus exclusively on this case in what follows. Letting  $a^*$  be, as before, a project supporting a Pareto efficient allocation, this condition says that at the efficient asset price  $M(c^*)$ , late producers default

$$M(c^*)a_2^* < a_1^*$$

and there is insufficient liquidity for the asset market to clear

$$\frac{\alpha a_1^*}{(1 - \alpha) a_2^*} < M(c^*).$$

<sup>11</sup>Since for  $\varepsilon$  small, by continuity, we still have  $\beta > \tilde{q}_1$ , the other constraint  $a_1 > \tilde{q}_1 a_2$  is redundant.

<sup>12</sup>The case where late producers default at  $\tilde{q}_1 = \beta$  but not at  $\tilde{q}_1$  is clearly impossible, since  $\tilde{q}_1 > \tilde{q}_1$ .

There are several ways to support the decentralization of the efficient allocation under these circumstances. We begin by considering the effect of introducing (firm-specific) contingent claims.

### Completing the market.

Markets are incomplete at date 0 because it is impossible to make trades contingent on the state of an individual firm at date 1. Suppose we introduce firm-specific securities that pay one unit of the good at date 1 if the entrepreneur turns out to be an early (resp. late) producer and nothing otherwise. These securities are traded at date 0 and their prices are denoted by  $q_{01}$  and  $q_{02}$ , respectively. Now each entrepreneur is able to purchase insurance against being an early or late producer and to avoid default if he wishes. With these additional markets, we can show that there exists a competitive equilibrium in which entrepreneurs choose the project  $a^*$  and the market-clearing price is  $q_1^* = M(c^*)$ .

Let  $b_{01}$  (respectively,  $b_{02}$ ) denote the demand for securities that pay out if the entrepreneur is an early (respectively, late) producer. For any project  $a$  and any face value  $d_0$  of the debt issued at date 0 to finance it, an appropriate portfolio  $(b_{01}, b_{02})$  can be found to ensure that the entrepreneur's total revenue in period 1 is equal to  $d_0$ :

$$d_0 = a_1 + b_{01} = M(c^*) a_2 + b_{02}. \quad (26)$$

In other words, default never occurs. The firm's debt is riskless in this case and pays one unit for sure. Let  $q_0$  be its unit price in period 0. Since all firm specific risk can be fully diversified, the prices for the contingent claims are fair:

$$(q_{01}, q_{02}) = (\alpha q_0, (1 - \alpha) q_0). \quad (27)$$

The firm's revenue at date 0 is now given by

$$q_0 d_0 - 1 - (q_{01} b_{01} + q_{02} b_{02})$$

We can then substitute from (27) for the prices of the contingent claims in the above expression and use (26) to rewrite  $d_0$  and  $b_{01}$  in terms of  $b_{02}$  when a portfolio of contingent claims is acquired offering full insurance against fluctuations in revenue. From this substitution we get the following expression:

$$\begin{aligned} q_0 [M(c^*) a_2 + b_{02} - 1 - \alpha (M(c^*) a_2 + b_{02} - a_1) - (1 - \alpha) b_{02}] = \\ = q_0 [\alpha a_1 + (1 - \alpha) M(c^*) a_2 - 1] \end{aligned}$$

It is clear that the project which maximizes the above expression, and hence constitutes the value maximizing choice of the firm<sup>13</sup>, is the efficient one,  $a^*$ .

Note also that the optimal level of  $b_{02}$  is indeterminate because the usual Modigliani-Miller argument implies that the firm's capital structure is indeterminate. Any portfolio

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<sup>13</sup>If the firm chooses not to fully insure, and hence default occurs, it is immediate to verify that its value is never higher.

$(b_{01}, b_{02})$  and debt level  $d_0$  that finance the project  $a^*$  and ensure no default, that is, satisfying (26), are a solution of the entrepreneur's problem. Without loss of generality, we can impose the additional condition that the portfolio  $(b_{01}, b_{02})$  must be self-financing, that is,

$$q_{01}b_{01} + q_{02}b_{02} = 0.$$

Using (27) this equation can be rewritten as

$$\alpha b_{01} + (1 - \alpha) b_{02} = 0,$$

that is, the market for contingent claims clears (with zero trades by consumers). We conclude that the project  $a^*$  and the bond price  $q_1^* = M(c^*)$  together constitute a competitive equilibrium.

A similar argument suffices to show that default *cannot* occur in equilibrium. For example, consider the inefficient equilibrium found in the previous sections in which early producers default. In the presence of markets for contingent claims, entrepreneurs could achieve a higher value of the debt issued at date 0 by choosing a portfolio  $(b_{01}, b_{02})$  that avoids the possibility of default and adjusting the face value of the debt  $d_0$  accordingly.

### Asset purchases with IOU's

The introduction of contingent claims allows entrepreneurs to fully insure against the variability in their firms' future cashflow and hence removes the need for default in equilibrium. An alternative way to attain efficiency is to remove the "cash in advance constraint" that restricts asset purchases in period 1 and generates distortions in the asset price in that period. In this case, default still occurs at date 1, but does not distort the entrepreneur's decisions at date 0.

