The Bellman Equation

1 Statement of the Problem

$$V(x) = \sup_{x,y} F(x,y) + \beta V(y)$$
$$s.t. \tag{1}$$
$$y \in \Gamma(x)$$

- Some terminology:
  - The Functional Equation (1) is called a Bellman equation.
  - x is called a state variable.
  - $G(x) = \{y \in \Gamma(x) : V(x) = F(x,y) + \beta V(y)\}$ is called a policy correspondence. It spells out all the values of y that attain the maximum in the RHS of (1).
  - If $G(x)$ is single-valued (i.e. there is a unique optimum), G is called a policy function.
- Questions:
  1. Does (1) have a solution?
  2. Is it unique?
  3. How do we find it?

2 The Bellman Equation as a Fixed-Point Problem

- Define the operator T by

$$T(f)(x) = \sup_{x,y} F(x,y) + \beta f(y)$$
$$s.t. \quad y \in \Gamma(x)$$

- V can be defined as a fixed point of T, i.e. a function such that $T(V)(x) = V(x) \quad \forall x$

- Does T have a fixed point? How do we find it?

Assumption 1. *(Assumption 4.3 in SLP) $X \subseteq \mathbb{R}^n$ is convex. $\Gamma : X \rightrightarrows X$ is nonempty, compact-valued and continuous.*

Assumption 2. *(Assumption 4.4 in SLP) $F : X \times X \to \mathbb{R}$ is bounded, i.e. $\exists \bar{F}$ such that $F(x, y) < \bar{F}$ for all $\{x, y\}$ with $x \in X$ and $y \in \Gamma(x)$.*

- What space is the operator T defined in?
- Define the metric space (S, ρ) by

$$S \equiv \{f : X \to \mathbb{R} \text{ continuous and bounded}\} \tag{2}$$

  with the norm

$$\|f\| = \sup_{x \in X} |f(x)|$$

  and thus the distance

$$\rho(f, g) = \|f - g\| = \sup_{x \in X} |f(x) - g(x)| \tag{3}$$

- Notice that $T : S \to S$, i.e. if f is continuous and bounded, then g is continuous and bounded.
- How do we know this?

1. $T(f)(x)$ is continuous
   - Recall Theorem of the Maximum:

     Proposition 1. *(Theorem of the Maximum). $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$. $f : X \times Y \to \mathbb{R}$ is continuous. $\Gamma : X \to Y$ is compact-valued and continuous. Then the function $h : X \to \mathbb{R}$ defined by*

$$h(x) = \max_{y \in \Gamma(x)} f(x, y)$$

     is continuous; and the correspondence $G : X \to Y$ defined by

$$G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$$

     is non-empty, compact-valued and upper hemi-continuous.
   - Applied to this problem:
     * $f(x, y)$ becomes $F(x, y) + \beta f(y)$
     * $h(x)$ becomes $T(f)(x)$
2. This is because F is bounded (Assumption 2) and f is bounded.

- Now we want to show that T has a unique fixed point. Two steps:
  1. Show that T is a contraction (Blackwell's sufficient conditions hold)
  2. Appeal to contraction mapping theorem

1. Blackwell's sufficient conditions:

   Proposition 2. *(Blackwell's sufficient conditions) $X \subseteq \mathbb{R}^l$ and $B(X)$ is the space of bounded functions $f : X \to \mathbb{R}$, with the sup norm. T is a contraction with modulus β if:*

   a. [Monotonicity] $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X \Rightarrow (Tf)(x) \leq (Tg)(x)$ for all $x \in X$;

   b. [Discounting] There exists some $\beta \in (0,1)$ such that $[T(f+a)](x) \leq (Tf)(x) + \beta a$ for all $f \in B(X)$, $a \geq 0$, $x \in X$.

