RESEARCH STATEMENT

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1. Introduction

My research interests lie within differential geometry, partial differential equations and geometric measure theory, with a focus on variational methods in geometric problems. A major theme of my research is to understand geometric and topological structures of manifolds via investigating stationary points of related geometric functionals.

An important geometric functional is the area (or volume) functional for hypersurfaces Σ^{n-1} in a Riemannian manifold (M^n, g). Suppose Σ carries a unit normal vector field N (that is, Σ is two-sided). The first variation of area along a normal deformation fN, $f \in C_c^\infty(M)$, is

$$\delta V_\Sigma(f) = \int_\Sigma fH,$$

where H is the **mean curvature** of $\Sigma \subset (M, g)$. Critical points of V are called **minimal hypersurfaces**, and they are characterized by $H \equiv 0$. The second variation of volume is given by

$$\delta^2 V_\Sigma(f) = \int_\Sigma |\nabla^\Sigma f|^2 - |A_\Sigma|^2 f^2 - \mathrm{Ric}_M(N,N) f^2 = \int_\Sigma fJ(f),$$

where ∇^Σ is the gradient operator on Σ, A_Σ is the second fundamental form, Ric_M is the Ricci curvature of M, and $J = -\Delta^\Sigma - |A_\Sigma|^2 - \mathrm{Ric}_M(N,N)$ is the Jacobi operator. Assume Σ is compact. By standard elliptic theory, the number of negative Dirichlet eigenvalues of J is finite, and is called the Morse index of $\Sigma \subset (M, g)$. A minimal hypersurface is called **stable**, if its Morse index is zero.

Many important problems in differential geometry require a deep understanding of $\mathcal{M}(M, g, I)$, the space of two-sided minimal hypersurfaces in (M, g) with Morse index bounded by I. Nontrivial elements of $\mathcal{M}(M, g, I)$ have been successfully constructed using minimization, gluing or min-max methods, see, for instance, [CDL02][MN17]. It is then a central question to understand the global structure of $\mathcal{M}$. For instance, is $\mathcal{M}(M, g, 0)$ non-empty? Is $\mathcal{M}(M, g, I)$ compact with respect to C^∞ topology?

As a consequence of my work on the geometric and topological structures of minimal hypersurfaces, I am able to answer the questions above when M is the Euclidean space $\mathbf{R}^n$. In particular, under the assumption that $\Sigma^{n-1} \subset \mathbf{R}^n$ is a minimal hypersurface with finite total curvature, the volume growth rate and the first Betti number of Σ can be bounded by the Morse index and nullity of its Jacobi operator. Combined with some earlier work in [CKM17], I obtained a strong compactness property for minimal hypersurfaces with bounded index in $\mathbf{R}^4$.

Another aspect of my research is centered on the interaction between scalar curvature and minimal hypersurfaces, especially on manifolds with metric singularities. Motivated by a question of Gromov [Gro14], my coauthor and I have been able to study the effect of codimension two edge singularities and codimension three uniformly Euclidean singularities on the Yamabe type. Recently, by a novel application of area minimizing capillary surfaces, I answered affirmatively the "dihedral rigidity conjecture" of Gromov [Gro14] and obtained a polyhedron comparison theorem in 3-manifolds with positive scalar curvature.

2. Research up to now

2.1. **Index estimate and compactness theorems for minimal hypersurfaces in $\mathbf{R}^n$.** The basic question regarding the structure of $\mathcal{M}(M,g,I)$ is to measure the complexity of minimal hypersurfaces with bounded index. When $M=\mathbf{R}^3$ and equipped with the flat metric, this question has been studied by many authors. Among them, Fischer-Colbrie-Schoen [FCS80], do Carmo-Peng [dCP79] and Pogorelov [Pog81] proved that the only two-sided stable minimal surfaces in $\mathbf{R}^3$ are affine planes; López and Ros [LR89] proved that the Catenoid and the Enneper surfaces are the only immersed two-sided index 1 surfaces; Chodosh and Maximo [CM16] proved an effective upper bound of the first Betti number and the number of ends by the Morse index. All of such theorems rely heavily on the conformal structure of minimal surfaces in $\mathbf{R}^3$ and are technically difficult to generalize to higher dimensions. To our best knowledge, the only index-topology type result for high dimensional minimal hypersurfaces with general index was from Li and Wang [LW02]. They proved that finite index implies finitely many ends, without any effective control.

