TOPICS IN DIFFERENTIAL GEOMETRY
MINIMAL SUBMANIFOLDS
MATH 286, SPRING 2014-2015
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Contents

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These are notes from Rick Schoen's topics in differential geometry course taught at Stanford University in the Spring of 2015. We would like to thank Rick Schoen for an excellent class. Please be aware that it is likely that we have introduced numerous typos and mistakes in our compilation process, and would appreciate it if these are brought to our attention.

This course will focus on applications of the theory of minimal submanifolds. Topics covered include the two dimensional mapping problem and its relevance to the study of positive isotropic curvature, minimal hypersurfaces and scalar curvature as well as the more general theory of marginally outer trapped surfaces (MOTS), and calibrated submanifolds and associated problems.

1. Background on the 2D mapping problem

The basic setup in the 2D mapping problem is:

Question 1.1. Given a map $u_0 : \Sigma^2 \to (M^n, h)$ from a closed surface to a compact Riemannian manifold, can we homotope u_0 to a map of least area? That is, does there exist $u : \Sigma \to M$ such that $\text{Area}(u) = \inf\{\text{Area}(v) : v \sim u_0\}$?

Recall that if u is sufficiently differentiable then by the area formula we have

$$\text{Area}(u) = \int_\Sigma \|u_{x^1} \wedge u_{x^2}\| \, dx^1 \, dx^2$$

where

$$\|u_{x^1} \wedge u_{x^2}\| = \sqrt{\|u_{x^1}\|^2 \|u_{x^2}\|^2 - \langle u_{x^1}, u_{x^2}\rangle^2}.$$

One drawback of working with the area functional is its diffeomorphism invariance, i.e. $\text{Area}(u) = \text{Area}(u \circ F)$ for all $F \in \text{Diff}(\Sigma^2)$, which makes it behave poorly from an analytic point of view. For example even if we're minimizing area, we cannot expect to get good regularity in the limit unless we take care to choose good parametrizations. In two dimensions one way to overcome this is to introduce the "energy functional."

Definition 1.2. The energy function of a C^1 map $u : (\Sigma, g) \to (M, h)$ is defined to be

$$\text{E}(u) = \int_\Sigma \|du\|^2 dV_g. \tag{1.1}$$

From Cauchy-Schwarz we have

$$\|u_{x^1} \wedge u_{x^2}\| \leq \frac{1}{2}\left(\|u_{x^1}\|^2 + \|u_{x^2}\|^2\right) = \frac{1}{2}\|du\|^2,$$

(assuming we're working at the center point of an exponential chart) with equality happen if and only if

$$u_{x^1} \perp u_{x^2}, \|u_{x^1}\| = \|u_{x^2}\|.$$

In other words, for every C^1 map $u : (\Sigma, g) \to (M, h)$ we always have the area bounded by half of the energy, with equality only if u is wealky conformal.

Definition 1.3. We call a map $u : (\Sigma, g) \to (M, h)$ harmonic if u is a critical point of the energy functional.

When u is simultaneously harmonic and conformal, then any variation $\{u_t\}$ produces two curves depending on the variation: one is the half of its energy, the other is its area. We know u_0 is critical point for energy, then the first curve has vanishing slope at $t = 0$, which forces the second curve, always lying below the first curve, to have vanishing slope at $t = 0$. That means u_0 is also a critical point for area functional. In conclusion, we observed the following

Fact 1.4. *If u_0 is harmonic and conformal, it's also a critical point for the area functional.*

This observation allows us to study conformal harmonic maps instead of minimizers for area functional. Now we regard energy E as a functional on both the map u and the metric g.

