
Econ 235, Spring 2013

• What determines leverage?
• What is deleveraging and why does it happen?
• How are leverage and asset prices jointly determined?
• From a modeling perspective: disagreement.

1 A two-period example

• Everyone has $a$ apples and one tree at $t = 0$
• People disagree about $\pi$
• "Agree to disagree"
• For GE theory, probabilities are preferences
  – Heterogeneity in risk aversion also gives different risk-neutral probabilities
• Everyone is risk neutral and no discounting
• Apples can be stored so w.l.o.g people consume at $t = 1$

1.1 Simple case with no borrowing
• The only market is for trading apples for trees at $t = 0$
• No shortselling
• Type $\pi$’s problem is

$$\max_x \pi c_H + (1 - \pi) c_L$$
$$s.t\ c_H = (a + p + x) - px$$
$$c_L = (a + p + dx) - px$$
$$px \leq a + p$$
$$x \geq 0$$

• Linear program. You buy if

$$\pi (1 - p) + (1 - \pi) (d - p) > 0$$
$$p < \pi + (1 - \pi) d$$

i.e. if you think the tree is underpriced
• The threshold type $\pi^*$ is indifferent

$$\pi^* (p) = \frac{p - d}{1 - d}$$

• According to this type, the price is right
• Demand:

$$[1 - F (\pi^* (p))] \frac{a + p}{p}$$

is decreasing in price because
marginal type stops demanding
- each buyer can afford fewer units

- Equilibrium:

\[
[1 - F(\pi^*(p))] \frac{a + p}{p} = 1
\]

- \(p\) is increasing in \(a\)
- A FOSD increase in \(F\) increases \(p\)
- For \(F\) uniform, \(a = 1, d = 0.2\)

\[p = 0.68\]
\[\pi^* = 0.6\]

1.2 Complete markets benchmark

- With pure complete markets, type \(\pi\) solves

\[
\max_{x_H,x_L} \pi c_H + (1 - \pi) c_L \\
\text{s.t. } p_H c_H + p_L c_L \leq a + p_H + d p_L \\
\quad c_L \geq 0 \\
\quad c_H \geq 0
\]

- Consume in state \(H\) iff

\[\pi > p_H\]

and in state \(L\) if

\[(1 - \pi) > p_L\]

- But due to arbitrage with storage,

\[p_H + p_L = 1 \tag{1}\]

so only one of those conditions holds

- The marginal type is

\[\pi^*(p_H) = p_H \tag{2}\]

- Demand for \(H\) apples:

\[
\frac{a + p_H + d p_L}{p_H} (1 - F(\pi^*))
\]
• Supply of $H$ apples:

$$1 + a$$

• $H$-market clearing:

$$\frac{a + p_H + dp_L}{p_H} (1 - F(\pi^*)) = 1 + a$$

$$\frac{a + p_H + d - dp_H}{p_H} (1 - F(\pi^*)) = 1 + a$$

$$\frac{a + d}{p_H} (1 - F(\pi^*)) + (1 - d) (1 - F(\pi^*)) = 1 + a$$

$$a + d = \frac{1 + a}{1 - F(\pi^*)} - (1 - d)$$

$$p_H = \frac{a + d}{1 + a - (1 - d)}$$

• We have three equations in three unknowns

• We can also price the trees. Notice that the same marginal type is pricing both the trees and the arrow securities

$$p = p_H + dp_L$$

$$= \pi^* + (1 - \pi^*) d$$

• For $F$ uniform, $a = 1$, $d = 0.2$

$$\pi^* = 0.44$$

$$p_H = 0.44$$

$$p_L = 0.56$$

$$p = 0.55$$

1.3 Riskless collateralized borrowing only

• Because we only allow riskless borrowing, the interest rate will be zero

• Borrowing is limited by the amount of collateral
• Type \( \pi' \)'s problem is

\[
\max_{x,b} \quad \pi c_H + (1 - \pi) c_L
\]

s.t

\[
\begin{align*}
    c_H &= (a + p + x) - px \\
    c_L &= (a + p + dx) - px \\
    px &\leq a + p + b \\
    b &\leq dx \\
    x &\geq 0
\end{align*}
\]

