Recursive Contracts and Endogenous Incompleteness

- Why are the markets for insuring against idiosyncratic risk imperfect/missing?
- Methodologically: recursive contracts (“dynamic programing squared”)
- How can we look for evidence about these mechanisms?

1 Frictionless benchmark

- One risk-neutral planner, one risk averse household
- Both discount the future at rate \( \beta \)
- States of the world: \( s \in \{1, 2, \cdots, S\} \)
- Output: \( y(s) \)
- Effort \( e_t \)
- Effort affects probability distribution over states: \( \Pr(s_t|e_t) \)
- Exogenous income is a special case
- Social planner wants to maximize household’s utility s.t. budget constraint
- (Dual problem): planner wants to maximize profits s.t. utility of the household
- Planner chooses a sequence of effort \( e \) and a sequence of consumption \( c \)
- Timing:
  1. Effort chosen
  2. State realized
  3. Consumption chosen
- Ways in which this setup can be generalized:
- \( \Pr(s_t|e') \) if effort has effects for more than on period
- \( \Pr(s_t|e_t, s^{t-1}) \) for the non iid case
- Multiple households
- Risk-averse planner
- Different discount rates between planner and household

• Planner’s problem:

\[
V(w_0) \equiv \max_{c,e} \sum_{s'} \Pr(s'|e) \beta^t (y(s_t) - c(s')) \\
\text{s.t. } \sum_{s'} \Pr(s'|e) \beta^t [u(c(s')) - \epsilon(s^{t-1})] = w_0
\]

• FOC:

\[
- \Pr(s'|e) \beta^t + \mu \Pr(s'|e) \beta^t u'(c(s')) = 0 \\
\Rightarrow u'(c(s')) = \frac{1}{\mu}
\]

• Full insurance (because planner is risk-neutral)

• Consumption constant over time (because planner discounts at the same rate as household)

• For many-households case:

\[
\frac{u'(c'(s'))}{u'(c(s'))} = \frac{\mu_j}{\mu_i}
\]

• Ratio of marginal utilities of two households constant over time

• Remember from the beginning of the course that this condition is what held when there were complete markets

2 Recursive representation

• Promised utility as a state variable on the planner’s problem

\[
V(w) = \max_{e,c(s),w'(s)} \sum_s \Pr(s|e) [(y(s) - c(s)) + \beta V(w'(s))] \\
\text{s.t. } \sum_s \Pr(s|e) [u(c(s)) - \epsilon + \beta w'(s)] = w
\]
• Constraint (4) is sometimes known as a “promise keeping” constraint: imagine that the planner has “promised” the household a level of utility $w$ and must now keep the promise.

• FOC:

\[- \Pr(s|e) + \mu \Pr(s|e) u'(c(s)) = 0\]

\[\Rightarrow u'(c(s)) = \frac{1}{\mu}\]

\[\Pr(s|e) \beta V'(w'(s)) + \mu \Pr(s|e) \beta = 0\]

\[V'(w'(s)) = -\mu\]

so continuation utilities are also equated across states

• Envelope condition:

\[V'(w) = -\mu\]

(5)

so continuation utility is the same as current utility. We know this from the sequence problem too.

• Promised utilities as state variables are especially useful when we introduce various constraints into program (1)

3 Limited commitment

• Suppose income is exogenous (no effort choice)

• Suppose household cannot commit to make payments to the planner and/or these payments cannot be enforced. Recall that first best allocation will sometimes dictate $c(s) < y(s)$.

• Timing:

  – Income realized
  – If allocation dictates $c(s) > y(s)$, the planner transfers $c(s) - y(s)$ to the household (assume for now that the planner can commit)
  – If allocation dictates $c(s) < y(s)$, the household decides how much to give to the planner
  – Consumption takes place (assume the household cannot save)

• Planner’s only means of enforcement is to threaten the household with no longer insuring it if it misses a payment
• Let the value of autarky be

\[ v_{\text{aut}} = \sum_{t=0}^{\infty} \beta^t \left( \sum_s \Pr(s) u(c(s)) \right) \]

\[ = \frac{\sum_s \Pr(s) u(c(s))}{1 - \beta} \]

• At every history, the allocation has to be better than autarky for the household

• Problem is

\[ V(w_0) \equiv \max_{c,e} \sum_{s',t} \Pr(s') \beta^t (y(s_t) - c(s_t)) \]

\[ s.t. \quad \sum_{s',t} \Pr(s') \beta^t u(c(s')) = w_0 \]

\[ u(c(s^j)) + \beta \sum_{s^j+t} \Pr(s^j+t|s^j) \beta^{t-1} u(c(s^j+t)) \geq u(y(s_j)) + \beta v_{\text{aut}} \quad \forall s^j \]

• In principle, this could be very complicated: history-dependent allocations, S constraints for each history, etc.

