1 Note on exponential-affine stock prices

The idea is to specify the dividend yield δ and short rate r to be affine in X where X is an affine diffusion under the risk-neutral measure. Then the stock price is guessed to be exponential affine. To show that the guess works, I have to show that the guess satisfies

$$P(t) = E_t^* \left[\int_t^{\infty} e^{-\int_t^s r(u)du} \delta(s) P(s) \, ds \right], \tag{1}$$

where E^* denotes expectation under the risk-neutral measure. The following simple example with a normally distributed dividend yield, zero short rate and zero market prices of risk illustrates that the functional form result applies. Backshi and Chen (1997, JFE) compute exponential-affine stock prices for the case where X is a square-root process.

PROPOSITION: Assume that the short rate is zero, $r = 0$ (in the notation of the paper $r_0 = 0$ and $r_X = 0$). The dividend yield is $\delta = X$ (in the notation of the paper, $\delta_0 = 0$ and $\delta_X = 1$) where X is an OU process which solves

$$dX(t) = k(\theta - X(t))dt + \sigma dW(t).$$

Suppose P is of the form

$$P(t) = \exp\left(at + \frac{X(t)}{k}\right) \tag{2}$$

with

$$a = -\theta - \frac{1}{2}\frac{\sigma^2}{k} < 0. \tag{3}$$

then P satisfies the pricing equation (1).

PROOF OF PROPOSITION (the proof refers to a series of facts stated in section 2 of this note): Assume that the price satisfies the guess (2). We need to show that

$$\begin{aligned} P(t) &= E_t \int_t^{\infty} \delta(s) P(s) \, ds \qquad (4) \\ &= E_t \left[\lim_{T\to\infty} \int_t^{T} \delta(s) P(s) \, ds \right] \end{aligned}$$

Define

$$P^T(t) = E_t \int_t^T \delta(s) P(s) \, ds$$

I will show (4) in two steps. Step (i) is

$$P(t) = \lim_{T \to \infty} P^T(t) \tag{5}$$

Step (ii) is

$$\lim_{T \to \infty} P^T(t) = E_t \left[\lim_{T \to \infty} \int_t^T \delta(s) P(s) \, ds \right] \tag{6}$$

Together, these two steps yield (4).

STEP (i) : I want to interchange expectation and integral,

$$P^T(t) = E_t \left[\int_t^T P(s) \delta(s) \, ds \right] = \int_t^T E_t [P(s) \delta(s)] \, ds \tag{7}$$

For Fubini to apply, I need that

$$E_t \int_t^T |P(s) \delta(s)| \, ds < \infty. \tag{8}$$

Tonelli's theorem says

$$E_t \int_t^T |P(s) \delta(s)| \, ds = \int_t^T E_t |P(s) \delta(s)| \, ds \tag{9}$$

The RHS is finite, because by FACT 3:

$$\begin{aligned} E_t [|P(s) \delta(s)|] &= E_t [|\delta(s)| P(s)] \\ &= E_t [|Z(s)|] \exp\left(as + m_t(s)/k + \frac{1}{2} v_t(s)/k^2 \right) \end{aligned} \tag{10}$$

where $Z(s) \sim N(m_t(s) + v_t(s)/k, v_t(s))$. This expression is continuous in s.

The term beneath the integral in (7) is given by FACT 2:

$$\begin{aligned} E_t\left[\delta\left(s\right)P\left(s\right)\right] &= E_t\left[X\left(s\right)\exp\left(as + X\left(s\right)/k\right)\right] \\ &= \left(m_t\left(s\right) + v_t\left(s\right)/k\right)\exp\left(as + m_t\left(s\right)/k + \frac{1}{2}v_t\left(s\right)/k^2\right) \end{aligned}$$

Using FACT 4,

$$P^T\left(t\right) = \int_t^T E_t\left[P\left(s\right)\delta\left(s\right)\right]ds$$

$$= \int_t^T \left(m_t\left(s\right) + v_t\left(s\right)/k\right)\exp\left(as + m_t\left(s\right)/k + \frac{1}{2}v_t\left(s\right)/k^2\right)ds$$

$$= -\exp\left(aT + m_t\left(T\right)/k + \frac{1}{2}v_t\left(T\right)/k^2\right) + \exp\left(at + m_t\left(t\right)/k + \frac{1}{2}v_t\left(t\right)/k^2\right).$$

Since $\lim_{T\to\infty} m_t\left(T\right) = \theta$ and $\lim_{T\to\infty} v_t\left(T\right) = \frac{\sigma^2}{2k}$, I have

$$\lim_{T\to\infty}\exp\left(aT + m_t\left(T\right)/k + \frac{1}{2}v_t\left(T\right)/k^2\right) = 0$$

as long as $a < 0$, which I assumed in (3). This leaves

$$\lim_{T\to\infty} P^T\left(t\right) = \exp\left(at + m_t\left(t\right)/k + \frac{1}{2}v_t\left(t\right)/k^2\right).$$

where I can note that $m_t\left(t\right)/k = X\left(t\right)/k$ ad $v_t\left(t\right)/k^2 = 0$, so that I indeed get equation (5) for our guess (2).