• \( b \) does not appear in the expressions for \( c_H \) and \( c_L \) because whatever you borrow, you pay back

• We don’t have the K&M issue that part of the marginal product of the asset cannot be pledged

• Note that the assumptions of

  − risk neutrality
  − only riskless borrowing
  − no shortselling

lead to:

− High-\( \pi \) types only consume in state \( H \) (because the hit borrowing constraints)

− Low-\( \pi \) types consume equal amounts in both states. They would like to shortsell the asset in order to shift consumption from the \( H \) state (which they think is unlikely) to the \( L \) state (which they think is likely), but we don’t let them.

• The threshold type is still defined by

\[
\pi^*(p) = \frac{p - d}{1 - d}
\]

• Each agent with \( \pi > \pi^* \) will buy the following amount of trees:

\[
\begin{align*}
    x &= \frac{a + p + dx}{p} \\
    &= \frac{a + p}{p - d}
\end{align*}
\]
which is the sort of expression from K&M. Total demand of trees will be

\[ [1 - F(\pi^*(p))] \frac{a + p}{p - d} \]

so in equilibrium:

\[ [1 - F(\pi^*(p))] \frac{a + p}{p - d} = 1 \] \hspace{1cm} (4)

- Prices are higher because of leverage
- For \( F \) uniform, \( a = 1, d = 0.2 \)

\[ p = 0.75 \]
\[ \pi^* = 0.69 \]

- The marginal type still thinks the price is right, but now a more optimistic person is the marginal buyer

### 1.4 Risky collateralized borrowing?

- Suppose you allowed contracts of the type:
  - You give me \( \frac{1}{1+r} \) apples today
  - I post one tree as collateral
  - I promise you \( j \) apples “unconditionally”
  - If the tree pays \( y > j \), I pay you back \( j \)
  - If the tree pays \( y < j \), I pay you back \( y \)

- Risky, nonrecourse, collateralized debt

- \( j = d \) is the riskless debt contract

- GE formulation:
  - Each contract \( j \) is a different commodity
  - There is an interest rate \( r \) for each contract \( j \), i.e. you can buy the contract \( j \) for \( \frac{1}{1+r(j)} \)
  - Agents take the prices as given and, subject to the amount of collateral they have, trade as much as they want in each contract
  - Markets clear
• Result: only riskless contracts are traded!

• Program:

$$\text{max}_{x,b(c)} \pi c_H + (1 - \pi) c_L$$

$$\text{s.t} \quad c_H = (a + p + x) - px + \sum_j b(j) [q(j) - \min \{j, 1\}]$$

$$c_L = (a + p + dx) - px + \sum_j b(j) [q(j) - \min \{j, d\}]$$

$$\sum_j \max \{b(j), 0\} \leq x$$

$$px \leq a + p + \sum_j b(j) q(j)$$

$$x \geq 0$$

where $b(j)$ denotes the number of $j$-contracts that the trader sells. So a negative number for $b(j)$ means the trader is buying these contracts.

• The first constraint is a collateral constraint

• The second constraint is a borrowing constraint

• Conjecture equilibrium:

- There is a cutoff type $\pi^*$ that prices every claim
- Therefore

$$p = \pi^* + (1 - \pi^*) d$$

$$q(j) = \pi^* \min \{j, 1\} + (1 - \pi^*) \min \{j, d\}$$

• Replace conjecture in budget constraint:

$$c_H = a + x + [\pi^* + (1 - \pi^*) d] (1 - x) + (\pi^* - 1) \sum_j b(j) [\min \{j, 1\} - \min \{j, d\}]$$

$$c_L = a + dx + [\pi^* + (1 - \pi^*) d] (1 - x) + \pi^* \sum_j b(j) [\min \{j, 1\} - \min \{j, d\}]$$

$$\Rightarrow \pi^* c_H + (1 - \pi^*) c_L = a + \pi^* + (1 - \pi^*) d$$

• NPV of consumption = NPV of endowment
• Types above $\pi^*$ want

\[
\begin{align*}
  c_L &= 0 \\
  c_H &= \frac{a + \pi^* + (1 - \pi^*) \, d}{\pi^*}
\end{align*}
\]