• Recursive representation:

\[ V(w) = \max_{c(s),w'(s)} \sum_s \Pr(s) \left[ (y(s) - c(s)) + \beta V(w'(s)) \right] \]

\[ s.t. \quad \sum_s \Pr(s) [u(c(s)) + \beta w'(s)] = w \]

\[ u(c(s)) + \beta w'(s) \geq u(y(s)) + \beta v_{\text{aut}} \]  

(6)  

(7)

• FOC:

\[ -\Pr(s) + \mu \Pr(s) u'(c(s)) + \lambda(s) u'(c(s)) = 0 \]

\[ u'(c(s)) = \frac{\Pr(s)}{\mu \Pr(s) + \lambda(s)} \]  

(8)

\[ \Pr(s) \beta V'(w'(s)) + \mu \Pr(s) \beta + \beta \lambda(s) = 0 \]

\[ V'(w'(s)) = -\left( \frac{\mu \Pr(s) + \lambda(s)}{\Pr(s)} \right) \]  

(9)
From (8) and (9):

\[ u'(c(s)) = -\frac{1}{V'(w'(s))} \]  \hspace{1cm} (10)

**Interpretation:**
- MRS between consumption and future promised utility: \(\frac{u'(c(s))}{\beta}\)
- MRT between consumption and future promised utility: \(\frac{1}{\beta V'(w'(s))}\)

**Envelope condition** (5) still holds.

Using (5) in (9):

\[ V'(w'(s)) = V'(w) - \frac{\lambda(s)}{Pr(s)} \]

\(V'(w)\) never goes up \(\Rightarrow\) \(w\) never goes down.

- (Concavity of the planner’s value function needs to be shown)

Given any \(w\), divide states \(s\) into those where (7) binds and those where it does not

1. **When (7) binds:** \(\lambda(s) > 0\)
   - \(V'(w'(s)) < V'(w) \Rightarrow w'(s) > w\)
   - Binding constraint implies \(c(s)\) and \(w'(s)\) solve:
     \[ u(c(s)) + \beta w'(s) = u(y(s)) + \beta v_{aut} \]
     \[ u'(c(s)) = -\frac{1}{V'(w'(s))} \]  \hspace{1cm} (11)

   - “Amnesia”: consumption and promised values do not depend on history! In particular, they do not depend on \(w\)

2. **When (7) does not bind:** \(\lambda(s) = 0\)
   - \(V'(w'(s)) = V'(w)\): promised value does not change
   - \(u'(c(s)) = -\frac{1}{V'(w'(s))}\): consumption is the same across states

- Note that (7) will bind for high states (if at all)

- Implied dynamics:
  1. As long as (7) does not bind, consumption is constant
2. As soon as (7) binds, set consumption and promised utility to whatever level is consistent with the system of equations (11). This level will be higher than at the beginning
3. Keep consumption constant at the new level until (7) binds again
4. Eventually, state $S$ will be realized. After that, consumption is constant forever

### 3.1 Some testable implications [Kocherlakota, 1996]
- Consumption will have the following pattern:

- Consumption positively correlated with current income
- Everything about the past that can help predict current income is summarized by promised utility $w$
- Because of (10), $\frac{1}{w(c_{t-1})}$ is a sufficient statistic for today’s promised utility, so nothing else about the past should help predict current consumption
Moreover, because of the amnesia property for households that increase consumption, *nothing* about the past should predict current consumption, but current income *should* predict consumption.

Instead, for the rest of the households, current income should *not* predict consumption.

### 3.2 Two-sided limited commitment [Thomas and Worrall, 1988]

- In the one-sided limited commitment problem, the principal will be making profits at first and losses later (assuming, e.g. \( V(w_0) = 0 \)).
- What if the principal can also not commit?

\[
V(w) = \max_{c(s), w'(s)} \sum_s \Pr(s) [(y(s) - c(s)) + \beta V(w'(s))] \\
\text{s.t.} \quad \sum_s \Pr(s) [u(c(s)) + \beta w'(s)] = w \\
u(c(s)) + \beta w'(s) \geq u(y(s)) + \beta v_{aut} \\
y(s) - c(s) + \beta V(w'(s)) \geq 0
\]

- FOC:

\[
- \Pr(s) + \mu \Pr(s) u'(c(s)) + \lambda(s) u'(c(s)) - \eta(s) = 0 \\
\quad u'(c(s)) = \frac{\Pr(s) + \eta(s)}{\mu \Pr(s) + \lambda(s)}
\]

\[
\Pr(s) \beta V'(w'(s)) + \mu \Pr(s) \beta + \beta \lambda(s) + \eta(s) \beta V'(w'(s)) = 0 \\
\quad V'(w'(s)) = -\frac{\mu \Pr(s) + \lambda(s)}{\Pr(s) + \eta(s)}
\]

- Equation (10) and the envelope condition (5) still hold.
- Using (5):

\[
V'(w'(s)) = \frac{V'(w) \Pr(s) - \lambda(s)}{\Pr(s) + \eta(s)}
\]

- Now \( V'(w'(s)) \) may go up or down depending on which (if any) of the constraints binds.
- General pattern
  - If household’s constraint binds, promised utility (and consumption) go up.
If planner’s constraint binds, promised utility (and consumption) go down
If no constraint binds, promised utility (and consumption) is constant
(it’s easy to show that both constraints won’t bind at once)
Amnesia: contract resets whenever any of the constraints binds

- Kehoe and Levine [1993], Alvarez and Jermann [2000]: competitive equilibrium approach (rather than optimal contracting approach) to the limited commitment problem. Welfare theorems. Implications for asset prices.