In the work [Li], I obtained an effective estimate of some geometric and topological invariants by the Morse index of minimal hypersurfaces in $\mathbf{R}^n$ with finite total curvature:

Theorem 2.1. *Let Σ^{n-1} be a complete connected two-sided minimal hypersurface in $\mathbf{R}^n$, $n\geq 4$. Suppose that Σ has finite total curvature, that is, $\int_\Sigma |A|^{n-1}$ is finite. Then we have*

$$\operatorname{index}(\Sigma)+\operatorname{nullity}(\Sigma)\geq \frac{2}{n(n-1)}(\#ends+b_1(\bar{\Sigma})-1),$$

where $\operatorname{nullity}(\Sigma)$ *is the dimension of the space of L^2 solutions of the Jacobi operator, and $b_1(\bar{\Sigma})$ is the first Betti number of the compactification of Σ.*

Through a careful rigidity analysis, we were able to get rid of the nullity term under the extra assumption that $n=4$, or that there exists a point on Σ where all the principal curvatures are distinct.

Theorem 2.2. *Let $\Sigma^{n-1}\subset\mathbf{R}^n$ be a complete connected two-sided minimal hypersurface with finite total curvature. Suppose that $n=4$, or that there exists a point on Σ where all the principal curvatures are distinct. Then*

$$\operatorname{index}(\Sigma)\geq \frac{2}{n(n-1)}(\#ends+b_1(\bar{\Sigma}))-\frac{4}{n}.$$

This is the first effective lower bound of Morse index in terms of geometric and topological invariants. It can be used to obtain several structural theorems of the space $\mathcal{M}(\mathbf{R}^n,g_{\mathrm{Euclid}},I)$. Combined with a recent result of Chodosh-Maximo [CKM17], I proved that $\mathcal{M}(\mathbf{R}^4,g_{\mathrm{Euclid}},I)$ has at most $N=N(I)$ diffeomorphism types.

Corollary 2.3. *There exists $N=N(I)$ such that there are at most N mutually non-diffeomorphic complete embedded minimal hypersurfaces Σ^3 in $\mathbf{R}^4$ with Euclidean volume growth and* $\operatorname{index}(\Sigma)\leq I$.

In another application, I obtained the following strong finiteness result when $I=1$, which is believed to characterize index 1 minimal hypersurfaces uniquely. See Example 1.6 in [Li].

Corollary 2.4. *There exists a constant R such that the following holds: for any complete connected embedded two-sided minimal hypersurface $\Sigma\subset\mathbf{R}^4$ with finite total curvature and index 1, normalized so that $|A_\Sigma|(0)=\max|A_\Sigma|=1$, Σ is a union of minimal graphs in $\mathbf{R}^4\setminus B_R(0)$.*

2.2. **Positive scalar curvature in singular spaces.** Given a closed boundaryless manifold M^n of dimension $n\geq 3$ and a conformal class $\mathcal{C}$ of Riemannian metrics, one defines the conformal Yamabe constant as

$$\mathcal{Y}(M,\mathcal{C})=\inf\left\{\int_M R(g): g\in\mathcal{C},\operatorname{Vol}_g(M)=1\right\},$$

where $R(g)$ defines the scalar curvature of g. The σ-invariant (or Schoen-invariant) is then defined as

$$\sigma(M) = \sup\{\mathcal{Y}(M, \mathcal{C}) : \mathcal{C} \text{ is a conformal class of metrics on } M\}.$$

The sign of $\sigma(M)$, called the **Yamabe type** of M, is an important invariant of M. Kazdan-Warner showed that every closed n-manifold with $n \geq 3$ carries a metric with negative scalar curvature. On the other hand, a manifold M carries a metric of positive scalar curvature if and only if $\sigma(M) > 0$. A rigidity-type theorem of Schoen [Sch89] characterizes the boarderline case: if g is a C^2 metric on M with $\sigma(M) \leq 0$ and $R(g) \geq 0$, then $\mathrm{Ric}(g) \equiv 0$ and $\sigma(M) = 0$.