Suppose, for example, that consumers are allowed to purchase assets using IOU's backed by their claims on defaulting firms. In that case the total amount of "cash" available to consumers in the bond market in sub-period  $B$  is equal to the revenue received from early producers,  $\alpha a_1$ , plus the value of the liquidated assets received from defaulting firms in sub-period  $C$ ,  $q_1 (1 - \alpha) a_2$ . The market-clearing condition now becomes

$$q_1 (1 - \alpha) a_2 \leq \alpha a_1 + q_1 (1 - \alpha) a_2,$$

with  $q_1 = M(c)$  if the inequality is strict. Of course, the inequality must be strict since  $\alpha a_1 > 0$ , so the equilibrium asset price will be at the efficient level  $q_1 = M(c)$  and entrepreneurs will make efficient decisions. Since the value of the liquidated assets is returned to creditors in sub-period  $C$ , the amount of "cash" available to buy assets will always be greater than the purchase price, which implies that the cash in advance constraint is never binding.

### Government intervention

Another way of preventing market failure is to use Pigovian taxes to offset the distortion caused by the liquidity-constrained level of the asset price. The distortion, as we saw,

consists in the adoption of projects with an inefficiently high payoff at date 1. Suppose the government adopts a tax policy that imposes a tax on returns from early producers and a subsidy on returns from late producers. Let  $\tau_1$  denote the tax on the income accruing to bondholders from early producers and  $-\tau_2$  the tax on the income they receive from late producers. We will show that  $\tau = (\tau_1, \tau_2)$  can be chosen so that the efficient project  $a^*$  is chosen in equilibrium.

All tax payments are collected (or paid out, in the case of negative taxes) in the initial sub-period  $A$  at the beginning of date 1. The policy is designed to raise zero revenue,

$$\alpha\tau_1a_1^* = (1 - \alpha)\tau_2q_1a_2^*, \quad (28)$$

where  $q_1$  is the market-clearing price. Because the tax imposed on the revenue consumers obtain from early producers is equal to the subsidy on the revenue consumers get from late producers, and both the tax and the subsidy<sup>14</sup> are paid and received at the beginning of date 1, the total liquidity in the market remains equal to  $\alpha a_1^*$ . In equilibrium we still have late producers defaulting so that the market-clearing price  $q_1$  is again given by

$$q_1 = \frac{\alpha a_1^*}{(1 - \alpha)a_2^*}. \quad (29)$$

Substituting from this condition into the budget balance equation (28), we see that

$$\alpha\tau_1a_1^* = (1 - \alpha)\tau_2a_2^*\frac{\alpha a_1^*}{(1 - \alpha)a_2^*},$$

which implies that  $\tau_1 = \tau_2 = \tau$ , say.

The entrepreneur's problem then is to choose  $a_1$  to maximize the value of the debt, now proportional to

$$(1 - \alpha)q_1\varphi(a_1)\left(1 + \tau\frac{M(c^*)}{q_1}\right) + M(c^*)\frac{\alpha a_1(1 - \tau)}{q_1}.$$

This expression takes into account the taxes bondholders must pay (or receive) on their revenue as well as the fact that the subsidy (negative tax) on the returns from the late producers is paid at the beginning of date 1, before the asset market opens, and hence can be invested in bonds and consumed at date 2. This is optimal because  $q_1 < q_1^*$ . With this formulation of the problem, the first-order condition for value maximization is

$$(1 - \alpha)q_1\varphi'(a_1)\left(1 + \tau\frac{M(c^*)}{q_1}\right) + M(c^*)\frac{\alpha(1 - \tau)}{q_1} = 0.$$

Evaluating the above expression at  $a^*$  and using the first-order condition for efficiency, (1) yields:

$$-\frac{\alpha q_1}{M(c^*)} - \alpha\tau + M(c^*)\frac{\alpha(1 - \tau)}{q_1} = 0,$$

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<sup>14</sup>Unlike the payments from entrepreneurs. Also, the fact that these are taxes on bondholders' income, not corporate taxes, imply that the conditions under which late producers default are unchanged.

which is satisfied if

$$\tau = 1 - \frac{q_1}{M(c^*)}. \quad (30)$$

The tax scheme described above is effective in deterring entrepreneurs from exploiting the arbitrage opportunity provided by the fact that defaulting firms sell at a low value. As a consequence, liquidity remains scarce but entrepreneurs' investment decisions are not distorted.

