Notes on Uhlenbeck Theorem

Dashen Yan

May 14, 2021

1 Main Theorem

We identify $\bar{B}^4$ with a hemisphere on S^4 via the standard stereographic map m. We fix the standard metic on (B^4, g) as the pull back metric of m. This is conformal to the flat metric. Therefore on B^4 we have point-wise equality:

$$|F|_g^2 dVol_g = |F|_{g_{\mathbb{R}^4}} dVol_{g_{\mathbb{R}^4}}$$

The use of this metric enables us to do computation on S^4 and bypasses the difficulties when dealing with the boundary condition.

Theorem 1.1 (Uhlenbeck). *There are constant $\epsilon_1, M > 0$ such that any connection A (Here we work with $U(n)$ connection) over the trivial bundle over $\bar{B}^4$ with $\|F_A\|_{L^2} \leq \epsilon_1$ is gauge equivalent to a connection $\tilde{A}$ over B^4 (the behavior near boundary may be bad) with*

1. $d^*\tilde{A} = 0$
2. $\|\tilde{A}\|_{H^1} \leq M\|F_{\tilde{A}}\|_{L^2} = M\|F_A\|_{L^2}$

The main ingredients of the proof is the following proposition:

Proposition 1.2 (Method of continuity). *There is a constant $\zeta > 0$ such that if $B'_t, (t \in [0,1])$ is a one-parameter family of connections on trivial bundle over S^4 with $\|F_{B'_t}\|_{L^2} < \zeta$ and B'_0 be the trivial connection d, then for each t there exists a gauge transformation u_t such that $u_t(B'_t) = B_t$ satisfies:*

1. $d^*B_t = 0$
2. $\|B_t\|_{H^1} < 2N\|F_{B_t}\|$. *$N$ can be chosen to equal to $2c_1$, the constant appear in the following Sobolev inequality which we shall prove later $\|B\|_{H^1} \leq c_1\|dB\|_{L^2}$.*

We break the proof of Uhlenbeck theorem into method of continuity and several regularity estimate. Let A as in the main theorem. Let $\delta_t : \mathbb{R}^4 \to \mathbb{R}^4, x \mapsto tx, t \in [0,1]$. Let $A_t = \delta_t^* A$ be a one parameter family of connections such that $A_0 = 0$. Let r be reflection of S^4 with respect to the plane $\{x_5 = 0\}$. We define $p : S^4 \to \bar{B}^4$ such that p is the stereographic projection of north(south) pole on south(north) hemisphere respectively. Roughly speaking if we let $\|F_{A_t}\|_{L^2}$ small enough, $B_t = p^*A_t$ and "apply" continuity method to B_t we can "prove" the theorem, but p is mere a Lipschitz map from S^4 to $\bar{B}^4$. To fix this problem we construct a family of smooth maps p_ϵ which converges to p in $W^{1,\infty}$ with ∇p_ϵ bounded and p_ϵ equal to p outside ϵ neighborhood of equatorial three-sphere. To do this we identify $S^4 \cap \{x_5 \leq 0\}$ with $\bar{B}^4$ in $\mathbb{R}^4$ and define:

$$f_\epsilon : \bar{B}^4 \to B^4, f_\epsilon(0) = 0, f_\epsilon(x) = \frac{x\phi_\epsilon(|x|)}{|x|}, |x| > 0$$

where ϕ_ϵ is a smooth non-decreasing function on $[0,1]$ such that

$$\phi_\epsilon : [0,1] \to [0,1], \phi_\epsilon(r) = r, r \le 1 - \frac{\epsilon}{2}; \phi_\epsilon \equiv 1 - \frac{\epsilon}{3}, r \ge 1 - \frac{\epsilon}{4}$$

We define p_ϵ on the open hemisphere by $p_\epsilon = p \circ f_\epsilon$ and extend to the whole sphere smoothly. Therefore we have:

$$\int_{S^4} |F(p_\epsilon^* A_t)|^2 dVol_g \le 2 \int_{B^4} |F(A)|^2 dVol_g + C(n)\epsilon \|F(A)\|_{L^\infty} \tag{1.1}$$

To apply continuity method, we let $\epsilon_1 < 2^{-\frac{1}{2}}\zeta$ and $B_t^{\epsilon}{}' = p_\epsilon^* A_t$. We then get a gauge equivalent connection $\tilde{B}^\epsilon$ of $p_\epsilon^*(A)$. By restricting back to B^4 we obtain A^ϵ satisfying $d^* A^\epsilon = 0, \|A^\epsilon\|_{H^1(B^4)} < 2N\|F(\tilde{A})\|_{L^2}$. Moreover A^ϵ is gauge equivalent to A on $B^4(1-\epsilon)$. Therefore it is suffice to proof the continuity method and study the behavior of $\tilde{B}^\epsilon$ when $\epsilon \to 0$.

