Patching arguments

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Prop $G = U(n)$, Ω^4 ori. Rmn. mfd. (not nec. cpt., simply cntd),
$[A_i] \in \mathcal{M}$ s.t. $\forall x \in \Omega$, $\exists$ nbhd $U_x \ni x$ s.t.

$$\|F_{A_i}\|_{L^2(U_x)} < \varepsilon \quad \forall i \text{ large} \qquad (\text{non concentrating})$$

Then $\exists$ conv. sbsq. in $\mathcal{M}$.

The pf in the book seems to have a gap. We give another pf.
(c.f. Uhlenbeck, Connections with L^p bounds on curvatures, Section 3, esp. Cor 3.3.)

Pf Take a f. cover $\{U_\alpha\}$ of any given cpt sbset $K \subset \Omega$ by small geo. balls s.t. $\|F_{A_i}\|_{L^2(U_\alpha)}$ small $\forall i$ ← wlog. w/ throw away some.

ETS $\exists$ conv. sbsq. of $[A_i|_K]$.
(If $K \hookrightarrow \Omega$ a deformation retract, then any g.t. on K ext. to Ω. Then use diagonal argument over exhausting K.)

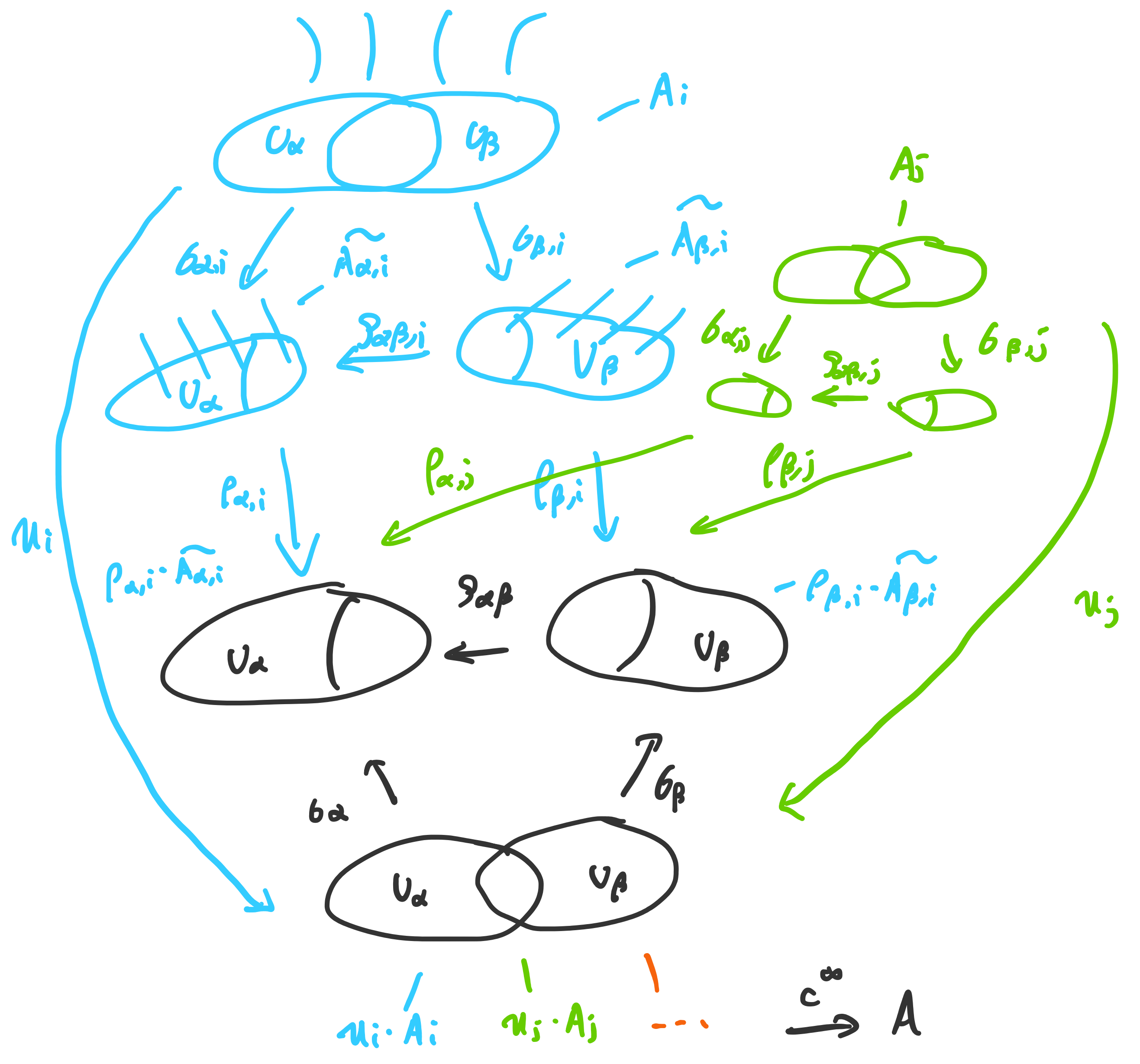
Spse trivialization $\sigma_{\alpha,i} : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^n$ puts $A_i|_{U_\alpha}$ in Coulomb gauge w/ $\|\hat{A}_{\alpha,i}\|_{L^{1,2}(U_\alpha)}$ small, $\hat{A}_{\alpha,i} = \sigma_{\alpha,i} \cdot A_{\alpha,i}$, a mtrx matrix.
$\xrightarrow[\text{Thm 2.3.8}]{\text{ell. reg.}}$ $\|\hat{A}_{\alpha,i}\|$ small for any Sobolev norm.