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# Appendix

## Proof of Proposition 2

First, we extend the definition of a competitive equilibrium given in Definition 1 to the case of a mixed equilibrium. In that case, a fraction  $\mu_D$  of entrepreneurs choose a project  $a^D$  such that late producers default, while the remaining fraction  $\mu_S = 1 - \mu_D$  choose a project  $a^S$  such that late producers are solvent. Consumption is then  $\bar{c} = \left( \sum_{i=D,S} \mu_i \alpha a_1^i, \sum_{i=D,S} \mu_i (1 - \alpha) a_2^i \right)$ , the market clearing condition (16) becomes:

$$\bar{q}_1 = \min \left\{ \frac{\mu_D \alpha a_1^D + \mu_S \alpha a_1^S}{\mu_D (1 - \alpha) a_2^D}, M(\bar{c}) \right\}, \quad (\text{A1})$$

and both  $a^D$  and  $a^S$  solve the entrepreneur's problem (4) at  $\bar{q}_1, M(\bar{c})$ .

Let then

$$X \equiv \{x \in [0, 1]^2 : x_S \leq x_D\}$$

and

$$X_\varepsilon \equiv \{x \in X : \varepsilon \leq x_S, x_D \leq 1 - \varepsilon\},$$

for some fixed but arbitrary  $\varepsilon > 0$ ,

$$M \equiv \left\{ \mu \in \mathbf{R}_+^2 : \sum_{i=S,D} \mu_i = 1 \right\},$$

and

$$M_\varepsilon \equiv \{\mu \in M : \mu_i \geq \varepsilon, \forall i\}.$$

For any small  $\varepsilon > 0$ , a correspondence

$$\Gamma_1 \times \Gamma_2 : M_\varepsilon \times X_\varepsilon \rightrightarrows M_\varepsilon \times X_\varepsilon$$

is constructed in Steps 1-4 below. Let  $(\mu, x)$  denote a generic element of  $M_\varepsilon \times X_\varepsilon$  and  $(\mu', x')$  a generic element of  $\Gamma_1(\mu, x) \times \Gamma_2(\mu, x)$ .

**Step 1.** First, the allocation  $c$  induced by  $(\mu, x)$  is given by:

$$c(\mu, x) = \sum_{i=S,D} \mu_i \cdot (\alpha x_i, (1 - \alpha) \varphi(x_i)).$$

**Step 2.** Secondly, define the map yielding the equilibrium price  $q_1$  induced by the allocation  $(\mu, x)$ :

$$q_1(\mu, x) = \min \left\{ \frac{u'_2(c_2(\mu, x))}{u'_1(c_1(\mu, x))}, \frac{\alpha \{\mu_S x_S + \mu_D x_D\}}{(1 - \alpha) \mu_D \varphi(x_D)} \right\}. \quad (\text{A2})$$

Note that, since  $x \in X_\varepsilon$ , both  $x_S > 0$  and  $x_D < 1$  and, hence,  $\varphi(x_D) > 0$ . Furthermore, since  $\mu \in M_\varepsilon$ , both  $\mu_S > 0$  and  $\mu_D > 0$ . Then  $0 < q_1 < \infty$  and the map defined by the

expression on the right hand side of (A2) is continuous at all  $(\mu, x)$ .

**Step 3.** Next, the following map gives the entrepreneurs' choice which is optimal (profit maximizing) among all projects such that late producers are solvent (for  $c = c(\mu, x)$ ,  $q_1 = q_1(\mu, x)$ ):

$$x'_S(\mu, x) = \arg \max_{\varepsilon \leq a_1 \leq q_1(\mu, x)\varphi(a_1)} \left\{ \frac{u'_2(c_1(\mu, x))}{q_1(\mu, x)} \alpha a_1 + u'_2(c_2(\mu, x))(1 - \alpha)\varphi(a_1) \right\}. \quad (\text{A3})$$

Note that  $x'_S$  is uniquely defined because of the strict concavity of  $\varphi$  and  $x'_S$  is continuous in  $(\mu, x)$  by the Maximum Theorem. The following map yields then the value of the objective function in (A3), evaluated at the optimum  $x'_S$ :<sup>15</sup>

$$V_S(\mu, x) = \left\{ \frac{u'_2(c_1)}{q_1(\mu, x)} \alpha x'_S(\mu, x) + u'_2(c_2(\mu, x))(1 - \alpha)\varphi(x'_S(\mu, x)) \right\}.$$

In a similar way, we define the optimal choice among all projects such that late producers default:

$$x'_D(\mu, x) = \arg \max_{q_1(\mu, x)\varphi(a_1) \leq a_1 \leq 1 - \varepsilon} \left\{ u'_1(c_1(\mu, x))(1 - \alpha)q_1(\mu, x)\varphi(a_1) + u'_2(c_2(\mu, x)) \frac{\alpha a_1}{q_1(\mu, x)} \right\}. \quad (\text{A4})$$

Again,  $x'_D$  is unique, because  $\varphi$  is strictly concave, and  $x'_D$  is continuous in  $(\mu, x)$ , because of the Maximum Theorem. Let then

$$V_D(\mu, x) = \left\{ u'_1(c_1(\mu, x))(1 - \alpha)q_1(\mu, x)\varphi(x'_D(\mu, x)) + u'_2(c_2(\mu, x)) \frac{\alpha x'_D(\mu, x)}{q_1(\mu, x)} \right\}$$

denote the associated value of the objective function in (A4).