In a recent paper [Gro14], Gromov proposed a notion of positive scalar curvature for metric measure spaces. This naturally leads us to the question of whether the validity of the above theorem of Schoen holds in singular settings:

Question 2.5. Suppose $\mathcal{S} \subset M$ is compact, and g is a bounded measurable metric on M that is C^2 away from $\mathcal{S}$. If $\sigma(M) \leq 0$ and $R(g) \geq 0$ on $M \setminus \mathcal{S}$, then is it true that $\sigma(M) = 0$, g extends smoothly across $\mathcal{S}$ and $\mathrm{Ric}(g) \equiv 0$?

The answer of Question 2.5 depends significantly on the structure of the singular set $\mathcal{S}$, and the behavior of g near $\mathcal{S}$. If $\mathcal{S}$ is a smooth hypersurface and g approaches smoothly from both sides of $\mathcal{S}$, Bartnik [Bar97] and Miao [Mia02] answered Question 2.5 affirmatively under a particular "positive jump of mean curvature" condition.[1] C. Mantoulidis and I began to research this question for more general singular sets $\mathcal{S}$ [LM17]: we gave an affirmative answer to Question 2.5 for some singular sets $\mathcal{S}$ of higher codimensions, justifying some crucial observations by Gromov [Gro14] and Schoen [Sch17].

Definition 2.6 (Edge singularities, [LM17])**.** *Let $U \subset M^n$ be open, g be a bounded measurable metric on U, and $N^{n-2} \subset U$ be a codimension-2 submanifold (without boundary). We say g is an edge metric on U with singularities along N if g is C^2 on $U \setminus N$ and near every $p \in N$ there are local coordinates $(x^1, x^2, x^3, \cdots, x^n)$ on U such that N is given by $x^1 = x^2 = 0$ and g can be locally expressed as*

$$g = \exp\left(\alpha(x) + 2\beta(x^3, \cdots, x^n)\log|z|\right)|dz|^2 + \sum_{\substack{i=1,2\\ j\geq 3}} F_{ij}(x)dx^i dx^j + \sum_{k,l\geq 3} G_{kl}(x)dx^k dx^l, \quad (2.1)$$

where $z = x^1 + \sqrt{-1}x^2$, and α, F_{ij}, G_{kl}, $\beta > -1$, are all C^2 functions. The (possibly nonconstant) function $2\pi(\beta+1)$ is called the cone angle along N.

Examples of edge metrics include orbifold metrics obtained as quotient metrics under discrete isometry groups with $n-2$ dimensional fixed submanifolds, and Kähler manifolds admitting cusps along $n-2$ dimensional divisors. We mention here that edge metrics have attracted profoud research recently, both in real (e.g. [AL13]) and in complex settings (e.g. [CDS15a][CDS15b][CDS15c][Tia15][JMR16]). We investigated the effect of edge singularities on $\sigma(M)$, and obtained:

Theorem 2.7. *Let M be a closed n-dimensional manifold with $\sigma(M) \leq 0$, and let g be a metric on M. If g is an edge metric on M with singularities with angles $\in (0, 2\pi]$ along a closed $(n-2)$-dimensional submanifold $\mathcal{S} \subset M$, and $R(g) \geq 0$ on $M \setminus \mathcal{S}$, then g is a smooth Ricci-flat metric everywhere on M.*

On the other hand, the statement is false for edge metrics with cone angles larger than 2π, see section 8 of [LM17]. Motivated by the surgery results of scalar curvature by Scheon-Yau [SY79] and Gromov-Lawson [GL80], we speculated that singularities along codimension 3 submanifolds should be "removable" for positive scalar curvature. Precisely, we proved:

[1] The results in [Bar97] and [Mia02] were stated as positive mass theorems with hypersurface singularities, but the techniques are applicable to answer Question 2.5

Theorem 2.8. *Let M be a closed three-dimensional manifold with $\sigma(M) \leq 0$. If g is C^2 with $R(g) \geq 0$ away from a finite subset $\{p_1, \ldots, p_k\}$, and g is bounded with bounded inverse at $\{p_1, \cdots, p_k\}$, then g is a smooth flat metric everywhere on M.*

The assumption that g is bounded across $p_1, \cdots, p_k$ is imperative. In fact, allowing the metric g to blow up will completely invalidate the statement, as illustrated in section 8 of [LM17].