STEP (ii) : From step (i), I know that

$$\lim_{T\to\infty} P^T\left(t\right) = \lim_{T\to\infty}\int_t^T E_t\left[P\left(s\right)\delta\left(s\right)\right]ds$$

I want to use Fubini to argue that the RHS of the last equation is equal to the RHS of (6). For Fubini to apply, I need condition (8) for $T = \infty$. The same arguments go through as before, and I know that $m_t\left(s\right)$ and $v_t\left(s\right)$ go to constants for $s \to \infty$, which means that the expression in (10) goes to zero because $a < 0$. This completes the proof that (6) holds.

2 Useful facts

FACT 1. Suppose X solves

$$dX(t) = k(\theta - X(t))dt + \sigma dW(t). \tag{11}$$

starting at $X(0) = x_0$ and for constants k, θ and σ. Then the solution to (11) is

$$\begin{aligned} X_s &= \exp(-k(s-t)) X_t + \theta(1 - \exp(-k(s-t))) \\ &\quad + \int_t^s \exp(-k(s-u)) \sigma dW(u). \end{aligned}$$

which is normal with mean

$$m_t(s) \equiv \exp(-k(s-t)) X_t + \theta(1 - \exp(-k(s-t))),$$

and variance

$$v_t(s) \equiv \frac{\sigma^2}{2k}(1 - \exp(-2k(s-t))).$$

FACT 2: Suppose $X \sim N(m, v)$. Then I have for any constant c

$$E\left[Xe^{cX}\right] = (m + cv)\exp\left(cm + \frac{1}{2}c^2 v\right).$$

This can be verified by direct computation

$$\begin{aligned} E\left[Xe^{cX}\right] &= \int X \exp(cX) \exp\left(\frac{-(X-m)^2}{0.5v}\right) \frac{1}{\sqrt{2\pi v}} dX \\ &= \int X \exp\left(\frac{-X^2 - m^2 + 2X(m+cv)}{2v}\right) \frac{1}{\sqrt{2\pi v}} dX \\ &= \int X \exp\left(\frac{-(X-(m+cv))^2 + 2mcv + c^2v^2}{2v}\right) \frac{1}{\sqrt{2\pi v}} dX \\ &= \int X \exp\left(mc + \frac{1}{2}c^2 v\right) \exp\left(\frac{-(X-(m+cv))^2}{2v}\right) \frac{1}{\sqrt{2\pi v}} dX \end{aligned}$$

FACT 3: Suppose $X \sim N(m, v)$. Then we have for any constant c

$$E\left[|X| e^{cX}\right] = E[|Y|] \exp\left(cm + \frac{1}{2}c^2 v\right).$$

where $Y \sim N(m + cv, v)$

FACT 4:

$$\begin{aligned} & \frac{d}{ds} \exp\left(as + m_t(s)/k + \frac{1}{2}v_t(s)/k^2\right) \\ = & -(m_t(s) + v_t(s)/k) \exp\left(as + m_t(s)/k + \frac{1}{2}v_t(s)/k^2\right) \end{aligned}$$

as long as

$$a = -\theta - \frac{1}{2}\frac{\sigma^2}{k^2}$$

PROOF OF FACT 4: Taking derivatives:

$$\begin{aligned} & \frac{d}{ds} \exp\left(as + m_t(s)/k + \frac{1}{2}v_t(s)/k^2\right) \\ = & \left(a + \frac{\partial m_t(s)}{\partial s}\frac{1}{k} + \frac{\partial v_t(s)}{\partial s}\frac{1}{2k^2}\right) \exp\left(as + m_t(s)/k + \frac{1}{2}v_t(s)/k^2\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial m_t(s)}{\partial s} &= -k \exp(-k(s-t))(X_t - \theta) \\ \frac{\partial v_t(s)}{\partial s} &= \sigma^2 \exp(-2k(s-t)) \end{aligned}$$

$$\begin{aligned} & a - \exp(-k(s-t))(X_t - \theta) + \sigma^2 \exp(-2k(s-t))\frac{1}{2k^2} \\ = & -\theta - \frac{1}{2}\frac{\sigma^2}{k^2} - \exp(-k(s-t))(X_t - \theta) + \sigma^2 \exp(-2k(s-t))\frac{1}{2k^2} \\ = & -(\theta + \exp(-k(s-t))(X_t - \theta)) - \left(\frac{1 - \exp(-2k(s-t))}{2k^2}\right)\sigma^2 \\ = & -(m_t(s) + v_t(s)/k) \end{aligned}$$

3 Remarks

Theorem 1 of the paper states a solution of the form

$$P(t) = \exp(A(t) - B(t)X(t))$$

with coefficients (10)-(12)

$$\begin{aligned} 0 &= A'(t) - \theta k B(t) + \frac{1}{2}\sigma^2 B(t)^2 \\ 0 &= 1 - B'(t) + kB \end{aligned}$$

for $\delta_0 = 0$ and $\delta_X = 1$ in $\delta = \delta_0 + \delta_X X$ and $r_0 = 0$ (because $r = 0$). Now use Restriction 2 from the paper, which sets $B'(t) = 0$. This implies

$$B(t) = -\frac{1}{k}.$$

and therefore

$$0 = A'(t) + \theta + \frac{1}{2}\frac{\sigma^2}{k^2}$$

This equation is solved for $A(t) = at$ where

$$a = -\theta - \frac{1}{2}\frac{\sigma^2}{k^2}.$$