**Step 4.** Finally, define the maps:

$$\Gamma_2(\mu, x) = \{x'_S(\mu, x), x'_D(\mu, x)\},$$

and

$$\Gamma_1(\mu, x) = \left\{ \mu' \in \arg \max_{\tilde{\mu} \in M_\varepsilon} \left[ \sum_{i=S, D} \tilde{\mu}_i V_i(\mu, x) \right] \right\}.$$

As established above, both  $x'_S$  and  $x'_D$  are continuous and hence so is  $\Gamma_2$ . The functions  $\{V_i\}_{i=D, S}$ , being the maximum values of the maximization problems (A4), (A3), are continuous by the Maximum Theorem. Also, the image set  $\Gamma_1(\mu, x)$  is compact and convex by construction and upper hemi-continuous by the Maximum Theorem and the continuity of  $\{V_i\}_{i=D, S}$ . Thus  $\Gamma_1 \times \Gamma_2$  satisfies the conditions of Kakutani's theorem, so there exists a fixed point  $(\mu, x)$  for every value of  $\varepsilon > 0$ .

<sup>15</sup>For  $\lambda = 1$ ,  $V_S(\mu, x)$  also equals the market value of the debt issued to finance  $a^S$ .

To prove the existence of an equilibrium, we use then a limiting argument as  $\varepsilon \rightarrow 0$ . Consider a sequence of positive numbers  $\{\varepsilon_n\}$  such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and let  $\{(\mu^n, x^n)\}$  denote the corresponding sequence of fixed points. Since the sequence is bounded there exists a convergent subsequence. By an abuse of notation we use the same notation for the subsequence and write  $\lim_{n \rightarrow \infty} (\mu^n, x^n) = (\mu^0, x^0)$ . We show next that  $(\mu^0, x^0)$  is an equilibrium.

Suppose that  $q_1(\mu^n, x^n)$ , the price corresponding to the fixed point  $(\mu^n, x^n)$ , converges to 0 as  $n$  tends to infinity. This requires, given the specification of the map  $q_1(\cdot)$  in (A2), that either

$$\lim_{n \rightarrow \infty} \mu_S^n x_S^n + \mu_D^n x_D^n = 0, \quad (\text{A5})$$

or  $\lim_{n \rightarrow \infty} u'_2(c_2(\mu^n, x^n)) / u'_1(c_1(\mu^n, x^n)) = 0$ , which in turn may only hold if  $c_1(\mu^n, x^n) \rightarrow 0$ , which is equivalent to, again, (A5) holding. Condition (A5) in turn implies, since  $x_S \leq x_D$ , that  $x_S^n \rightarrow 0$  and that  $\lim_{n \rightarrow \infty} x_D^n > 0$ ,  $\lim_{n \rightarrow \infty} \mu_D^n = 0$ .<sup>16</sup> Hence from

$$\begin{aligned} V_S(\mu^n, x^n) &= \frac{u'_2(c_2(\mu^n, x^n))}{q_1(\mu^n, x^n)} \alpha x_S^n + u'_2(c_2(\mu^n, x^n)) (1 - \alpha) \varphi(x_S^n) \\ &= \frac{u'_2(c_2(\mu^n, x^n)) (1 - \alpha) \varphi(x_D^n)}{\{\mu_S^n / \mu_D^n + x_D^n / x_S^n\}} + u'_2(c_2(\mu^n, x^n)) (1 - \alpha) \varphi(x_S^n) \end{aligned}$$

and

$$V_D(\mu^n, x^n) = u'_1(c_1(\mu^n, x^n)) q_1(\mu^n, x^n) (1 - \alpha) \varphi(x_D^n) + u'_2(c_2(\mu^n, x^n)) \frac{\alpha x_D^n}{q_1(\mu^n, x^n)}$$

we get  $\lim_{n \rightarrow \infty} V_S(\mu^n, x^n) < \infty$  while  $V_D(\mu^n, x^n) \rightarrow \infty$ , which contradicts the previous implication that  $\mu_D^n \rightarrow 0$ . Thus we conclude that  $\lim_{n \rightarrow \infty} q_1(\mu^n, x^n)$  is bounded away from zero.

Then it is straightforward to verify that  $(\mu^0, x^0)$  satisfies the equilibrium conditions:

- (i) the values  $\{x_i^0\}_{i=S,D}$  are the entrepreneurs' optimal choices subject to the constraint that late producers are solvent, respectively default, that is solve (A3) and (A4), for  $\varepsilon = 0$ , at  $q_1^0 = \lim_{n \rightarrow \infty} q_1(\mu^n, x^n)$ ;
- (ii) the distribution  $\mu^0$  is concentrated on the profit-maximizing values among  $\{x_i^0\}_{i=S,D}$ ;
- (iii) the asset market clears at  $t = 1$  because (A2) is satisfied in the limit. ■

## Proof of Lemma 2

Given the properties of  $a_1^*(\beta)$  established in the text, it is clear that there must be a unique value of  $\beta^{**}$  such that

$$\beta \geq \frac{a_1^*(\beta)}{a_2^*(\beta)} \text{ as } \beta \geq \beta^{**}.$$

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<sup>16</sup>Since  $x_D^n$  must be a solution of problem (A4), this is obviously true if  $\lim_{n \rightarrow \infty} u'_1(c_1(\mu^n, x^n)) < \infty$ . But also when  $\lim_{n \rightarrow \infty} u'_1(c_1(\mu^n, x^n)) = \infty$ , since  $q_1 \geq M(c)$ , from the first order conditions of problem (A4) we get  $\lim_{n \rightarrow \infty} \varphi'(x_S^n) = -\infty$  and hence  $\lim_{n \rightarrow \infty} x_D^n > 0$ .