The basic idea to prove Theorem 2.7 and Theorem 2.8 is perturbative in nature: assuming the existence of a singular metric g with $R(g) > 0$, we were able to construct a family of smooth metrics $\{g_\delta\}$ approaching g as $\delta \to 0$, such that $R(g_\delta) > 0$. The construction of g_δ, however, requires a novel application of area minimizing surfaces in asymptotically Euclidean spaces. Our approach was also applicable for more robust singular sets in dimension 3, namely skeleton singularities - a network of piecewise smooth curves intersecting transversely.

Theorem 2.9. *Let M be a closed three-dimensional manifold with $\sigma(M) \leq 0$. If $\mathcal{S} \subset M$ is a compact, nondegenerate 1-skeleton, and g is an edge metric on $M \setminus \operatorname{sing} \mathcal{S}$ with singularities with angles $\in [\delta, 2\pi]$ along $\operatorname{reg} \mathcal{S}$, where $\delta > 0$ and $R(g) \geq 0$ on $M \setminus \mathcal{S}$, then g is a smooth flat metric everywhere on M.*

2.3. **A polyhedron comparison theorem for 3-manifolds with positive scalar curvature.** A fundamental question in differential geometry is to understand metric properties under global curvature conditions, and study weak notions of curvature lower bounds by ways of comparison geometry. This has been successfully carried out for *sectional* curvature lower bounds (Alexandrov spaces; see, e.g. [ABN86][BBI01]) and *Ricci* curvature lower bounds (Cheeger-Colding-Naber theory [CC97][CC00a][CC00b][CN12], or an optimal transport approach [LV09][Stu06a][Stu06b][Stu06c]). The case of *scalar* curvature lower bounds is not as well established, possibly due to a lack of relevant geometric comparison theory. Gromov [Gro14] proposed a geometric comparison result for cubes in Riemannian 3-manifolds with nonnegative scalar curvature.

Definition 2.10. *Let P be a polyhedron in $\mathbf{R}^3$. A closed Riemannian manifold M^3 with nonempty boundary is called a P-type polyhedron, if it admits a Lipschitz diffeomorphism $\phi : M \to P$, such that ϕ^{-1} is smooth when restricted to the interior and the faces and the edges of P. We consequently define the faces, edges and vertices of M as the image of ϕ^{-1} when restricted to the corresponding objects in P.*

The first case that Gromov investigated was cube-type polyhedrons in three manifolds with nonnegative scalar curvature ($P = [0, 1]^3 \subset \mathbf{R}^3$). He proposed that if (M^3, g) is a cube-type polyhedron, then it cannot simultaneously satisfy: interior $R(g) > 0$; the faces of M are strictly mean convex; the dihedral angles along M's edges are less than $\pi/2$. The tentative proof he suggested relid on the fact that cubes are the fundamental domains of a $\mathbf{Z}^3$ action on $\mathbf{R}^3$, hence not applicable to general polyhedra. In section 2.2 of [Gro14], Gromov gave a "Dihedral rigidity conjecture": a similar comparison result holds for general polyhedron types, together with the rigidity statement - if a P-type polyhedron (M^3, g) has $R(g) \geq 0$, mean convex faces and dihedral angles less or equal than those of P, then (M^3, g) is isometric to an Euclidean polyhedron.