2 Rearrangement argument

Lemma 2.1. *Let B be a connection on the trivial bundle over S^4 in Coulomb gauge relative to the trivial connection d ($d^*B = 0$). There are constant $N, \eta > 0$ such that if $\|B\|_{L^4} < \eta$ then $\|B\|_{H^1} \le N\|F_B\|_{L^2}$.*

Proof. Since $d^* + d$ is an elliptic operator, we then have the following elliptic estimate:

$$\|B\|_{H^l} \le \tilde{c}_1(\|dB\|_{H^{l-1}} + \|d^*B\|_{H^{l-1}} + \|B\|_{L^2}) \tag{2.1}$$

one the other hand since $H^1(S^4) = 0$, Hodge theory implies:

$$\Omega^1 = Im(\Delta)$$

Therefore given $d^*B = 0$ we will have $\|B\|_{L^2} \le C\|dB\|_{L^2}$ for some constant independent on B. Otherwise $\exists B_k$ such that $\|B_k\|_{L^2} = 1, \|dB\|_{L^2} \le \frac{1}{k}$. The elliptic estimate and the Rellich compact embedding theorem implies B_k are uniformly H^l bounded and hence admit an H^{l-1} convergent subsequence (W.L.O.G. let B_k be the subsequence itself). We denote the limit as B_∞. It is easy to see $\|B_\infty\|_{L^2} = 1$. For arbitrary smooth section η

$$(B_\infty, \eta) = (B_\infty, \Delta\xi) = \lim_{k\to\infty}(B_k, \Delta\xi) = \lim_{k\to\infty}(dB_k, d\xi) = 0$$

Which draws contradiction. We obtain in particular:

$$\|B\|_{H^1} \le c_1\|dB\|_{L^2} \tag{2.2}$$

Using Cauchy Schwartz and Sobolev embedding we will have:

$$\|B \wedge B\|_{L^2} \le c_2\|B\|_{L^4}\|B\|_{H^1} \tag{2.3}$$

So

$$\|B\|_{H^1} \le c_1\|F(B)\|_{H^1} + c_1c_2\|B\|_{L^4}\|B\|_{H^1} \tag{2.4}$$

If $\|B\|_{L^4} \le \frac{1}{2c_1c_2}$, $\|B\|_{H^1} \le 2c_1\|F(B)\|_{L^2}$. □

Now we are able to deduce a higher order estimate of $\|B\|_{H^{l+1}}, l \ge 1$ in terms of the L^∞ and H^l norm of $F(B)$. It should be remarked here that the gauge action on the curvature form does not change $|F(B)|$.

For a smooth connection connection B put:

$$Q_l(B) = \|F(B)\|_{L^\infty} + \sum_{i=1}^{l} \|\nabla_B^{(i)} F(B)\|_{L^2}$$

Lemma 2.2. *There are constant $\eta' > 0$ such that if the connection matrix B of Lemma 2.1 has $\|B\|_{L^2} < \eta'$ then for each $l \geq 1$ a bound,*

$$\|B\|_{H^{l+1}} \leq f_l(Q_l(B)) \tag{2.5}$$

where $f_l(0) = 0$ are nondecreasing, smooth and are independent on B.

Proof. When $l \geq 3$ multiplication by B induce a bounded map from $H^s \to H^s, s \leq l$ with norm less than $C\|B\|_{H^l}$, where C is a constant depends only on l and the based compact manifold. The outline of the proof is that we first apply P.O.U. and use Sobolev inequality, Morrey inequality, Hölder inequality to get a local estimate. Then gathering the local estimate we obtain the global estimate of the norm.

$$\|B \wedge B\|_{H^l} \leq const\|B\|^2_{H^l} \tag{2.6}$$

and

$$\|F\|_{H^l} \leq P_l(\|B\|_{H^l}) \sum_{i=0}^{l} \|\nabla^{(i)}_B F(B)\|_{L^2} \tag{2.7}$$

For some polynomial P_l with $P_l(0) = 0$. Therefore:

$$\|B\|_{H^{l+1}} \leq const\bigg(\|B \wedge B\|_{H^l} + P_l(\|B\|_{H^l}) \sum_{i=0}^{l} \|\nabla^{(i)}_B F(B)\|_{L^2}\bigg) \tag{2.8}$$

Which prove the case when $l \geq 3$

We now consider the case when $l = 1$. Since $\nabla F = \nabla_B F - [B_k, F_{ij}] \otimes dx^k \otimes dx^i \otimes dx^j$

$$\|B\|_{H^2} \leq \tilde{C}_2\bigg(\|F(B)\|_{H^1} + \|B \wedge B\|_{H^1}\bigg)$$

$$\leq \tilde{C}_2\bigg(\|\nabla_B F(B)\|_{L^2} + \|F(B)\|_{L^2} + \|F\|_{L^\infty}\|B\|_{L^4} + \|B\|_{H^2}\|B\|_{L^4}\bigg)$$