A similar argument shows that there is a unique value of  $\beta^*$  such that

$$\beta \geq \frac{\alpha a_1^*(\beta)}{(1-\alpha) a_2^*(\beta)} \text{ as } \beta \geq \beta^*.$$

By the definition of  $\beta^*$  and  $\beta^{**}$  we then readily see that  $\beta^* < \beta^{**}$  if and only if  $\alpha < 1/2$ . The rest of the claim follows by an immediate application of Proposition 3. ■

## Proof of Proposition 4

Using (20), the specification given in (A4) of the program yielding the optimal production plan among those such that late producers default simplifies to:

$$\max_{q_1 \varphi(a_1) \leq a_1 \leq 1} \left\{ (1-\alpha) q_1 \varphi(a_1) + \beta \alpha \frac{a_1}{q_1} \right\}, \quad (\text{A6})$$

while the one in (A3) giving the optimal project such that late producers are solvent becomes:

$$\max_{0 \leq a_1 \leq q_1 \varphi(a_1)} \left\{ \beta \frac{\alpha a_1}{q_1} + \beta(1-\alpha) \varphi(a_1) \right\}. \quad (\text{A7})$$

Let  $a_1^D(q_1, \beta)$  denote the solution of problem (A6) and  $v_D(q_1, \beta)$  the corresponding value of the objective function, while the corresponding expressions for problem (A7) are  $a_1^S(q_1, \beta)$  and  $v_S(q_1, \beta)$ .

When  $q_1 = \beta$ , as already argued in the proof of Proposition 3, the expressions in (A6) and (A7) are identical and equal to  $\alpha a_1 + \beta(1-\alpha)\varphi(a_1)$ , whose maximum is attained at the efficient project  $a^*(\beta)$ . For any  $\beta < \beta^{**}$ , from Lemma 2 we know that  $a_1^*(\beta) > \beta\varphi(a_1^*(\beta))$ , hence  $a_1^D(\beta, \beta) = a_1^*(\beta)$  and  $v_D(\beta, \beta) > v_S(\beta, \beta)$ . Moreover, since  $\alpha a_1 + \beta(1-\alpha)\varphi(a_1)$  is strictly concave in  $a_1$  and, when we maximize this expression, the constraint  $a_1 \leq \beta\varphi(a_1)$  is binding, the optimal project  $a^S(\beta, \beta)$  such that late producers are solvent is attained at the value of  $a_1$  that is the closest as possible to  $a_1^*$ , i.e. is the largest possible:  $\hat{a}_1$  such that  $\hat{a}_1 = \beta\varphi(\hat{a}_1)$ . More generally, let  $\hat{a}_1(q_1)$  denote the solution of  $a_1 = q_1\varphi(a_1)$ . The following lemma extends the argument to prices  $q_1 < \beta$ :

**Lemma 3** *When  $\beta < \beta^{**}$ , for any  $q_1 < \beta$  we have  $a_1^S(q_1, \beta) = \hat{a}_1(q_1)$  and  $a_1^D(q_1, \beta) \in \arg \max (1-\alpha) q_1 \varphi(a_1) + \beta \alpha \frac{a_1}{q_1}$ , and  $a_1^D(q_1, \beta)$  is decreasing in  $q_1$ .*

**Proof.** It is immediate to see that if the maximal admissible value of  $a_1$ ,  $\hat{a}_1(q_1)$ , solves problem (A7) when  $q_1 = \beta$ , by the same argument the same is true for  $q_1 < \beta$ . Consider next the first order conditions for an interior maximum of problem (A6):

$$(1-\alpha) q_1 \varphi'(a_1) + \beta \frac{\alpha}{q_1} = 0. \quad (\text{A8})$$

As argued above  $a_1^*(\beta)$  solves this equation when  $q_1 = \beta$  and is admissible:  $a_1^*(\beta) > \beta\varphi(a_1^*(\beta))$ . Let  $a_1^{Di}(q_1, \beta)$  denote the solution of (A8) for arbitrary  $q_1$ . Note that  $a_1^{Di}(q_1, \beta)$  is decreasing in  $q_1$ :

$$\frac{\partial a_1^{Di}}{\partial q_1} = \frac{2\beta\alpha}{(q_1)^3 (1-\alpha) \varphi''} < 0.$$