• This is achievable by choosing:

\[
\begin{align*}
  x &= b(d) = \frac{a + p}{p - d} \\
  b(j) &= 0 \quad \forall j \neq d
\end{align*}
\]

• For these types, solution is not unique, they would be indifferent to borrow with $j > d$ (risky debt) to buy more of the asset

• Types below would like to consume as much as possible in state $L$

• Since there is no shortselling, they cannot engineer $c_L > c_H$ (there are no contracts for doing this), so they look for $c_L = c_H$

• This is achievable only if you choose

\[
\begin{align*}
  x &= 0 \\
  b(d) &\geq 0 \quad \text{only if } j \leq d
\end{align*}
\]

(i.e. don’t hold the asset or make risky loans)

• Making a risky loan

  - Increases $c_H$ by $(1 - \pi^*) \,(j - d)$
  - Decreases $c_L$ by $\pi^*(j - d)$

  - Net effect on utility:

\[
\left[ \pi \,(1 - \pi^*) - (1 - \pi) \, \pi^* \right] (j - d) = (\pi - \pi^*) \,(j - d) < 0
\]

• This confirms equilibrium:

  - Type $\pi^*$ prices every contract
  - Types above $\pi^*$ hold the asset, consume only in high state, borrow up to the limit with riskless debt
  - Types above $\pi^*$ lend with riskless debt
– No other contract gets traded

• This result is NOT general

• In particular, it does not generalize beyond the binary case

• [Simsek, 2012] looks at the case where payoffs are not binary.

1.5 Complete markets but collateral constraints

• We showed that if
  – Payoffs are binary
  – We only allow (risky or riskless) debt contracts

  then agents will trade the riskless contracts only

• But this is NOT equal to the complete markets allocation

• What if we did “complete markets subject to collateral constraints”?  
  – Would it look like pure complete markets?
  – Would it look like the riskless debt allocation?

• Answer: it would be like pure complete markets
  – Optimists would buy all the trees and all the \( t = 0 \) apples
  – They pay for these by selling \( L \)-contingent apples
  – They set aside both the trees and the apples as collateral for the \( L \)-contingent claims (assume apples can be used as collateral)
  – Pessimists spend all their wealth on \( L \)-contingent claims
  – (or trees and apples can be held by anyone as long as they are stripped into different state-contingent claims)
  – If apples cannot be pledged as collateral, then that would make a difference (see problem set)

• Conclusion: the limited set of contracts we allow makes a difference!

• Intrepretation:
  – Shortselling?
  – Writing CDS?
2 A three-period example

- People still disagree about $\pi$
- At an intermediate date, we learn whether we are in state $H$ or $L$
- People don’t update their opinion about $\pi$ given where they are in the tree: they have no uncertainty about the true value of $\pi$.
- Payoffs can be interpreted as debt-like
  - A lot of things must go wrong for you not to be paid
- Bad news increases uncertainty and disagreement over eventual payoffs
  - This is very special to this example
  - But perhaps realistic
- Assume still that the only contracts are riskless, one-period collateralized debt
- Let $p_0$ be the price of the tree at $t = 0$ and $p_1$ be the price at $t = 1$ in state $L$ (in state $H$ the price is 1)

\footnote{For the zero-measure of people who think $\pi = 1$, observing state $L$ can be a real problem. Bayes’ rule tells them to divide by zero and their head explodes.}
Type $\pi$ solves

$$\max_{x_0, b_0, x_1, b_1} \pi c_H + (1 - \pi) \pi c_{LH} + (1 - \pi)^2 c_{LL}$$

subject to

- $c_H = a + p_0 + x_0 - p_0 x_0$
- $c_{LH} = a + p_0 + x_1 - p_0 x_0 - p_1 (x_1 - x_0)$
- $c_{LL} = a + p_0 + dx_1 - p_0 x_0 - p_1 (x_1 - x_0)$
- $p_0 x_0 \leq a + p_0 + b_0$
- $b_0 \leq p_1 x_0$
- $p_1 x_1 \leq a + p_0 + (p_1 - p_0) x_0 + b_1$
- $b_1 \leq dx_1$