4 Moral hazard

- Now suppose everyone can commit to a contract
- But the household exerts unobservable effort.

\[
V(w) = \max_{\epsilon, \epsilon'(s), w'(s)} \sum_s \Pr(s|\epsilon) \left[ (y(s) - c(s)) + \beta V(w'(s)) \right] \\
\text{s.t. } \sum_s \Pr(s|\epsilon) \left[ u(c(s)) - \epsilon + \beta w'(s) \right] = w \\
\sum_s \Pr(s|\epsilon) \left[ u(c(s)) - e + \beta w'(s) \right] \geq \sum_s \Pr(s|\tilde{\epsilon}) \left[ u(c(s)) - \tilde{\epsilon} + \beta w'(s) \right] \quad \forall \tilde{\epsilon}
\]

- Special case: \( e \in \{0, 1\} \) and the planner wants to implement \( e = 1 \) always
- FOC:

\[
\Pr(s|1) + \mu \Pr(s|1) u'(c(s)) + \lambda \left[ \Pr(s|1) - \Pr(s|0) \right] u'(c(s)) = 0 \\
\Rightarrow \frac{1}{u'(c(s))} = \mu + \lambda \left[ 1 - \frac{\Pr(s|0)}{\Pr(s|1)} \right] \quad (14)
\]

\[
\Pr(s|1) \beta V'(w'(s)) + \mu \Pr(s|1) \beta + \lambda \beta \left[ \Pr(s|1) - \Pr(s|0) \right] = 0 \\
\Rightarrow V'(w'(s)) = - \left( \mu + \lambda \left[ 1 - \frac{\Pr(s|0)}{\Pr(s|1)} \right] \right) \quad (15)
\]

- Equation (10) still holds: no distortion in the optimal tradeoff between incentives in the form of current consumption versus promised future utility
• The envelope condition (5) still holds, so using (15):

\[ V'(w'(s)) = V'(w) - \lambda \left[ 1 - \frac{\Pr(s|0)}{\Pr(s|1)} \right] \]  

(16)

• Note that

\[ \mathbb{E}\left( 1 - \frac{\Pr(s|0)}{\Pr(s|1)} \mid c = 1 \right) = \sum_s \Pr(s|1) \left( 1 - \frac{\Pr(s|0)}{\Pr(s|1)} \right) = 0 \]

so \( V'(w) \) is a Martingale

• Furthermore, using (10) and (16):

\[ \mathbb{E}\left[ \frac{1}{u'(c_{t+1})} \right] = \frac{1}{u'(c_t)} \]  

(17)

• This equation is known as the “inverse Euler equation”, as opposed to the regular Euler equation, which is \( \mathbb{E}[u'(c_{t+1})] = u'(c_t) \) for the case of \( \beta R = 1 \). It first shows up in Rogerson [1985].

4.1 The meaning of the inverse Euler equation

• The usual Euler equation equates the expected marginal utility of a dollar over time. The inverse Euler equates the expected marginal cost of a unit of utility over time.

• There is always a way for the planner to change the timing of utility provision, leaving the household indifferent and with the same incentives

• Provide \( \Delta \) utils using today’s consumption - reduce \( \Delta \) utils using tomorrow’ consumption (in every state)

• Cost of providing \( \Delta \) utils today:

\[ \frac{\Delta}{u'(c_t)} \]

• Expected savings from reducing \( \Delta \) utils in every state of the world tomorrow:

\[ \beta \sum_{s_{t+1}} \Pr(s_{t+1}) \frac{\Delta}{\beta u'(c(s_{t+1}))} = \Delta \mathbb{E}\left[ \frac{1}{u'(c_{t+1})} \right] \]

• Because this scheme changes utility by the same amount in every state, it does not change incentives

• For there to be no \( \Delta \) that makes this a good idea, the IEE must hold
• Note that the household is savings-constrained. Jensen’s inequality implies

$$E \left[ \frac{1}{u'(c_{t+1})} \right] > \frac{1}{E[u'(c_{t+1})]}$$

Therefore

$$\frac{1}{u'(c_t)} > \frac{1}{E[u'(c_{t+1})]}$$

$$E[u'(c_{t+1})] > u'(c_t)$$

• Under the optimal allocation, if the household could save at a rate $R = \frac{1}{\beta}$ (which is the rate the planner can give it), it would want to do so.

• However, for incentive reasons, it is optimal not to let it save.

• Note that for this to work it has to be possible for the planner to observe consumption/savings. Otherwise, we have a hidden savings problem.

• Exercise: where in the maths do we see that if incentive constraints don’t bind the standard Euler equation holds?

4.2 “Immiseration”

Proposition 1. *Consumption converges to zero almost surely*

*Proof.*

• IEE implies that $\frac{1}{u'(c_t)}$ follows a nonnegative Martingale.

• By the Martingale Convergence Theorem, this implies that $\frac{1}{u'(c_t)}$ converges to a random variable a.s. (i.e. it converges to a constant but we don’t know what constant)

• The constant cannot be positive because that would mean full insurance, violating incentive constraints

• $\Rightarrow \frac{1}{u'(c_t)} \rightarrow_{a.s.} 0$

• $\Rightarrow u'(c_t) \rightarrow_{a.s.} \infty$

• $\Rightarrow c_t \rightarrow_{a.s.} 0$ (or to subsistence level under Stone-Geary preferences)

• Note: this is optimal!
• Note that commitment is required: the household would prefer to quit the scheme eventually.

• Atkeson and Lucas [1992] show the immiseration result in general equilibrium for an economy a fixed aggregate endowment and hidden preference shocks (rather than hidden effort). Special cases solved explicitly.