Hence, for all  $q_1 < \beta$ ,  $a_1^{Di}(q_1, \beta)$  is greater than  $a_1^*(\beta)$ , thus still admissible and constitutes so a solution of (A6). ■

Using the properties of  $a_1^S(q_1, \beta)$  and  $a_1^D(q_1, \beta)$  established in the previous Lemma, we can determine the pattern of  $v_S(q_1, \beta)$  and  $v_D(q_1, \beta)$  :

**Lemma 4** *Assume  $\varphi$  is symmetric. Then for all  $\beta < \beta^{**}$ , either we have (i)  $v_D(q_1, \beta) \gg v_S(q_1, \beta)$  for all  $q_1 < \beta$ , or (ii) we can find  $q'_1 < q''_1 < \beta$  such that  $v_D(q_1, \beta) > v_S(q_1, \beta)$  for all  $q_1 < q'_1$  and all  $q_1 > q''_1$ , while  $v_D(q_1, \beta) < v_S(q_1, \beta)$  for all  $q_1 \in (q'_1, q''_1)$ .*

**Proof.** It immediately follows from Lemma 3 that  $v_S(q_1, \beta) = \beta \frac{\hat{a}_1(q_1)}{q_1} = \beta \hat{a}_2(q_1)$ . By the symmetry property of  $\varphi$ ,  $\hat{a}_2(q_1)$  must also be a solution, for all  $q_1$ , of the equation  $\hat{a}_2 = (1/q_1)\varphi(\hat{a}_2)$ . In the rest of the proof of this lemma it is convenient to write  $\theta$  to denote  $1/q_1$ . Hence:

$$\frac{d\hat{a}_2}{d\theta} = \frac{\varphi}{1 - \theta\varphi'}$$

which is always positive, and so is  $\frac{\partial v_S}{\partial \theta} = \beta \frac{d\hat{a}_2}{d\theta}$ . Furthermore,

$$\frac{\partial^2 \hat{a}_2}{\partial (\theta)^2} = \frac{[\varphi'(1 - \theta\varphi') + \theta\varphi''\varphi][\varphi/(1 - \theta\varphi')] + \varphi'\varphi}{(1 - \theta\varphi')^2},$$

always negative since the numerator is the sum of three terms, and they are all negative. Thus  $\hat{a}_2$  - and  $v_S$  - are both strictly concave and increasing functions of  $\theta$ . On the other hand, as shown in that same lemma,  $v_D(\theta, \beta) = (1 - \alpha) \frac{\varphi(a_1^{Di}(\theta, \beta))}{\theta} + \beta\alpha\theta a_1^{Di}(\theta, \beta)$ . Hence, using the envelope theorem:

$$\frac{\partial v_D}{\partial \theta} = - (1 - \alpha) \frac{\varphi(a_1^{Di}(\theta, \beta))}{\theta^2} + \beta\alpha a_1^{Di}(\theta, \beta),$$

which is positive - and hence  $v_D$  is also increasing in  $\theta$  - as long as  $\theta$  is sufficiently large (and increasing for all  $\theta \geq 1/\beta$  if  $(1 - \alpha)\beta\varphi(a_1^*(\beta)) < \alpha a_1^*(\beta)$ , that is if  $\beta < \beta^*$ ). Moreover,

$$\frac{\partial^2 v_D}{\partial (\theta)^2} = \left[ \beta\alpha - \frac{(1 - \alpha)\varphi'}{\theta^2} \right] \frac{\partial a_1^{Di}}{\partial \theta} + 2(1 - \alpha) \frac{\varphi(a_1^{Di}(\theta, \beta))}{\theta^3} = 2(1 - \alpha) \frac{\varphi(a_1^{Di}(\theta, \beta))}{\theta^3} > 0,$$

where the second equality sign follows by (A8). positive for all  $\theta$ , hence  $v_D$  is always convex. Finally, for  $\theta$  large  $v_D(\theta, \beta) - v_S(\theta, \beta) \cong \beta\alpha\theta (a_1^D(\theta, \beta) - \hat{a}_1(\theta, \beta)) > 0$ . Recalling that we have shown the same inequality holds for  $\theta = 1/\beta$ , the result follows from the monotonicity and concavity/convexity properties of  $v_S$  and  $v_D$  established above. ■

On the basis of the previous findings we can now characterize the properties of competitive equilibria in the region  $\beta < \beta^{**}$ . We know from Lemma 2 that an equilibrium with  $q_1 = \beta$

only exists when  $\beta \leq \beta^*$ , since for  $\beta \in (\beta^*, \beta^{**})$  (19) holds. Also, by Definition 1 at a symmetric equilibrium with  $q_1 < \beta$  the market clearing condition

$$q_1 = \frac{\alpha a_1}{(1 - \alpha) \varphi(a_1)} \quad (\text{A9})$$

has to hold and, in addition,  $a_1 = a_1^D(q_1, \beta)$  which, by Lemma 3, is equivalent to equation (??) being satisfied.

