1 The Permanent Income Hypothesis

- (This is mostly repeated from the first class)
- Household consumption decisions characterized by
  
  1. Euler equation:

\[
\frac{d}{dt} u(c_t) = \beta \frac{p_t}{p_{t+1}} u(c_{t+1}) = \beta R_{t+1} u(c_{t+1})
\]

  2. Budget constraint with equality:

\[
\sum_{t=0}^{\infty} p_t c_t = \sum_{t=0}^{\infty} p_t y_t
\]

- Cleanest formulation of the hypothesis is for the case of $\beta R = 1$ (but principle is more general)
- For $\beta R = 1$, Euler becomes

\[c_{t+1} = c_t\]

and budget becomes

\[\sum_{t=0}^{\infty} R^{-t} (c_t - c_t) = 0\]

so

\[c_t = \frac{R - 1}{R} \left( \sum_{s=0}^{\infty} R^{-s} y_s \right)\]

- Interpretation:
  
- Now assume that:
    
    - the income process is uncertain
the household acts as though it wasn’t (certainty equivalent behaviour)

- Certainty equivalent behaviour is not optimal in general
- But it’s optimal if preferences are quadratic - we’ll see this later
- Consumption becomes

$$c_t = \frac{R - 1}{R} \left( \mathbb{E}_t \left[ \sum_{s=0}^{\infty} R^{-s} y_s \right] \right)$$

- Notice timing
- Implicit: evolution of household’s assets

1.1 Contrast with traditional Keynesian view

- Keynes (1936):

“The fundamental psychological law, upon which we are entitled to depend with great confidence both a priori from our knowledge of human nature and from the detailed facts of experience, is that men are disposed, as a rule and on the average, to increase their consumption as their income increases but not by as much as the increase in the income.”

- Keynesian consumption function:

$$c_t = \alpha_0 + \alpha_1 y_t$$

- Keynesian MPC: $\alpha_1 < 1$

- PIH MPC

- Marignal propensity to consume out of wealth: $\frac{\partial c_t}{\partial (\mathbb{E}_t \left[ \sum_{s=0}^{\infty} R^{-s} y_s \right])} = \frac{R-1}{R} \approx 0.04$
- Marginal propensity to consume out of purely temporary income: $\frac{\partial c_t}{\partial y_t} = \frac{R-1}{R}$ (holding $\mathbb{E}_t \left[ \sum_{s=1}^{\infty} R^{-s} y_s \right]$ constant)
- Marginal propensity to consume out of permanent income $\frac{\partial c_t}{\partial y_t} = 1$ (assuming $\frac{\partial \mathbb{E}_t y_t}{\partial y_t} = 1$)
- In general, MPC out of shocks to current income depends on income process. It could be greater than 1 if income growth is highly serially correlated. See below.
1.2 The random walk prediction

- Take first difference:
  \[ \Delta c_t = \frac{R - 1}{R} \left( \sum_{s=0}^{\infty} R^{-s} \mathbb{E}_t y_{t+s} - \mathbb{E}_{t-1} y_{t+s} \right) \]  
  \hspace{1cm} (1)

- Therefore
  \[ E_{t-1} (\Delta c_t) = 0 \]

- Testable!

1. Regress \( \Delta c_t \) on information known at \( t - 1 \)

2. Relax \( \beta R = 1 \) but keep the certainty-equivalence assumption:
  \[ c_t = a_0 + a_1 c_{t-1} + \varepsilon_t \]

Regress \( c_t \) on \( c_{t-1} \) and information known at \( t - 1 \). This is like the previous case but you allow \( a_0 \neq 0 \) and \( a_1 \neq 1 \).