Recently I [Li17] was able to answer this conjecture affirmatively for a large collection of polyhedron types. Let us define two polyhedron types in $\mathbf{R}^3$:

Definition 2.11. (1) *Let $k \geq 3$ be an integer. In $\mathbf{R}^3$, let $B \subset \{x^3 = 0\}$ be a convex k-polygon, and $p \in \{x_3 = 1\}$ be a point. Call the set*

$$\{tp + (1-t)x : t \in [0, 1], x \in B\}$$

a (B, p)-cone. Call B the base face and all the other faces side faces.

(2) *Let $k \geq 3$ be an integer. In $\mathbf{R}^3$, let $B_1 \subset \{x^3 = 0\}, B_2 \subset \{x_3 = 1\}$ be two <u>similar</u> convex k-polygons with parallel corresponding edges (namely, they are congruent up to scaling, but not rotations). Call the set*

$$\{tp + (1-t)q : t \in [0, 1], p \in B_1, q \in B_2\}$$

a (B_1, B_2)-prism. Call B_1, B_2 the base faces and all the other faces side faces.

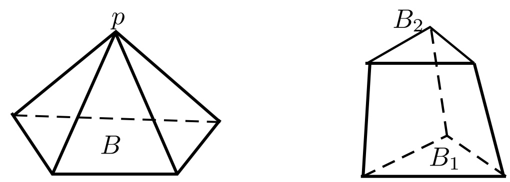

FIGURE 1. A (B, p)-cone and a (B_1, B_2)-prism.

Theorem 2.12. *Let P be a (B, p)-cone or a (B_1, B_2)-prism with side faces $F_1, \cdots, F_k$. Denote γ_j the dihedral angle between the base and F_j of p. Let (M^3, g) be a P-type polyhedron with side faces $F_1', \cdots, F_k'$. Assume that*

$$|\gamma_j + \gamma_{j+1} - \pi| < \measuredangle(F_j', F_{j+1}') < \pi - |\gamma_j - \gamma_{j+1}|. \tag{2.2}$$

Assume also that $R(g) \geq 0$ in the interior of M, and that the faces of M are mean convex. Then the dihedral angles of M along its edges cannot be everywhere less than those of P.

We remark that the angle assumption (2.2) can be viewed as a mild regularity assumption of the polyhedron M: it is satisfied for small C^1 perturbations of the flat polyhedron P, and is vacuous if all the base angles $\gamma_1, \cdots, \gamma_k$ are $\pi/2$. I was also able to obtain the rigidity statement:

Theorem 2.13. *Let P be a (B, p)-cone or a (B_1, B_2)-prism such that the dihedral angles between the base and the side faces lie within $(0, \pi/2]$. Let (M^3, g) be a P-type polyhedron satisfying (2.2). Assume that $R(g) \geq 0$ in the interior of M, that the faces of M are mean convex, and that the dihedral angles of M along its edges are everywhere no larger than those of P. Then (M^3, g) is isometric to an Euclidean polyhedron.*

The proof of Theorem 2.12 relies on the study of a variational problem that captures the relationship between the interior scalar curvature, the face mean curvatures and the edge dihedral angles. Let us sketch the proof when P is a (B, p)-cone and (M^3, g) is a P-type polyhedron. Let $q \in M$ be the vertex of the cone. Consider the variational problem

$$\mathcal{I} = \inf \Big\{ \mathcal{H}^2(\partial U \cap \mathring{M}) - \sum_{j=1}^{k} (\cos \gamma_j) \mathcal{H}^2(\partial U \cap F_j') : \\ U \text{ is a contratible open subset of } M \text{ containing } q, \text{ and } U \cap B = \emptyset. \Big\}. \tag{2.3}$$

Then:

- Assume, for the sake of contradiction, that M has dihedral angle everywhere less than P, we conclude $\mathcal{I} < 0$.
- An interior maximum principle [SW89] and a new boundary maximum principle guarantees that $\mathcal{I}$ is achieved by an open set U. Denote $\Sigma = \partial U \cap \mathring{M}$. Σ is a **capillary** surface: it is minimal and meets F_j' at constant angle γ_j.
- Σ is a $C^{1,\alpha}$ surface up to its corners for some $\alpha > 0$ ([Lie88]). This is the only place where the angle assumption (2.2) is used.
- A contradiction can be deduced from a second variation inequality and the Gauss-Bonnet formula.