   These conditions hold in our problem because

   (a) For any x

$$F(x,y) + \beta f(y) \leq F(x,y) + \beta g(y)$$
$$\sup_{y \in \Gamma(x)} F(x,y) + \beta f(y) \leq \sup_{y \in \Gamma(x)} F(x,y) + \beta g(y)$$
$$T(f)(y) \leq T(g)(y)$$

   (b)

$$T(f+a)(x) = \sup_{y \in \Gamma(x)} F(x,y) + \beta [f(y) + a]$$
$$= T(f)(x) + \beta a$$

2. Contraction Mapping Theorem

   Proposition 3. *(Contraction Mapping Theorem). If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then:*

   a. T has exactly one fixed point V in S;

   b. For any $V_0 \in S$, $\rho(T^n V_0, V) \leq \beta^n \rho(V_0, V)$, $n = 0, 1, 2, ...$

- The only missing step is to show that (S, ρ) defined by (2) and (3) indeed constitutes a complete metric space. (SLP Thm 3.1). Notice that if we used

$$\rho(f,g) = \int |f(x) - g(x)| \, dx$$

then (S,ρ) would NOT be a complete metric space. (SLP exercise 3.6.a., due this week).

Proposition 4. *(SLP 4.6) If Assumptions refregular and 2 hold, then T has a unique fixed point in S, i.e. there is a unique continuous bounded function that solves (1).*

Proof. From Contraction Mapping Theorem, knowing that Blackwell's sufficient conditions are met. □

Proposition 5. *The policy correspondence $G(x)=\{y\in\Gamma(x):V(x)=F(x,y)+\beta V(y)\}$ is compact-valued and u.h.c.*

Proof. From the Theorem of the Maximum □

Proposition 6. *If Assumptions refregular and 2 hold, then V is the value function of the sequence problem.*

Proof. V solves (1) and, because V is bounded, then $\lim_{T\to\infty}\beta^T V(x_T)=0 \quad \forall\tilde{x}\Pi(x_0),\forall x_0\in X$, so the sufficient conditions for Theorem SLP 4.3 hold. □

3 Proving Properties of V

Proposition 7. *If (S,ρ) is a complete metric space and S' is a closed subset of S, then S' is a complete metric space*

Proof. SLP Exercise 3.6.b. (due this week) □

Proposition 8. *(SLP Corollary 1, page 52). Let (S,ρ) be a complete metric space and $T:S\to S$ be a contraction mapping with fixed point $V\in S$.*

1. *If $S'$is a closed subset of S and $T(f)\in S'$ for all $f\in S'$, then $V\in S'$*
2. *If in addition $S''\subseteq S'$ and $T(f)\in S''$ for all $f\in S'$, then $V\in S''$*

Proof. □

1. *Choose $V_0\in S'$. $T^n(V)$ is a sequence in S' converging to V. Since S' is closed, $V\in S'$.*
2. *Since $V\in S'$, then $T(V)\in S''$. But $T(V)=V$ so $V\in S''$*

- Example:
  - S: all continuous functions $f : [a, b] \to \mathbb{R}$
  - S': all increasing functions $f : [a, b] \to \mathbb{R}$
  - S'': all strictly increasing functions $f : [a, b] \to \mathbb{R}$
- Note: we require the subset S' to be closed but not the sub-subset S''
- In our example:
  1. If T maps increasing functions into increasing functions, then the fixed point must be an increasing function
  2. If T maps increasing functions into strictly increasing functions, then the fixed point must be a strictly increasing function

  - What the result does not say is that if T maps strictly increasing functions into strictly increasing functions, then the fixed point must be a strictly increasing function (because the set of strictly increasing functions is not closed)

3.1 V increasing

Assumption 3. *(Assumption 4.5 in SLP) $F(x, y)$ is strictly increasing in x.*

Assumption 4. *(Assumption 4.6 in SLP) $x \le x'$ implies $\Gamma(x) \subseteq \Gamma(x')$*

- Do Assumptions (3) and (4) hold in the Neoclassical model?