Let $g_{\alpha\beta,i} : U_{\alpha\beta} \to G$ denote the transition fns. for $\{\sigma_{\alpha,i}\}$.

Claim $g_{\alpha\beta,i}$ is bdd. in any Sobolev norm for i large (dep. on norm)

Pf $\hat{A}_{\alpha,i} = g_{\alpha\beta,i} \cdot \hat{A}_{\beta,i} \Rightarrow dg_{\alpha\beta,i} = g_{\alpha\beta,i}\hat{A}_{\beta,i} - \hat{A}_{\alpha,i} g_{\alpha\beta,i}$
$\xrightarrow{\text{bootstrap}}$ $\|g_{\alpha\beta,i}\|_{L^{\ell+1,2}} \lesssim \|g_{\alpha\beta,i}\|_{L^2}$ $\forall \ell > 0$, if $\hat{A}_{\alpha,i}, \hat{A}_{\beta,i}$ small.
Smallness is unif. in i for fixed Sobolev $\Rightarrow$ ✓. □

Thus $\exists$ conv. sbsq. $g_{\alpha\beta,i} \to g_{\alpha\beta}$ in C^∞.
Clearly $(g_{\alpha\beta})$ satisfies cocycle condition, and defs $E \to U = \bigcup_\alpha U_\alpha \supset K$.



Claim $\exists\ \rho_{\alpha,i} : U'_\alpha \to G$ w/ $g_{\alpha\beta,i} = \rho_{\beta,i}\, g_{\alpha\beta}\, \rho_{\alpha,i}^{-1}$,
w/ $\rho_{\alpha,i} \to 1$ in any norm. Here U'_α is any shrinked U_α.

Pf (c.f. Uhlenbeck Cor 3.3)
Let one $\rho_{\alpha,i} = 1$ & inductively defs the rest by interpolating.
Note C^0-small $\Rightarrow$ all lies in some exp. nbhd of $1 \in G$
$\Rightarrow$ no extension problem. □

Now fix triv. $\sigma_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^n$ w/ trans. fns. $= (g_{\alpha\beta})$
(e.g. $\sigma_\alpha = \rho_{\alpha,1} \circ \sigma_{\alpha,1}$) and def. $u_{\alpha,i} = \sigma_\alpha^{-1} \circ \rho_{\alpha,i} \circ \sigma_{\alpha,i}$.
Then $(u_{\alpha,i})$ patches to global g.t. u_i on $E|_U$.

Claim $\exists$ sbsq $u_i \cdot A_i$ conv. in $C^\infty(K)$.

Pf In triv. σ_α, $u_i \cdot A_i$ is $\rho_{\alpha,i} \cdot \hat{A}_{\alpha,i} = \rho_{\alpha,i}\hat{A}_{\alpha,i}\rho_{\alpha,i}^{-1} - (d\rho_{\alpha,i})\rho_{\alpha,i}^{-1}$
is small for any norm, for large i dep. on norm. $\Rightarrow$ conv. sbsq. □