3. Relax certainty-equivalence

- Test Euler equation:
  \[ \mathbb{E}_{t-1} \left( \beta R_t \frac{u'(c_t)}{u'(c_{t-1})} \right) = 1 \]

- CRRA case with riskless interest rate
  \[ \mathbb{E}_t \left( \beta R \left( \frac{c_t}{c_{t-1}} \right)^{-\sigma} \right) = 1 \]
  \[ c_t^{-\sigma} = c_{t-1}^{-\sigma} (\beta R)^{-1} + \varepsilon_t \]

- Assume values for \( \sigma \)

- Regress \( c_t^{-\sigma} \) on \( c_{t-1}^{-\sigma} \) and information known at \( t - 1 \)

- Evidence from Hall [1978]:
• As long as income is somewhat predictable, this distinguishes PIH from Keynesian or more generally from:

\[ c_t = D(L)y_t \]

1.3 Test of consumption smoothing

• Start from (1), estimate stochastic process for income and check.

• Campbell and Deaton [1989]:

\[
\Delta y_t = 8.2 + 0.442 \Delta y_{t-1} + \varepsilon_t, \quad \sigma_\varepsilon = 25.2. \quad (6)
\]

• Income growth positively correlated ⇒ permanent income more volatile than current income.

• Using (1):

\[
\Delta c_t = \frac{R}{0.558 + R - 1}\varepsilon_t = 1.78\varepsilon_t
\]

for \( R = 1.01 \) quarterly.

• Standard deviation of \( \Delta c_t \) should be 1.78 times \( \sigma_\varepsilon \).

• Data: 27.3 for total; 12.4 for nondurables, less than \( \sigma_\varepsilon = 25.2 \): “excess smoothness”!

• One explanation: slow adjustment to innovations in permanent income
1.4 Test of Euler equation (without assuming values for $\sigma$)

- Derive log-linear version of Euler equation [Hansen and Singleton, 1983]:

$$
E_t \left( \beta R_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \right) = 1
$$

- Define

$$z_{t+1} \equiv R_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma}$$

- Assume $z_{t+1}$ is lognormal given time-$t$ information:

$$
\log z_{t+1} = \log R_{t+1} - \sigma \Delta \log c_{t+1} \sim N(\mu_t, v_t)
$$

- Define

$$
\varepsilon_{t+1} \equiv \log z_{t+1} - \mu_t
$$

and note

$$
\varepsilon_{t+1} \sim N(0, v_t)
$$

- Using the properties of the lognormal distribution:

$$
E_t (z_{t+1}) = \exp \left[ \mu_t + \frac{v_t}{2} \right]
$$

- Now use the Euler equation:

$$
E_t (z_{t+1}) = \frac{1}{\beta} \exp \left[ \mu_t + \frac{v_t}{2} \right] = \frac{1}{\beta} \exp \left[ - \log \beta \right]
$$

$$
\mu_t + \frac{v_t}{2} = - \log \beta
$$

$$
\log (z_{t+1}) - \varepsilon_{t+1} + \frac{v_t}{2} = - \log \beta
$$

$$
\log R_{t+1} - \sigma \Delta \log c_{t+1} - \varepsilon_{t+1} + \frac{v_t}{2} = - \log \beta
$$

$$
\Delta \log c_{t+1} = \left[ \frac{\log \beta}{\sigma} + \frac{v_t}{2\sigma} \right] - \frac{\log R_{t+1}}{\sigma} - \frac{\varepsilon_{t+1}}{\sigma}
$$

(2)

- $\Delta \log c_{t+1}$ should not be predictable with time-$t$ information other than interest rates.

- Campbell and Mankiw [1990] estimate:

$$
\Delta \log c_{t+1} = \mu + \lambda \Delta \log y_{t+1} + \theta r_{t+1} + \varepsilon_{t+1}
$$
using lagged values of consumption or income growth or interest rates as instruments for \( \Delta \log y_{t+1} \).