We prove next that equations (A9) and (A8) have a solution with respect to  $q_1$  and  $a_1$  if and only if  $\beta > \beta^*$ , and in that case the solution is unique. As shown in the proof of Lemma 3,  $a_1^D(q_1, \beta)$  is always continuous and decreasing in  $q_1$ . The solution of (A9) with respect to  $a_1$  is also continuous but increasing in  $q_1$ . For  $q_1 \approx 0$ ,  $a_1^D(q_1, \beta) \approx 1$  while the solution of (A9) for  $a_1$  yields  $a_1 \approx 0$ . On the other hand, for  $q_1 = \beta$ , we have seen that  $a_1^D(\beta, \beta) = a_1^*(\beta)$  and it is easy to verify<sup>17</sup> that the solution of (A9) obtains at a value  $a_1 < a_1^*(\beta)$  iff  $\beta < \beta^*$ . Hence if  $\beta < \beta^*$  equations (A8) and (A9) have no solution (such that  $q_1 \leq \beta$ ), which implies there is no symmetric inefficient equilibrium with  $q_1 < \beta$ . In contrast, if  $\beta > \beta^*$  there is a unique pair  $(a_1, q_1)$ , say  $(\bar{a}_1^D(\beta), \bar{q}_1^D(\beta))$ , that satisfies both equations.

The ordered pair  $(\bar{a}_1^D(\beta), \bar{q}_1^D(\beta))$  identifies a candidate symmetric, inefficient equilibrium, which is indeed an equilibrium if the entrepreneurs' optimality condition  $v_D(\bar{q}_1^D(\beta), \beta) \geq v_S(\bar{q}_1^D(\beta), \beta)$  is satisfied.

In addition, a mixed equilibrium might also exist at some price  $q_1 \leq \beta$ . At such an equilibrium, a positive fraction of entrepreneurs choose project  $\hat{a}_1$  where late producers are solvent and another positive fraction chooses  $a_1^D = a_1^D(q_1, \beta)$  where late producers default. In addition, the entrepreneurs' optimality condition,  $v_D(q_1, \beta) = v_S(q_1, \beta)$ , and the following market clearing condition must hold:  $q_1 > \frac{\alpha a_1^D}{(1 - \alpha) \varphi(a_1^D)}$ .<sup>18</sup> Such inequality is satisfied if, and only if<sup>19</sup>,  $\bar{q}_1^D(\beta) < \beta$ , for all  $q_1 \in (\bar{q}_1^D(\beta), \beta)$ . Since, as shown above,  $\bar{q}_1^D(\beta) < \beta$  exists if and only if  $\beta > \beta^*$ , a mixed equilibrium in the region  $\beta < \min\{\beta^*, \beta^{**}\}$  does not exist either, thus establishing claim (ii).

Recalling the pattern of  $v_D, v_S$  established in Lemma 4, we can now characterize the set of equilibria when  $\beta^* < \beta < \beta^{**}$ , thus completing the proof of Proposition 4:

1. If  $v_D(q_1, \beta) \geq v_S(q_1, \beta)$  for all  $q_1 < \beta$ , or if  $\bar{q}_1^D(\beta) > q_1''$  (which is, recall, the highest value of  $q_1$  for which  $v_D(q_1, \beta) = v_S(q_1, \beta)$ ), so that  $v_D(\bar{q}_1^D(\beta), \beta) > v_S(\bar{q}_1^D(\beta), \beta)$ , a unique equilibrium exists at  $\bar{q}_1^D(\beta), \bar{a}_1^D(\beta)$ .
2. If  $q_1' < \bar{q}_1^D(\beta) < q_1''$  there is a unique mixed equilibrium at  $q_1''$ . In this case  $v_D(\bar{q}_1^D(\beta), \beta) < v_S(\bar{q}_1^D(\beta), \beta)$  so  $\bar{q}_1^D(\beta)$  is not an equilibrium.

<sup>17</sup>This follows from the fact, established in Lemma 2, that  $\beta < \alpha a_1^*(\beta) / [(1 - \alpha) \varphi(a_1^*(\beta))]$  iff  $\beta < \beta^*$ .

<sup>18</sup>If, and only if, this inequality is satisfied we can always find  $\mu_S > 0$  and  $\mu_D = 1 - \mu_S > 0$  such that the bond market clearing condition (A1) holds.

<sup>19</sup>The claims follows from the facts that, at  $q_1 = \bar{q}_1^D(\beta)$  the equality  $q_1 = \frac{\alpha a_1^D}{(1 - \alpha) \varphi(a_1^D)}$  holds and that  $a_1^D$  is decreasing in  $q_1$ .

3. If  $\bar{q}_1^D(\beta) < q_1'$  (and hence  $\bar{q}_1^D(\beta) < q_1''$ ), we have a symmetric equilibrium at  $\bar{q}_1^D(\beta)$  as well as two mixed equilibria, one at  $q_1'$  and one at  $q_1''$ .