The rigidity theorem 2.13 requires a further analysis of the minimizer for the variational problem (2.3). We combine the ideas pioneered by Bray-Brendle-Neves [BBN10] and by Ye [Ye91] in the case of area-minimizing capillary surfaces.

3. Future research

3.1. Morse index of minimal hypersurfaces.

(1) **General index estimate.** The quest to characterize minimal surfaces with bounded index has attracted many interesting works. A general conjecture in this direction is

Conjecture 3.1 (Schoen, Marques-Neves[Mar14][Nev14])**.** Let (M^n, g) be a closed boundaryless Riemannian manifold with $\mathrm{Ric}(g) > 0$. Then there exists a constant $C = C(M, g)$ such that for any two-sided minimal hypersurface Σ^{n-1},

$$\mathrm{index}(\Sigma) \geq Cb_1(\Sigma).$$

In addition to Theorem 2.1 and 2.2, there have been several results (see, e.g. [Sav10] [ACS16][ACS17][Sar17]) providing partial answers to Conjecture 3.1. All the existing results depend essentially on:

(a) the conformal structure of a minimal surface when $n = 3$, or
(b) some nontrivial isometry groups of the ambient space M^n.

My long-term course of action is to combine these ideas in the works of index estimates in higher dimensions. In particular, a directly related question is to characterize embedded minimal hypersurfaces of $\mathbf{R}^n$ with index 1. Conjecturally the only example is the higher dimensional Catenoid. See [TZ09] for a calculation of the Catenoid's index, and [Sch83] for a characterization of the Catenoid.

(2) **A dynamical characterization of index.** Minimal surfaces are stationary points for the mean curvature flow. Motivated by the "unstable submanifold theory" in ODE, one may relate the Morse index of a minimal surface and the dimension of an "unstable manifold" of the mean curvature flow. For strictly stable submanifolds, this has been carried out by M.T. Wang and his coauthors [TW17]. In general, one may observe the toy case:

Fact 3.2. *Let γ be a closed simple curve on S^1. Then the curve shortening flow emanating from γ converges to an equatorial S^1 if and only if γ bisects S^2 into two components with equal area.*

I intend to investigate to what extent this fact may be generalized to minimal surfaces with general index in higher dimensions.

3.2. Scalar curvature.

(1) **Codimension 3 singularities and σ-invariant.** Altough the removable singularity theorem (Theorem 2.8) has only been proven in dimension 3, the surgery result of Schoen-Yau [SY79] and Gromov-Lawson [GL80] suggest a similar statement in higher dimensions. The currents methods of proof have three steps:

(a) "blowing-up" the singularity by the Green's function of the conformal Laplacian;
(b) excising the asymptotically Euclidean end by cutting along a particular minimal surface;
(c) "filling in" the holes generated in the previous two steps with regions of positive scalar curvature.

Among them, step (a) and (b) may be carried out in higher dimensions. Step (c), however, uses several theories specific to minimal surfaces in dimension three: the topology of area-minimizing surfaces [MY80], the classification of stable minimal surfaces [FCS80], and the existence of a proper filling-in [Man17]. Generally, none of these theories hold in higher dimensions. However, given the special asymptotic structure of the blow-up metric, I intend to work toward a more precise understanding of the minimization procedure, especially in four dimensions, where it is more likely to have a strong "cut-and-fill" operation along 4-manifolds.

(2) **Comparison theorem for positive scalar curvature.** A natural quest is to generalize the comparison theorem in higher dimensions. The geometry argument follow from a Schoen-Yau [SY81] dimension reduction procedure. The only remaining question is the regularity of the minimizer of (2.3) at the corners. I speculate that the following general theories should hold:
   - A curvature estimate for stable capillary surfaces in corners.
   - An Allard-type theorem for varifolds with constant contact angle along the boundary.

   Motivated by the recent work of the positive mass theorem [SY17] in all dimensions, I also intend to pursue a similar general approach for the comparison theorem in dimension above 7.

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