Thus if $\|B\|_{L^4} \leq \frac{1}{2\tilde{C}_2}$ There is an independent constant C_2 such that:

$$\|B\|_{H^2} \leq C_2 Q_1(B) \tag{2.9}$$

Similarly

$$\|B\|_{H^3} \leq \tilde{C}_3\bigg(\|B\|_{L^4}\|B\|_{H^3} + \|B\|^2_{W^{1,4}} + \|F(B)\|_{H^2}\bigg)$$

$$\leq \tilde{C}_3\bigg(\|B\|_{L^4}\|B\|_{H^3} + \|B\|^2_{H^2} + \|\nabla^2_B F(B)\|_{L^2} + \|B \otimes \nabla_B F\|_{L^2} + \|(\nabla_B B) \otimes F\|_{L^2}$$
$$+\|B \otimes B \otimes F(B)\|_{L^2} + \|\nabla_B F(B)\|_{L^2} + \|F(B)\|_{L^2} + \|F\|_{L^\infty}\|B\|_{L^4}\bigg)$$

Since the following inequality holds for some constant A_3, we will have:

$$\|F(B)\|_{H^2} \leq A\bigg(\|B \otimes F(B)\|_{L^2} + \|B \otimes B \otimes F(B)\|_{L^2} + \|(\nabla_B B)F(B)\|_{L^2} + \|B\|_{L^4}\|F(B)\|_{H^2}\bigg)$$
$$+\|\nabla^2_B F(B)\|_{L^2} + \|F(B)\|_{H^1}$$

Given $\|B\|_{L^4} \leq min(\frac{1}{2A}, \frac{1}{2\tilde{C}_3})$, we will have

$$\|B\|_{H^3} \leq C_3\bigg(\|B\|^2_{H^2} + \sum_{j=1}^{2} \|\nabla^j_B F(B)\|_{L^2} + \|F(B)\|_{L^\infty}(\|B\|_{H^1} + 1)\bigg) \tag{2.10}$$

Therefore we complete the whole proof. Here $\eta' \leq min(\eta, \frac{1}{2\tilde{C}_2}, \frac{1}{2\tilde{C}_3}, \frac{1}{2A})$ □

3 Proof of Method of Continuity

Let S be set of $t \in [0,1]$ such that u_t exists. S is nonempty.

3.1 S is closed

Proposition 3.1. *If A_i, B_i are C^∞-bounded sequences of connections on a unitary bundle over a compact manifold X, and if A_i, B_i are gauge equivalent connection for each i, then there are subsequences converging to limiting connection A_∞, B_∞, and A_∞ is Gauge equivalent to B_∞.*

Proof. W.L.O.G. We can assume the vector bundle is trivial. By AA lemma we may assume $A_i \to A_\infty, B_i \to B_\infty$ and the compactness of the structure group implies $u_i \to u_\infty$ uniformly as continuous map.

$$du_i = u_i A_i - B_i u_i \tag{3.1}$$

Suppose u_i is C^r convergent then we can deduce that u_i is C^{r+1} convergent. □

We can now get down to the proof of S is closed. We choose ζ so that $2CN\zeta$ is less than η', η, where C is a Sobolev constant. Then if t lies in S we have:

$$\|B_t\|_{L^4} \leq C\|B_t\|_{H^1} \leq 2NC\|F(B_t)\|_{L^2} \leq 2NC\zeta \leq min(\eta, \eta') \tag{3.2}$$

We conclude from lemma 2.1 that $\|B\|_{H^1}$ is uniformly bounded We now prove that $Q_l(B)$ is invariant under Gauge transformation.

Lemma 3.2.

$$|\nabla_B^{(j)} F(B)| = |\nabla_{u(B)}^{(j)} F(u(B))| \tag{3.3}$$

Proof. When $j = 0$, $F(u(B)) = uF(B)u^{-1}$ which is obvious. We assume the conclusion is true for $j \leq l$ we proof that the case $j = l+1$ is also correct.

$$\begin{aligned} \nabla_{u(B)}(\nabla_{u(B)}^{(l)} F(u(B))) &= \nabla_{u(B)} u(\nabla_B^{(l)} F(B))u^{-1} \\ &= \nabla_{u(B)}(uF_{ij;k_1\cdots k_l} u^{-1} dx^i dx^j dx^{k_1} \cdots dx^{k_l}) \\ = (\nabla_{u(B)} uF_{ij;k_1\cdots k_l} u^{-1}) dx^i dx^j dx^{k_1} \cdots dx^{k_l} &+ uF_{ij;k_1\cdots k_l} u^{-1} \nabla^{TM} x^i dx^j dx^{k_1} \cdots dx^{k_l} \\ &= u(\nabla_B^{(l+1)} F(B))u^{-1} \end{aligned}$$