<table>
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<tr>
<th>Row</th>
<th>Instruments ((z))</th>
<th>(\Delta c)</th>
<th>(\Delta y) or (\Delta x)</th>
<th>(r) (standard error)</th>
<th>(\lambda) (standard error)</th>
<th>(\theta) (standard error)</th>
<th>Test of restrictions</th>
</tr>
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<td>1</td>
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<td>.049</td>
<td>.467</td>
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<td>2</td>
<td>(\Delta i_{1:2}), (\Delta i_{1:4}) (r_{1:2}), (r_{1:4})</td>
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<td>.026</td>
<td>.448</td>
<td>.668</td>
<td>-.022</td>
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A. Real interest rates

- “Excess sensitivity”
- Campbell and Mankiw [1990]: a fraction of consumers just consume their income. Liquidity constraints?
- Campbell and Deaton [1989]: delayed response of consumption to innovations. Would also explain “excess smoothness”
- Statistically: predictable changes in consumption
- Why the difference with Hall [1978]? 1950:1 (large payments to WWII veterans)?
- Carroll [1997]: don’t ignore the \(v_t\) term in (2)!
- “Buffer-stock” behaviour predicts systematic relation between \(v_t\) and wealth/income ratio
- More generally: “precautionary savings”

2 Precautionary savings: two-period example

- No complete markets
- Only riskless borrowing-saving
- Uncertain future income

\[
\max_a u(y_0 - a) + \beta \sum_s \Pr(s) u(Ra + y_1(s))
\]
• FOC:

\[-u'(y_0 - a^*) + \beta R \sum_s \Pr(s) u'(Ra^* + y_1(s)) = 0 \]  

(3)

• Define \( \hat{a} \) as the solution to

\[-u'(y_0 - \hat{a}) + \beta R u' \left( R\hat{a} + \sum_s \Pr(s) y_1(s) \right) = 0 \]

• Is it the case that \( a^* = \hat{a} \) (certainty equivalence)?

**Proposition 1.** \( a^* > \hat{a} \) if \( u'(\cdot) \) is convex

*Proof.* Assume the contrary:

\[
a^* \leq \hat{a} \\
\Rightarrow u'(Ra^* + y_1(s)) \geq u'(R\hat{a} + y_1(s)) \quad \forall s \\
\sum_s \Pr(s) u'(Ra^* + y_1(s)) \geq \sum_s \Pr(s) u'(R\hat{a} + y_1(s)) \\
> u' \left( R\hat{a} + \sum_s \Pr(s) y_1(s) \right) \quad (u' \text{ convex}) \\
= \frac{u'(y_0 - \hat{a})}{\beta R} \quad \text{by definition} \\
\geq \frac{u'(y_0 - a^*)}{\beta R} \quad (a^* \geq \hat{a})
\]

which contradicts (3) \( \square \)

• “Precautionary savings” if \( u'''(\cdot) > 0 \)

• Inevitable as \( c \to \infty \)

• True for CRRA:

\[
\begin{align*}
u(c) &= \frac{c^{1-\sigma}}{1-\sigma} \\
u'(c) &= c^{-\sigma} \\
u''(c) &= -\sigma c^{-\sigma-1} \\
u'''(c) &= \sigma (1 + \sigma) c^{-\sigma-2} > 0
\end{align*}
\]

• For quadratic preferences: certainty equivalence

• We talked about this as one example of a dynamic optimization problem under uncertainty. Now we’ll analyze it in detail.

• Recursive representation:

\[ V(x, s) = \max_{c, x'(s')} u(c) + \beta \sum_{s'} \Pr(s'|s)V(x'(s'), s') \]

s.t. \[ x'(s') \leq R[x - c] + y(s') \]
\[ c \leq x + b \]

•

\[ u'(c) - \sum_s \lambda(s) R - \mu = 0 \]
\[ \beta \Pr(s'|s) \frac{\partial V(x'(s'), s')}{\partial x} - \lambda(s) = 0 \]
\[ \Rightarrow u'(c) \geq \beta R \sum_{s'} \Pr(s'|s) \frac{\partial V(x'(s'), s')}{\partial x} \]

with equality if \( c < x + b \)

• Envelope condition

\[ \frac{\partial V(x, s)}{\partial x} = u'(c(x, s)) \]

• Euler equation:

\[ u'(c) \geq \beta R \sum_{s'} \Pr(s'|s)u'(c(x'(s'), s')) \]

with equality if \( c < x + b \)

3.1 The \( \beta R = 1 \) case

• For \( \beta R = 1 \), the Euler equation is

\[ u'(c) \geq \sum_{s'} \Pr(s'|s)u'(c(x'(s'), s')) \]

so marginal utility follows a supermartingale.