Equilibrium has the following structure

- Super-optimists buy the tree at $t = 0$ with maximum leverage. The cutoff is denoted by $\pi^{**}$
- In state L, their collateral is just enough to pay their debt, so they go bankrupt
- The tree is bought by semi-optimists $\pi \in [\pi^* \pi^{**}]$ in state L
- They also get maximum leverage
- The maximum leverage at $t = 0$ is $\frac{p_0}{p_0 - p_1}$
- The maximum leverage at $t = 1$ is $\frac{p_1}{p_1 - d}$. This involves deleveraging whenever $\frac{p_1}{p_0} > \frac{d}{p_1}$, i.e. when a little bit of bad news is not too bad but further bad news would be a big problem. Whether this happens depends on parameters.
- As with K&M, we define leverage as $(\text{fraction downpayment})^{-1}$.

Equilibrium conditions

The payoffs of each possible strategy are as follows:

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$c_H$</th>
<th>$c_{LH}$</th>
<th>$c_{LL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 0</td>
<td>$\frac{a+p_0}{p_0-p_1}$</td>
<td>$a + p_0 + \frac{a+p_0}{p_0-p_1} (1 - p_0) = (a + p_0) \frac{1-p_1}{p_0-p_1}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Buy 1</td>
<td>$\frac{a+p_0}{p_1-d}$</td>
<td>$a + p_0 + \frac{a+p_0}{p_1-d} (1 - p_1) = (a + p_0) \frac{1-d}{p_1-d}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Don’t</td>
<td>0</td>
<td>$a + p_0$</td>
<td>$a + p_0$</td>
<td>$a + p_0$</td>
</tr>
</tbody>
</table>

1. Indifference for type $\pi^{**}$

$$\pi^{**} \frac{1-p_1}{p_0-p_1} = \pi^{**} + (1-\pi^{**}) \pi^{**} \frac{1-d}{p_1-d}$$
2. Indifference for type $\pi^*$

$$1 = \pi^* + (1 - \pi^*) \pi^* \frac{1 - d}{p_1 - d}$$

$$p_1 = \pi^* + (1 - \pi^*) d$$

3. Market clearing at $t = 0$

$$(1 - F(\pi^{**})) \frac{a + p_0}{p_0 - p_1} = 1$$

4. Market clearing at $t = 1$

$$[F(\pi^{**}) - F(\pi^*)] \frac{a + p_0}{p_1 - d} = 1$$

- Properties:
  - $p_1 < p_0$
  - $p_0 < [1 - (1 - \pi^{**})^2] + d (1 - \pi^{**})^2 = \mathbb{E}[y|\pi^{**}]$
    - Price is below the expected dividend according to the marginal buyer
    - There is an arbitrage opportunity, but it might widen!
    - This effect is present also in Shleifer and Vishny [1997]
  - The decrease in price includes three effects:
    - The expected dividend decreases (according to everyone)
    - The optimists go bankrupt, which means someone more pessimistic else is pricing the asset
    - Leverage decreases (if whenever $\frac{p_1}{p_0} > \frac{d}{p_1}$)
  - For $F$ uniform, $a = 1$, $d = 0.2$

    $$p_0 = 0.947$$
    $$\pi^{**} = 0.871$$
    $$\mathbb{E}[y|\pi^{**}] = 0.986$$
    $$p_1 = 0.693$$
    $$\pi^* = 0.616$$

- Would people want to use long-term debt?
No: you can only borrow \( d = 0.2 \) long-term but \( p_1 = 0.693 \) short term with one unit of collateral.

Theory of maturity mismatch!

References