• Farhi and Werning [2007] characterize optimal allocations with OLG and Pareto weights on future generations. Immiseration result goes away.

4.3 Some testable implications

• Consumption positively correlated with current income (assuming MLRP). This follows from (14)

• Everything about the past that can help predict current income is summarized by promised utility \( w \)

• Because of (10), \( \frac{1}{w(c_{t-1})} \) is a sufficient statistic for today’s promised utility, so nothing else about the past should help predict current consumption

• These predictions are also true under limited commitment, so it’s not so easy to distinguish the two environments.

5 Hidden income [Thomas and Worrall, 1990]

• Go back to the case where there is no effort choice and contracts are enforceable

• Assume however that the planner cannot observe the realizations of \( y(s) \)

• Appeal to revelation principle: ask households to report income and check for incentive compatibility.

• Implement by a series of transfers: \( \tau(s) \equiv c(s) - y(s) \). A household whose income is \( y(s) \) can declare that its income was \( y(\tilde{s}) \) and consume \( y(s) + \tau(\tilde{s}) \). The planner will never know the household lied!

• The planner’s program is:

\[
V(w) = \max_{\tau(s), w(s)} \sum_s \Pr(s) \left[ -\tau(s) + \beta V(w'(s)) \right] \\
\text{s.t.} \quad \sum_s \Pr(s) \left[ u(y(s) + \tau(s)) + \beta w'(s) \right] = w \\
\Delta(s, \tilde{s}) \equiv u(y(s) + \tau(s)) + \beta w'(s) - \left[ u(y(s) + \tau(\tilde{s})) + \beta w'(\tilde{s}) \right] \geq 0 \quad \forall s, \tilde{s}
\]
• More complicated problem: \( S(S-1) \) incentive constraints

• Note: the formulation assumes that the planner cannot observe consumption (because he could then back out \( y = c - \tau \)) but he can observe savings (and w.l.o.g forbid them), because he knows that, whatever income is, \( c = y + \tau \)

Lemma 1. \( \tau(s) \leq \tau(s-1) \) and \( u'(s) \geq u'(s-1) \)

Proof. From (19)

\[
\begin{align*}
  u(y(s) + \tau(s)) + \beta w'(s) &\geq u(y(s) + \tau(s-1)) + \beta w'(s-1) \\
  u(y(s-1) + \tau(s-1)) + \beta w'(s-1) &\geq u(y(s-1) + \tau(s)) + \beta w'(s) \\
  \Rightarrow u(y(s) + \tau(s)) + u(y(s-1) + \tau(s-1)) &\geq u(y(s) + \tau(s-1)) + u(y(s-1) + \tau(s))
\end{align*}
\]

so \( \tau(s) \leq \tau(s-1) \) follows from the concavity of \( u \). Then \( u'(s) \geq u'(s-1) \) follows from incentive compatibility.  

Lemma 2. Local incentive constraints imply global incentive constraints

Proof. Let \( \tilde{s} < s \). We show first that if \( \Delta(s, \tilde{s}) \geq 0 \) and \( \Delta(s + 1, s) \geq 0 \), then \( \Delta(s + 1, \tilde{s}) \geq 0 \).

\( \tilde{s} < s \) implies \( \tau(s) \leq \tau(\tilde{s}) \). From the concavity of \( u \) we know that:

\[
\begin{align*}
  u(y(s+1) + \tau(s)) + u(y(s) + \tau(\tilde{s})) &\geq u(y(s+1) + \tau(\tilde{s})) + u(y(s) + \tau(s)) \\
\end{align*}
\]

(20)

Add (19) and (20):

\[
\begin{align*}
  u(y(s) + \tau(s)) + \beta w'(s) + u(y(s+1) + \tau(s)) + u(y(s) + \tau(\tilde{s})) &\geq \\
  u(y(s) + \tau(\tilde{s})) + \beta w'(\tilde{s}) + u(y(s+1) + \tau(\tilde{s})) + u(y(s) + \tau(s)) \\
  \Rightarrow \beta w'(s) + u(y(s+1) + \tau(s)) &\geq \beta w'(\tilde{s}) + u(y(s+1) + \tau(\tilde{s}))
\end{align*}
\]

Use that \( \Delta(s + 1, s) \geq 0 \):

\[
\begin{align*}
  \Rightarrow \beta w'(s+1) + u(y(s+1) + \tau(s+1)) &\geq \beta w'(s) + u(y(s+1) + \tau(s)) \geq \beta w'(\tilde{s}) + u(y(s+1) + \tau(\tilde{s}))
\end{align*}
\]

so \( \Delta(s + 1, \tilde{s}) \geq 0 \).

The result then follows by induction. The same logic applies to upward incentive constraints.

Lemma 3. Downwards local incentive constraints always bind

Proof. Assume, to the contrary, that \( \Delta(s, s-1) > 0 \).
Because $\tau(s - 1) \geq \tau(s)$, the incentive compatibility condition (19) implies that $w'(s) > w'(s - 1)$ (strict inequality).