**Theorem 1** (Doob). Let \( \{Z_t\} \) be a nonnegative supermartingale. Then \( Z_t \rightarrow_{a.s.} Z \), where \( Z \) is a random variable with \( \mathbb{E}[Z] < +\infty \)
Example 1. Suppose

\[ Z_t = \begin{cases} 
Z_{t-1} - u_t & \text{if } t \leq 7 \\
Z_{t-1} & \text{otherwise}
\end{cases} \]

where \( u_t \sim U[0,1] \), and \( Z_0 = 8 \). Then \( Z_t \) is a nonnegative supermartingale. It converges to some random variable \( Z \) with support in \([1,8]\)

**Proposition 2.** For \( \beta R = 1 \), the solution of the household problem has \( \Pr \left[ \lim_{t \to \infty} c(s^t) = +\infty \right] = 1 \), i.e. consumption diverges to infinity almost surely

**Proof.**

- By the Martingale Convergence Theorem, the marginal utility of consumption converges almost surely to a finite number. (What finite number could depend on the sequence of income realizations).
- Suppose than number is greater than zero. Then consumption converges almost surely to a finite number.
- Since after any sequence \( s^t \) the realization of \( y_t \) could be \( y(s_1) \) forever, then the maximum possible constant consumption after history \( s^t \) is \( c(s^{t+n}) = \frac{y(s_1)}{R} + \frac{R-1}{R} x(s^t) \) for all \( n \geq 0 \).
- But the household can improve upon this constant consumption by ratcheting up consumption by \( \frac{R-1}{R} (y(s_t) - y(s_1)) \) forever whenever an income level other than \( y(s_1) \), which contradicts optimality.
- Therefore \( u'(c_t) \) must converge to zero
- Therefore (assuming \( u'(c) > 0 \forall c \)), \( c_t \) must diverge to \( \infty \)

(See Chamberlain and Wilson [2000] for a rigorous version of this proof and exact conditions under which it holds)

- Strong precautionary motive
- Does not depend on \( u''(c) \) (but in a way it does)

### 3.2 The iid-CARA case in detail (without making assumptions on \( \beta \) and \( R \))

- Suppose \( u(c) = -\exp(-\gamma c) \)
- Do not impose a borrowing limit (just a no-Ponzi condition)
• Allow $c < 0$

• Suppose $y$ is iid

• Useful properties of CARA:
  \[
  u'(c) = \gamma \exp(-\gamma c) = -\gamma u(c)
  \]
  \[
  u(a + b) = -u(a)u(b)
  \]
  \[
  u^{-1}(v) = -\frac{1}{\gamma} \log(-v)
  \]
  \[
  \frac{1}{u(c)} = u(-c)
  \]

• Household solves
  \[
  V(x) = \max_c u(c) + \beta \sum_{s'} \Pr(s') V(R[x - c] + y(s'))
  \]

• Guess:
  \[
  V(x) = Au(\lambda x)
  \]
  with $\lambda = \frac{R - 1}{R}$

• Using guess:
  \[
  V(x) = \max_c u(c) + \beta A \mathbb{E} \left[ u \left( R - 1 \right) \left( x - c \right) + \frac{R - 1}{R} y \right]
  \]