Note, finally, that if case 1 above holds for some  $\beta' \in (\beta^*, \beta^{**})$ , the same is true, a fortiori, for all higher<sup>20</sup> values of  $\beta \in (\beta', \beta^{**})$ . Conversely, if case 3 holds, the same is true for all lower values of  $\beta \in (\beta^*, \beta')$ . ■

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<sup>20</sup>This follows from the fact that both  $\bar{q}_1^D(\beta)$  and  $\bar{a}_1^D(\beta)$  are increasing in  $\beta$  (since (A9) does not depend on  $\beta$  while it is immediate to verify from (A8) that  $a_1^D$  is increasing in  $\beta$ ).

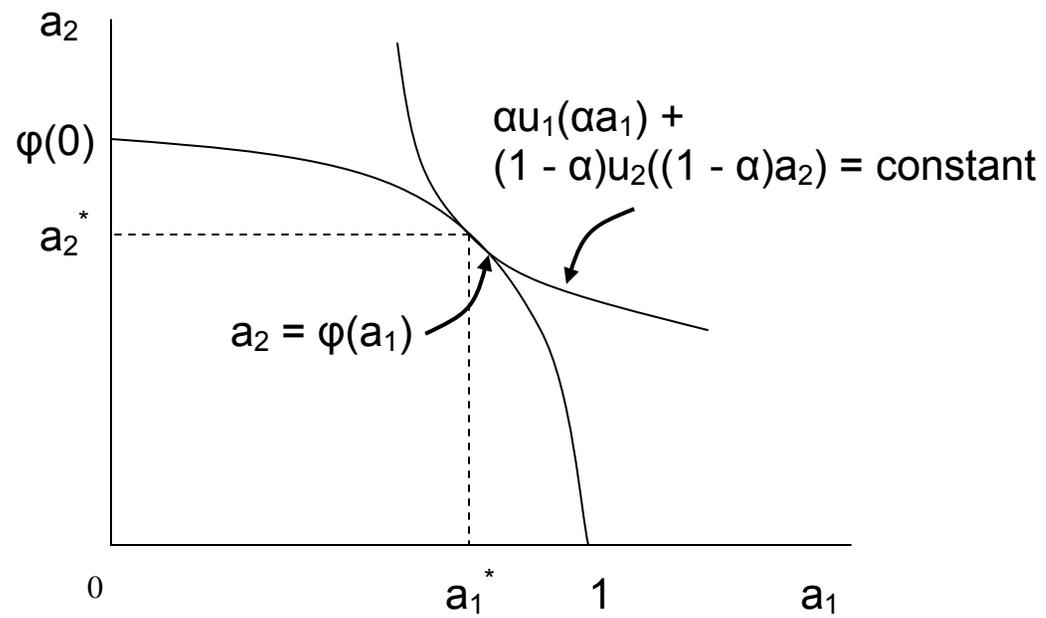
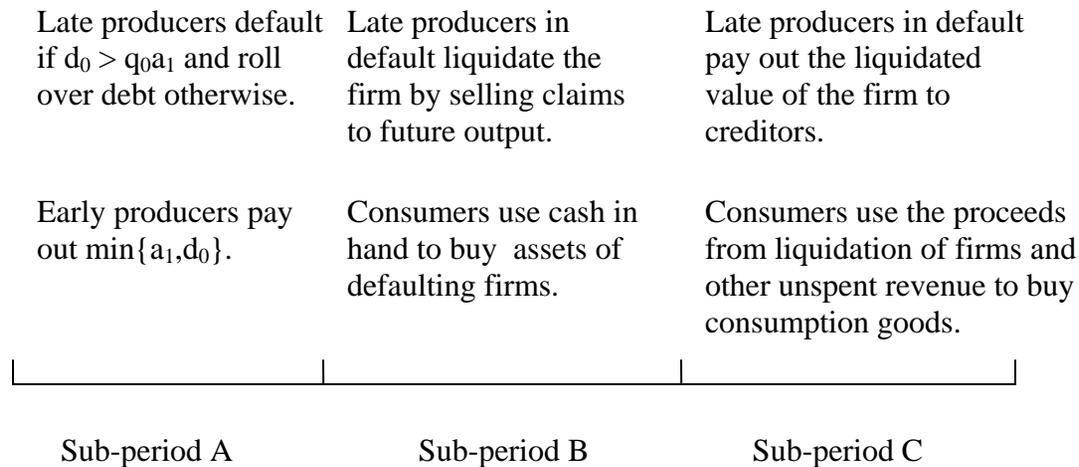


Figure 1

The efficient allocation  $(a_1^*, a_2^*)$  occurs where the indifference curve defined by  $\alpha u_1(\alpha a_1) + (1 - \alpha)u_2((1 - \alpha)a_2) = \text{constant}$  is tangent to the production possibility frontier  $a_2 = \varphi(a_1)$



**Figure 2**

**The resolution of default at date 1 occurs in three stages. In sub-period A the producers repay their debt, roll it over, or default. In sub-period B defaulting producers sell their claims to future output. In sub-period C the producers use the proceeds of liquidation to pay their creditors.**