Consider the following improvement on the allocation:

- Keep $w'(1)$ fixed
- Reduce $w'(2), w'(3), \ldots, w'(S)$ one by one until the incentive compatibility constraints hold exactly
- Add a constant to all $w'(s)$ until the promise-keeping constraint (18) holds exactly

Incentive compatibility and promise-keeping constraints hold by construction

Due to the concavity of the objective function (which must be proved too...), this improves the planner’s objective

Now we have reduced the problem to:

$$V(w) = \max_{\tau(s), w'(s)} \sum_s \Pr(s) \left[ -\tau(s) + \beta V'(w'(s)) \right]$$

subject to

$$\sum_s \Pr(s) \left[ u(y(s) + \tau(s)) + \beta w'(s) \right] = w$$

$$\Delta(s, s - 1) \equiv u(y(s) + \tau(s)) + \beta w'(s) - \left[ u(y(s) + \tau(s - 1)) + \beta w'(s - 1) \right] \geq 0 \quad \forall s > 1$$

**FOC:**

$$- \Pr(s) + \mu \Pr(s) u'(y(s) + \tau(s)) + \lambda(s) u'(y(s) + \tau(s)) - \lambda(s + 1) u'(y(s + 1) + \tau(s)) = 0$$

$$\Rightarrow \Pr(s) \left[ 1 - \mu u'(y(s) + \tau(s)) \right] = \lambda(s) u'(y(s) + \tau(s)) - \lambda(s + 1) u'(y(s + 1) + \tau(s))$$

$$\Pr(s) \beta V'(w'(s)) + \mu \Pr(s) \beta + \lambda(s) \beta - \lambda(s + 1) \beta = 0$$

$$\Rightarrow \Pr(s) \left[ V'(w'(s)) + \mu \right] = \lambda(s + 1) - \lambda(s)$$

- $\lambda(1) = 0$ (there is no state lower than 1 for this type to misreport)
- $\lambda(S + 1) = 0$ (state $S + 1$ does not exist)
- The envelope condition is still

$$V'(w) = -\mu$$
• Sum (24) over all states and using (25):

\[
\sum_s \Pr(s) [V'(w'(s)) + \mu] = \sum_s [\lambda(s + 1) - \lambda(s)] + \mu + \sum_s \Pr(s)V'(w'(s)) = -\lambda(1) + \lambda(S + 1) = 0
\]

\[
\sum_s \Pr(s)V'(w'(s)) = V'(w)
\]

so the planner’s marginal cost of providing a unit of utility to the household follows a Martingale

**Lemma 4.** \( w'(1) < w < w'(S) \)

**Proof.** Assume the contrary.

• By Lemma 1, \( w' \) is weakly increasing in \( s \)

• Since \( V'(w) \) is a Martingale, if the statement is not true, then \( w'(s) = w \) for all \( s \)

• Therefore \( V'(w'(s)) = V'(w) \) for all \( s \)

• Using (25) and (24), this implies \( \lambda(s + 1) - \lambda(s) = 0 \) for all \( s \)

• (23) then implies that households get full insurance

• But this requires \( \tau(s) < \tau(s - 1) \)

• With constant \( w'(s) \), this violates incentive compatibility

\[ \square \]

**Lemma 5.** The following are upper and lower bounds for the planner’s value:

\[
-\frac{\bar{\tau}(w)}{1 - \beta} \leq V(w) \leq \sum_s \Pr(s)[y(s) - \bar{c}(w)]\frac{1}{1 - \beta}
\]

where \( \bar{\tau}(w) \) solves

\[
\sum_s \Pr(s)u(y(s) + \bar{\tau}(w))\frac{1}{1 - \beta} \equiv w
\]

and \( \bar{c}(w) \) solves

\[
\sum_s \Pr(s)u(\bar{c}(w))\frac{1}{1 - \beta} \equiv w
\]

**Proof.** A contract that pays \( \bar{\tau}(w) \) in every period in every state is incentive compatible and gives the planner a value of \(-\frac{\bar{\tau}(w)}{1 - \beta}\). A first-best insurance contract gives the planner a value \( \sum_s \Pr(s)[y(s) - \bar{c}(w)]\frac{1}{1 - \beta} \)

\[ \square \]
5.1 Immiseration again

Proposition 2. Assume \( u(c) \) is bounded above by \( \bar{u} \), \( \lim_{c \to a^-} u(c) = -\infty \) and \( \lim_{c \to a^+} u'(c) = \infty \) for some \( a \). Then \( V'(w) \to_{a.s.} 0 \)

Proof.

• From (26) and (27)

\[
\lim_{w \to \frac{\bar{u}}{1-\beta}^-} \bar{\tau} = \lim_{w \to \frac{\bar{c}}{1-\beta}^-} \bar{c} = \infty
\]

so

\[
\lim_{w \to \frac{\bar{u}}{1-\beta}^-} V(w) = -\infty
\]

which implies

\[
\lim_{w \to \frac{\bar{u}}{1-\beta}^-} V'(w) = -\infty
\]

• Furthermore, (26) and (27) imply that

\[
\lim_{w \to -\infty} V'(w) = 0
\]

• Therefore \( V'(w) \) is a strictly decreasing function (this part we haven’t proved) whose limits are 0 and \( -\infty \)

• By the Martingale Convergence Theorem, \( V'(w) \) must converge to a constant

• But by Lemma 4, \( w \) always keeps spreading out

• This means that \( V'(w) \) cannot converge to a strictly negative constant, it must converge to zero

\[ \square \]

• As a consequence of this \( w \) must diverge to \( -\infty \)