Let
  \[
  \alpha \equiv x - c - \frac{1}{R} x
  \]
  \[
  \Rightarrow c = \frac{R - 1}{R} x - \alpha
  \]
  \[
  (R - 1)(x - c) = (R - 1)\alpha + \frac{R - 1}{R} x
  \]

so
  \[
  V(x) = \max_\alpha \left[ u \left( \frac{R - 1}{R} x - \alpha \right) + \beta A \mathbb{E} \left[ u \left( (R - 1)\alpha + \frac{R - 1}{R} x + \frac{R - 1}{R} y \right) \right] \right]
  \]
  \[
  = -u \left( \frac{R - 1}{R} x \right) \max_\alpha \left[ u(-\alpha) + \beta A \mathbb{E} \left[ u \left( (R - 1)\alpha + \frac{R - 1}{R} y \right) \right] \right]
  \]
Therefore

\[ A = -\max_{\alpha} \left[ u(-\alpha) + \beta A \mathbb{E} \left[ u\left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] \right] \]  

which verifies the guess because \( A \) is a constant.

- Solve by taking FOCs:

\[ -u'(-\alpha) + \beta A (R - 1) \mathbb{E} \left[ u'\left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] = 0 \]

\[ u(-\alpha) = \beta A (R - 1) \mathbb{E} \left[ u\left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] = 0 \]  

(6)

- Recall (5):

\[ A = -u(-\alpha) - \beta A \mathbb{E} \left[ u\left( (R - 1) \alpha + \frac{R - 1}{R} y \right) \right] \]

and using (6):

\[ A = -u(-\alpha) - \frac{u(-\alpha)}{R - 1} \]

\[ = -\frac{R}{R - 1} u(-\alpha) \]
• Replace back in the FOC (6)

\[ u(-\alpha) = -\beta Ru(-\alpha)\mathbb{E}\left[u\left((R-1)\alpha + \frac{R-1}{R}y\right)\right] \]

\[ 1 = -\beta R\mathbb{E}\left[u\left((R-1)\alpha + \frac{R-1}{R}y\right)\right] \]

\[ = \beta Ru\cdot((R-1)\alpha)\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right] \]

\[ u\left((-1)\alpha\right) = \beta R\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right] \]

\[ - (R-1)\alpha = u^{-1}\left(\beta R\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right]\right) \]

\[ \alpha = -\frac{u^{-1}\left(\beta R\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right]\right)}{R-1} \]

\[ = \frac{1}{\gamma} \log\left[\beta R\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right]\right] \]

\[ = \frac{1}{\gamma} \log(\beta R) + \log\left[\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right]\right] \]

\[ = \frac{1}{\gamma} \log(\beta R) - \frac{u^{-1}\left(\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right]\right)}{R-1} \]

(7)

• The consumption function is therefore

\[ c = \frac{R-1}{R} x - \alpha \]

\[ = \frac{R-1}{R} x - \frac{1}{\gamma} \frac{\log(\beta R)}{R-1} + \frac{u^{-1}\left(\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right]\right)}{R-1} \]

• Special case of \( \beta R = 1 \) and no uncertainty about \( y \):

\[ c = \frac{R-1}{R} x + \frac{y}{R} \]

• The term:

\[ \frac{1}{\gamma} \frac{\log(\beta R)}{R-1} \]

adjusts for \( \beta R \neq 1 \)

• The (constant) term:

\[ \frac{u^{-1}\left(\mathbb{E}\left[u\left(\frac{R-1}{R}y\right)\right]\right)}{R-1} \]

adjusts for the precautionary savings effect.
• $u$ concave means

$$
\mathbb{E} \left[ u \left( \frac{R-1}{R} y \right) \right] < u \left( \frac{R-1}{R} \mathbb{E} y \right)
$$

$$
u^{-1} \left( \mathbb{E} \left[ u \left( \frac{R-1}{R} y \right) \right] \right) < u^{-1} \left( u \left( \frac{R-1}{R} \mathbb{E} y \right) \right)
$$

$$
= \frac{R-1}{R} \mathbb{E} y
$$

• Drift: consumption will tend to increase iff

$$
c < \frac{R-1}{R} x + \frac{\mathbb{E} y}{R}
$$

$$
- \frac{1}{\gamma} \log (\beta R) + \frac{u^{-1} \mathbb{E} y \left( \frac{R-1}{R} y \right)}{R-1} < \frac{\mathbb{E} y}{R}
$$