Suppose $\|\nabla_{B_t'}^{(j)} F(B_t')\|_{L^2} \leq K_j$ is uniformly bounded we have $\|B_t\|_{H^l}$ for each $l \geq 0$ is uniformly bounded. By AA lemma and that lemma 2.1 is preserved under limit, we know S is closed. □

Remark 3.3. We kwon from lemma 2.1 that $\|B\|_{H^1} \leq 2N\|F(B)\|_{L^2}$ implies $\|B\|_{H^1} \leq N\|F(B)\|_{L^2}$

3.2 S is open

Proof. Let $t_0 \in S$, W.L.O.G. we can assume $B_{t_0} = B'_{t_0}$ which we will just write B. Let $Ad(\mathfrak{g}) = P \times_G \mathfrak{g}$. We define: F_l be the space of H^l section of $\Omega^1(Ad(\mathfrak{g}))$ respectively, and E_l to the space of H^l section of $Ad(\mathfrak{g})$ with zero integral. The map H:

$$H : E_l \times F_{l-1} \to E_{l-2} : H(\chi, b) = d^*(e^\chi (B+b)e^{-\chi} - (de^\chi)e^{-\chi}) \tag{3.4}$$

We have $H(0,0) = 0$. Let $(DH)_0$ be the linearization of H at $(0,0)$

$$(DH)_0(\chi, b) = -d^* d_B \chi + d^* b \tag{3.5}$$

To prove the openness, it is suffice to show that: $d^*d_B\chi$ is surjective. Since d^*d_B is elliptic and hence Fredholm, by Fredholm alternative, assuming d^*d_B were not surjective, there would be a nonzero smooth section η such that:

$$(d^*d_B\chi, \eta) = 0, \forall\chi \tag{3.6}$$

set $\chi = \eta$, because $\int \eta = 0$ we have for some Sobolev constant:

$$\|d\eta\|^2_{L^2} \leq |([B, \eta], d\eta)| \leq const\|d\eta\|^2_{L^2}\|B\|_{H^1} \tag{3.7}$$

which gives a universal lower bound of the H^1 norm of B. So we deduce that if ζ is small then the set S is open. □

4 Proof of the Main Theorem

We claim that for any $D \Subset S^4 - S^3$ and $l \geq 1$ there is a constant depends only on D, l such that.

$$\|\tilde{B}^\epsilon\|_{H^l(D)} \leq N_{l,D} \tag{4.1}$$

The case $l = 1$ follows by the second conclusion of the method of continuity. When $l > 1$ we can apply the elliptic estimate for compact manifold with boundary:

$$\|\tilde{B}^\epsilon\|_{H^{l+1}(D)} \leq C(\|\tilde{B}^\epsilon\|_{H^l(D)} + \|d\tilde{B}^\epsilon\|_{H^l(D)}) \tag{4.2}$$

Using arguments appeared in lemma 2.2, we can complete the proof of the first and the third conclusion in the theorem 1.1.

5 Application

Intuitively we want to solve the ASD equation:

$$F(A)^+ = d^+A + (A \wedge A)^+ \tag{5.1}$$

The problem is that the operator d^+ is not elliptic, since $F^+(A) = 0$ is invariant under gauge transformation. Uhlenbeck theorem enable us to choose a suitable gauge under which the ASD solution satisfies a regularity estimate:

Theorem 5.1. *There is a constant $\epsilon_2 > 0$ such that if $\tilde{A}$ is any ASD connection on the trivial bundle over B^4 which satisfies the Coulomb gauge condition $d^*\tilde{A} = 0$ and $\|\tilde{A}\|_{L^4} \leq \epsilon_2$, then for any interior domain $D \Subset B^4$ and $l \geq 1$ we have*

$$\|\tilde{A}\|_{H^l(D)} \leq M_{l,D}\|F(A)\|_{L^2(B^4)} \tag{5.2}$$

for a constant $M_{l,D}$ depending only on l, D

Combined with Uhlenbeck theorem, one can prove the following result:

Corollary 5.2. *There exists a constant ϵ for any sequence of ASD connection A_a over $\bar{B}^4$ with $\|F(A_a)\| \leq \epsilon$ there exists a subsequence a' and gauge equivalent connections $\tilde{A}_{a'}$ which converge in C^∞ in the open ball.*

References

[1] The Geometry of Four-Manifold, S.K Donaldson, P.B. Kronheimer