6 Hidden income plus hidden saving (and perhaps borrowing)

• In addition to hidden income, suppose borrowing-saving is unobservable by planner and unconstrained

• (Subject only to the natural borrowing constraint / no-Ponzi condition)
• Given a transfer scheme $\tau(s)$ as a function of the history of announcements, household solves:

$$\max_{c(s^t), A(s^{t+1}), \tilde{s}(s^t)} \sum_{t=0}^{\infty} \beta^t \Pr(s^t)u(c(s^t))$$

$$s.t. \sum_{t=0}^{\infty} \beta^t \left[ y(s_t) + \tau_t \left( \tilde{s}^t(s^t) \right) - c(s^t) \right] \geq 0 \quad \forall s^t \quad (28)$$

so household just chooses $\tilde{s}$ to maximize the NPV of the transfers it will receive

• First pointed out by Allen [1985]

• No insurance is possible

• Conventional Euler equation holds

• This can rationalize the assumption of “pure borrowing and lending”

• Cole and Kocherlakota [2001] show that if the household can secretly save but not borrow, this does not increase the planner’s ability to provide insurance. The best the planner can do is undo the borrowing constraint, but not provide insurance.

7 Is there insurance at all? [Townsend, 1994]

• How much insurance is there?

• Data from southern Indian villages

• High idiosyncratic income risk - especially because weather has different impact depending on each household’s source of income

• Insurance includes gifts, loan forgiveness, etc.

• Runs the following regression separately on each household:

$$c_{it} = \alpha_i + \beta_i C_t + \delta_i A_{it} + \zeta_i X_{it} + \epsilon_{it}$$

where $C_t$ is average consumption, $A_{it}$ are demographic controls and $X_{it}$ are any other variables, which under full insurance should not matter (e.g. household income)

• CARA preferences and equal risk aversion for all households implies that, if allocations are Pareto optimal:

$$- \beta_i = 1$$ for all households and
- $\zeta_i = 0$ for all households

- With CRRA, the same should be true for $\log c_{it}$ instead of $c_{it}$
TABLE III

<table>
<thead>
<tr>
<th>Population</th>
<th>Coeff</th>
<th>N</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>$\alpha$</td>
<td>133</td>
<td>64.8840</td>
<td>424.1715</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>133</td>
<td>0.7386</td>
<td>1.9168</td>
</tr>
<tr>
<td></td>
<td>$\delta$</td>
<td>133</td>
<td>-171.8203</td>
<td>2364.43</td>
</tr>
<tr>
<td>Aurepalle</td>
<td>$\alpha$</td>
<td>44</td>
<td>-21.1252</td>
<td>316.4395</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>44</td>
<td>0.9410</td>
<td>1.2367</td>
</tr>
<tr>
<td></td>
<td>$\delta$</td>
<td>44</td>
<td>244.5828</td>
<td>2450.87</td>
</tr>
<tr>
<td>Shirapur</td>
<td>$\alpha$</td>
<td>45</td>
<td>17.4952</td>
<td>257.3634</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>45</td>
<td>0.9410</td>
<td>1.2026</td>
</tr>
<tr>
<td></td>
<td>$\delta$</td>
<td>45</td>
<td>-371.5519</td>
<td>1525.96</td>
</tr>
<tr>
<td>Kanzara</td>
<td>$\alpha$</td>
<td>44</td>
<td>150.1918</td>
<td>580.7951</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>44</td>
<td>0.4654</td>
<td>2.7933</td>
</tr>
<tr>
<td></td>
<td>$\delta$</td>
<td>44</td>
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TABLE III Continued

<table>
<thead>
<tr>
<th>Population</th>
<th>$X_i$</th>
<th>N</th>
<th>Mean</th>
<th>Std. Dev.</th>
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</thead>
<tbody>
<tr>
<td>Aurepalle</td>
<td>All Income</td>
<td>44</td>
<td>0.1107</td>
<td>0.6774</td>
</tr>
<tr>
<td></td>
<td>Crop Income</td>
<td>44</td>
<td>-0.0549</td>
<td>1.0683</td>
</tr>
<tr>
<td></td>
<td>Labor Income</td>
<td>44</td>
<td>0.3588</td>
<td>0.8839</td>
</tr>
<tr>
<td></td>
<td>Profit from Trade and Handicrafts</td>
<td>44</td>
<td>0.2289</td>
<td>2.8739</td>
</tr>
<tr>
<td></td>
<td>Profit from Animal Husbandry #Household Members</td>
<td>44</td>
<td>-34.8736</td>
<td>9.7365</td>
</tr>
<tr>
<td></td>
<td>#Adults #Children</td>
<td>35</td>
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<td>8.2518</td>
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<tr>
<td></td>
<td></td>
<td>35</td>
<td>-7.4447</td>
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<tr>
<td>Shirapur</td>
<td>All Income</td>
<td>45</td>
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<tr>
<td></td>
<td>Crop Income</td>
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<td>Labor Income</td>
<td>45</td>
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<td>0.8650</td>
</tr>
<tr>
<td></td>
<td>Profit from Trade and Handicrafts</td>
<td>45</td>
<td>0.0642</td>
<td>1.5072</td>
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<tr>
<td></td>
<td>Profit from Animal Husbandry #Household Members</td>
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<td>-11.2844</td>
<td>11.0306</td>
</tr>
<tr>
<td></td>
<td>#Adults #Children</td>
<td>33</td>
<td>-18.3510</td>
<td>9.4325</td>
</tr>
<tr>
<td></td>
<td></td>
<td>33</td>
<td>-12.7787</td>
<td>9.8409</td>
</tr>
<tr>
<td>Kanzara</td>
<td>All Income</td>
<td>44</td>
<td>0.1106</td>
<td>0.8024</td>
</tr>
<tr>
<td></td>
<td>Crop Income</td>
<td>44</td>
<td>0.2213</td>
<td>1.7041</td>
</tr>
<tr>
<td></td>
<td>Labor Income</td>
<td>44</td>
<td>0.0960</td>
<td>1.0223</td>
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<tr>
<td></td>
<td>Profit from Trade and Handicrafts</td>
<td>44</td>
<td>-0.2830</td>
<td>4.0115</td>
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<tr>
<td></td>
<td>Profit from Animal Husbandry #Household Members</td>
<td>44</td>
<td>-57.2679</td>
<td>25.5271</td>
</tr>
<tr>
<td></td>
<td>#Adults #Children</td>
<td>36</td>
<td>-12.2616</td>
<td>9.7776</td>
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<tr>
<td></td>
<td></td>
<td>36</td>
<td>6.5228</td>
<td>8.2398</td>
</tr>
</tbody>
</table>