• It could drift off to infinity even with $\beta R < 1$, if there is sufficient risk and risk aversion

• Evolution of wealth:

$$
x_{t+1} = R (x_t - c) + y_{t+1}
$$

$$
= R \left( x_t - R \frac{R-1}{R} x_t + \alpha \right) + y_{t+1}
$$

$$
= x_t + y_{t+1} + \alpha R
$$

so wealth is a random walk with drift. The drift is given by (7)

• Implication: the variance of the wealth distribution in the population diverges to infinity!

### 3.3 $\beta R < 1$ and decreasing absolute risk aversion

• Recall that the coefficient of absolute risk aversion is defined as

$$
\gamma(c) = - \frac{u''(c)}{u'(c)}
$$

• Now assume that $\beta R < 1$ and

$$
\lim_{c \to +\infty} \gamma(c) = 0
$$

• Focus on the iid case for simplicity (but the result is more general)

**Proposition 3.** Assume $\beta R < 1$ and $\lim_{c \to +\infty} \gamma(c) = 0$. Let $c(x)$ be the solution to program (4) and let $x'_{\text{max}}(x) = R [x - c(x)] + y(s_n)$ and $x'_{\text{min}}(x) = R [x - c(x)] + y(s_1)$. There exists a value $x^*$ such that $x'_{\text{max}}(x) \leq x$ for all $x \geq x^*$. 

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Proof. Below is part of the proof. There are three things missing from it (left as an exercise)

1. Showing that the borrowing constraint does not bind for sufficiently high $x$
2. Showing that $c(x)$ is increasing
3. Showing that $\lim_{x \to \infty} c(x) = \infty$

For sufficiently high $x$ the borrowing constraint will not bind, so the Euler equation is

\[
\frac{u'(c(x))}{u'(c(x_{\text{max}}))} = \beta R \sum_{s'} \Pr(s') u'(c(R[x - c(x)] + y(s'))) = \beta R \sum_{s'} \Pr(s') u'(c(R[x - c(x)] + y(s'))) \frac{u'(c(x_{\text{max}}(x)))}{u'(c(x'_{\text{max}}(x)))}
\]

(8)

Note that since $c(x)$ is increasing, the following inequalities hold:

\[
1 \leq \frac{\sum_{s'} \Pr(s') u'(c(R[x - c(x)] + y(s')))}{u'(c(x'_{\text{max}}(x)))} \leq \frac{u'(c(x'_{\text{min}}(x)))}{u'(c(x'_{\text{max}}(x)))} \leq 1 - \int_{c(x'_{\text{min}}(x))}^{c(x'_{\text{max}}(x))} \frac{u''(c)}{u'(c)} dc
\]

The fact that $\lim_{c \to \infty} \gamma(c) = 0$ implies that

\[
\lim_{x \to \infty} \frac{\sum_{s'} \Pr(s') u'(c(R[x - c(x)] + y(s')))}{u'(c(x'_{\text{max}}(x)))} = 1
\]

(9)

Replacing (9) in (8) and taking limits:

\[
\lim_{x \to \infty} [u'(c(x)) - \beta Ru'(c'_{\text{max}}(x))] = 0
\]

Since $\beta R < 1$, this implies that for high enough $x$, consumption is expected to fall for any realization of $s'$. Given that $c(x)$ is increasing, the result follows.

- Consumption does not grow to infinity
- There is an upper bound on wealth
- There is an invariant distribution of wealth for any given individual
• If we think of the population as a collection of independent individuals, there is a steady state wealth distribution

References