Proposition 9. *(SLP 4.7). Suppose Assumptions (1)-(4) hold. Then V is strictly increasing.*

Proof. Let $x' > x$ and $f \in S$.

$$\begin{aligned} T(f)(x) &= \max_{y \in \Gamma(x)} F(x, y) + \beta f(y) \\ &\le \max_{y \in \Gamma(x')} F(x, y) + \beta f(y) \\ &< \max_{y \in \Gamma(x')} F(x', y) + \beta f(y) \\ &= T(f)(x') \end{aligned}$$

This implies that T maps any continuous bounded function into a strictly increasing function. Proposition 8 gives the result. □

3.2 V concave

Assumption 5. *(Assumption 4.7 in SLP) $F(x, y)$ is strictly concave.*

Assumption 6. *(Assumption 4.8 in SLP) Γ is convex*

Proposition 10. *(SLP 4.8) Suppose Assumptions (1), (2), (5) and (6) hold. Then V is strictly concave and G is continuous and single-valued.*

Proof. We want to show that T maps concave functions into strictly concave functions. Strict concavity of G follows by Proposition 8.

- Let $x_0 \neq x_1$ and $x_\theta = \theta x_0 + (1-\theta) x_1$ for $\theta \in (0, 1)$.
- Let $y_0 \in \Gamma(x_0)$ be such that $T(f)(x_0) = F(x_0, y_0) + \beta f(y_0)$ and similarly $y_1 \in \Gamma(x_1)$ be such that $T(f)(x_1) = F(x_1, y_1) + \beta f(y_1)$
- Then:

$$
\begin{aligned}
T(f)(x_\theta) \geq\ & F(x_\theta, y_\theta) + \beta f(y_\theta) \\
& (\Gamma \text{ concave makes } x_\theta, y_\theta \text{ feasible}) \\
& > [\theta F(x_0, y_0) + (1-\theta) F(x_1, y_1)] + \beta [\theta f(y_0) + (1-\theta) f(y_1)] \\
& (f \text{ concave and } F \text{ strictly concave}) \\
& = \theta [F(x_0, y_0) + \beta f(y_0)] + (1-\theta) [F(x_1, y_1) + \beta f(y_1)] \\
& (\text{rearranging}) \\
& = T(f)(x_0) + T(f)(x_1) \\
& (\text{by assumption})
\end{aligned}
$$

- G single-valued follows from strict concavity
- G continuous follows from Theorem of the Maximum

□

4 Is V differentiable? (Benveniste & Scheinkman, 1979)

- Cannot use same proof technique:
  - Space of differentiable functions is not closed
  - T does not necessarily map f into a differentiable function
- Instead, rely on the following result

Proposition 11. *Suppose $V : X \to \mathbb{R}$ is concave. Let x_0 be an interior and D be a neighborhood around x_0. Suppose exists $w : D \to \mathbb{R}$ such that:*

1. $w(x) \leq V(x)$

2. $V(x_0) = w(x_0)$

3. *w is differentiable at x_0*

Then V is differentiable at x_0

Proof. Any subgradient p of V at x_0 must satisfy;

$$p \cdot (x - x_0) \geq V(x) - V(x_0) \geq w(x) - V(x_0) \geq w(x) - w(x_0)$$

but since w is differentiable, then p is unique, which implies V differentiable. □

- (Graph)
- This result is useful to establish the following:

Proposition 12. *(SLP 4.11) Suppose Assumptions (1), (2), (5) and (6) hold and F is continuosly differentiable. Then V is differentiable and*

$$V_i(x_0) = F_i(x_0, g(x_0))$$

Proof. Define

$$w(x) = F(x, g(x_0)) + \beta V(g(x_0))$$

□

- *w is concave, differentiable and satisfies*

$$\begin{aligned} w(x) &\leq F(x, g(x)) + \beta V(g(x)) \quad \forall x \\ \Rightarrow w(x) &\leq \max_{y \in \Gamma(x)} F(x, g(x)) + \beta V(g(x)) \\ &= V(x) \end{aligned}$$

  and

$$w(x_0) = V(x_0)$$

- *The result then follows from (11)*