$^b$The variables in lines 1–5, 9–13, and 17–21 are measured in units of 1975 rupees per adult equivalent. The units for lines 6–8, 14–16, and 22–24 are simply unweighted head counts. Reported means and standard deviations are of studentized ordinary least squares estimates of (14) with each of the $X_i$ added in turn as an additional independent variable, weighted to correctly reflect the proportion of landless households in the population. The measure of consumption for these regressions is the value of consumed grains. All years (1975–1984) are used.
• $\beta = 1$ is rejected in favour of $\beta < 1$ in 16% of households (and in favour of $\beta > 1$ in 3%)

• Note: with 10 years of data, tests have low power. $\beta = 0$ is rejected in favour of $\beta > 0$ for only 52% of households.

• $\zeta = 0$ is rejected for $\zeta > 0$ in 15% of households (and in favour of $\zeta < 0$ in 13%)

• $\zeta = 1$ is rejected for $\zeta < 1$ for 87% of households

8 Is the data consistent with optimal insurance under moral hazard? [Ligon, 1998]

• Pure borrowing/saving (or hidden income plus hidden borrowing/saving) implies standard Euler equation. Under CRRA and $\beta R = 1$:

$$\mathbb{E}_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} - 1 \right] = 0$$

• Optimal insurance with moral hazard implies Inverse Euler Equation. Under CRRA and $\beta R = 1$:

$$\mathbb{E}_t \left[ \left( \frac{c_t}{c_{t+1}} \right)^{-\sigma} - 1 \right] = 0$$

• Estimate:

$$\mathbb{E}_t \left[ \left( \frac{c_{t+1}}{c_t} \right)^b - 1 \right] = 0$$

(30)

• Since we think $\sigma > 0$

  - $b > 0 \iff$ IEE $\iff$ optimal insurance with moral hazard
  - $b < 0 \iff$ EE $\iff$ pure borrowing/saving

• Full insurance would imply both EE and IEE should hold.

  - Literally, this requires constant consumption
  - Allowing for measurement error, aggregate shocks, etc, if we can forecast $c_{t+1}$ just as well using EE or IEE, this is evidence of full insurance

• Equation (30) can be estimated by GMM, using any information known at time t as instruments, i.e. estimate:

$$\mathbb{E} \left[ \left( \frac{c_t}{c_{t+1}} \right)^b - 1 \right] x_t = 0$$
• Data from three villages in India

• $x$: income, landholdings, family size, rainfall, village consumption

### TABLE III

*Estimates of $b_0$. Starred estimates are significant at the 95% level. Standard errors are reported in parentheses. Statistics reported under the columns labelled $J$ are scaled to have an asymptotic $\chi^2$ distribution.*

<table>
<thead>
<tr>
<th>Instrument vector</th>
<th>Aurepalle $b$ ($\sigma(b)$)</th>
<th>$J$</th>
<th>Shirapur $b$ ($\sigma(b)$)</th>
<th>$J$</th>
<th>Kanzara $b$ ($\sigma(b)$)</th>
<th>$J$</th>
<th>All $b$ ($\sigma(b)$)</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Income</td>
<td>$-0.4896$ ($0.3721$)</td>
<td>0</td>
<td>$1.3450^*$ ($0.3753$)</td>
<td>0</td>
<td>$0.8015^*$ ($0.3349$)</td>
<td>0</td>
<td>$0.3016^*$ ($0.0773$)</td>
<td>0.6354</td>
</tr>
<tr>
<td>Income, land</td>
<td>$-0.3660$ ($0.3357$)</td>
<td>$0.0142$</td>
<td>$1.3153^*$ ($0.3942$)</td>
<td>$0.0044$</td>
<td>$0.7710^*$ ($0.3729$)</td>
<td>$0.0000$</td>
<td>$1.1056^*$ ($0.2135$)</td>
<td>$0.8318$</td>
</tr>
<tr>
<td>Family size, income, land</td>
<td>$-0.6396^*$ ($0.3106$)</td>
<td>$0.0707$</td>
<td>$1.2932^*$ ($0.4289$)</td>
<td>$0.1544$</td>
<td>$0.6018$ ($0.3792$)</td>
<td>$0.0603$</td>
<td>$1.5063^*$ ($0.2143$)</td>
<td>$0.5041$</td>
</tr>
<tr>
<td>Family size, income, land, rain, avg consumption</td>
<td>$-0.7820^*$ ($0.3389$)</td>
<td>$0.0309$</td>
<td>$1.1314^*$ ($0.4156$)</td>
<td>$0.1532$</td>
<td>$0.6728$ ($0.3616$)</td>
<td>$0.0457$</td>
<td>$1.2458^*$ ($0.1868$)</td>
<td>$1.0769$</td>
</tr>
</tbody>
</table>

9 Is the data consistent with other forms of constrained optimal insurance? [Kinnan, 2011]

• Data from Thai villages

• Tests some of the predictions of constrained-optimal insurance models:
  
  1. Limited commitment
  2. Moral Hazard
  3. Hidden Income
  4. Pure borrowing and saving with no insurance

• Who cares? Example: India has a “National Rural Employment Guarantee Act”. Does this crowd out existing insurance?
  
  – Under limited commitment, maybe yes (autarky is less painful)
  – Under moral hazard, perhaps too (lower penalty for low effort)
– Under hidden income, it might even help (because it rules out very low income realization)

• Under CRRA utility:

\[ u'(c_t) = c_t^{-\sigma} \]

so

\[ \log c_t = \frac{1}{\sigma} \log \frac{1}{u'(c_t)} \]

so predictions about \( \frac{1}{u'(c_t)} \) can be tested by looking at \( \log(c_t) \)

### 9.1 Is insurance imperfect?

\[ \log c_{it} = \alpha \log y_{it} + \beta_i + \epsilon_{it} \]

<table>
<thead>
<tr>
<th>Table 2: Consumption smoothing at the individual and village level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>log household income</td>
</tr>
<tr>
<td>avg log household income</td>
</tr>
<tr>
<td>Village-year fixed effect?</td>
</tr>
<tr>
<td>Village-year F statistic</td>
</tr>
<tr>
<td>P value</td>
</tr>
<tr>
<td>Observations</td>
</tr>
<tr>
<td>R-squared</td>
</tr>
</tbody>
</table>

Notes: Household-level variables in columns (1), (2), (4) and (5) are deviations from individual means. Standard errors in brackets. All variables are in 2002 Thai baht. F-statistic tests the joint significance of the village-year effects. In columns (4) and (5) income is instrumented with quarterly rainfall deviations from average province-level quarterly rainfall, and deviations, and deviations and squared deviations interacted with 11 occupation dummies. In column (6) income is instrumented with quarterly rainfall deviations and squared deviations. Rainfall data is available for 1999-2003. *p<.1, ** p<.05, *** p<.01

• Households bear idiosyncratic risk: a 1% increase in income is associated with 0.06% to 0.21% increase in consumption
• But villages do provide insurance (village-year fixed effects highly significant), consistent with the findings in Townsend [1994].

9.2 Is \( \frac{1}{u'(c_{t-1})} \) a sufficient statistic for \( \frac{1}{u'(c_t)} \)?

\[
\log c_{it} = \gamma \log c_{it-1} + \zeta X_{it-1} + \epsilon_{it}
\]

• \( X \) is any other variable, for instance past income

<table>
<thead>
<tr>
<th>Table 4: Testing sufficiency of lagged inverse marginal utility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>ln(LIMU)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Lagged log income</td>
</tr>
</tbody>
</table>

Village-year fixed effects? | Yes | Yes | Yes | Yes
R-squared                  | 0.6645 | 0.6687 | 0.6200 | 0.6299
Observations               | 3186 | 2845 | 2874 | 2573

Notes: Robust standard errors in brackets. ln(LIMU) is proportional to \( ln(c_{t-1}) \). LIMU is lagged inverse marginal utility.

• This would suggest that limited commitment or moral hazard alone cannot account for the data

• Same sort of regression rejects both the Euler equation and the inverse Euler equation

• This pattern could be consistent with optimal insurance under hidden income

• (Or with other explanations)

9.3 Is there amnesia?

• Strictest test: for anyone that has consumption growth above village minimum, the past should not matter
• Slightly more lenient test: estimate

\[
\log c_{it} = \gamma \log c_{it-1} + \zeta Y_{it-1} + \epsilon_{it}
\]

allowing interaction coefficients with what quartile of the consumption-growth distribution a household is in, i.e.

\[
\log c_{it} = \gamma \log c_{it-1} + \sum_{q=2}^{4} \gamma_q \log c_{it-1}(q) + \zeta Y_{it-1} + \sum_{q=2}^{4} \zeta_q \log Y_{it-1}(q) + \epsilon_{it}
\]

• The strict amnesia prediction says that you should find

\[
\gamma + \gamma_4 = 0
\]

or at least

\[
\gamma_4 < \gamma_3 < \gamma_2 < 0
\]

• (The argument is that high-consumption-growth households are those for whom the reneging constraints are more likely to be binding, so the past should matter less for them)

• Instead, we find

\[
\gamma_4 > \gamma_3 > \gamma_2 > 0
\]

opposite of what the limited-commitment model would say
## References