Proposition 1.5. *The energy functional* $\mathrm{E}(u,g)=\int_\Sigma \|du\|_g^2 dV_g$ *has the following properties:*

(1) *Conformal invariance:* $\mathrm{E}(u,e^2\lambda g)=\mathrm{E}(u,g)$. *This is so because*

$$\mathrm{E}(u,g)=\int g^{ij}\langle u_{x^i},u_{x^j}\rangle_h\sqrt{\det g}dx^1dx^2$$

and a conformal change of the metric g *transforms* g^{ij} *and* $\sqrt{\det g}$ *inversely.*

(2) *Diffeomophism invariance: For any diffeomorphism* $F:\Sigma\to\Sigma$, $\mathrm{E}(u\circ F,F^*g)=\mathrm{E}(u,g)$.

1.1. **Hopf Differential.** Assume $u:(\Sigma,g)\to(M,h)$ is harmonic, X is a vector field on Σ and F_t is the flow generated by X. By diffeomorphism invariance, we have, for small t,

$$\mathrm{E}(u\circ F_t,F_t^*g)=\mathrm{E}(u,g).$$

Take the differential both sides at $t=0$. Since u is critical point of energy functional, the differential in the u component is 0. Therefore

$$0=\left.\frac{d}{dt}\right|_{t=0}\mathrm{E}(u\circ F_t,F_t^*g)=0+\left.\frac{d}{dt}\right|_{t=0}\mathrm{E}(u,F_t^*g).$$

In local coordinates this is

$$0=\left.\frac{d}{dt}\right|_{t=0}\int_\Sigma g_t^{ij}\langle u_{x^i},u_{x^j}\rangle\sqrt{\det g_t}dx.$$

Now $\dot g=L_Xg=\nabla_iX_j+\nabla_jX_i$, so above gives

$$0=\int\{-(X_{i,j}+X_{j,i})\langle u_{x^i},u_{x^j}\rangle\sqrt{\det g}+g^{ij}\langle u_{x^i},u_{x^j}\rangle(\operatorname{div}X)\sqrt{\det g}\}dx.$$

Definition 1.6. We define the (stress-energy) tensor to be

$$T_{ij}=\langle u_{x^i},u_{x^j}\rangle-\frac{1}{2}\|du\|^2g_{ij}.$$

Then the computation above implies

$$0=2\int_\Sigma\langle X_{i,j},T_{ij}\rangle dx,\quad\forall X.$$

Therefore we conclude

$$\nabla_jT_{ij}=0,\quad i=1,2.$$

And by definition $\mathrm{tr}_g(T)=0$. So the (stress-energy) tensor T is a transverse traceless tensor.

In local coordinates normal at one point, we can write T as a two-by-two matrix:

$$(T_{ij})=\begin{pmatrix}\frac{1}{2}(\|u_{x^1}\|^2-\|u_{x^2}\|^2) & \langle u_{x^1},u_{x^2}\rangle\\ \langle u_{x^1},u_{x^2}\rangle & -\frac{1}{2}(\|u_{x^1}\|^2-\|u_{x^2}\|^2)\end{pmatrix}.$$

This reveals the interplay between T and the so called Hopf differential on a surface. Write $g=\lambda^2|dz|^2$, where $z=x^1+\sqrt{-1}x^2$ is a local holomorphic coordinate. We define Hopf differential to be $\phi=\langle u_z,u_z\rangle_hdz^2$. It's straightforward to check

$$T \text{ is transverse traceless } \Leftrightarrow\ \phi \text{ is holomorphic}$$

and

$$T=0\Leftrightarrow\phi=0$$

Note also that $T=0$ means u is weakly conformal. So from above we conclude the following

Theorem 1.7. *The map* u *is minimal for the area functional if and only if* $\mathrm{E}(u,g)$ *is a critical point jointly in* (u,g), *where* (u,g) *takes values in* $W^{1,2}(\Sigma,M)\times\mathcal{T}_r$, *where* $\mathcal{T}_r$ *is the Teichmüller space of genus* r.

1.2. **General existence theorem.** Now we state the general existence theorem of a minimal map. This is a major analytic tool we use in this course. We'll omit most of the proof due to analytic complexity. Instead, we'll focus on geometric applications.

Theorem 1.8. *Given $u_0 : \Sigma \to (M, h)$, denote $A_0 = \inf\{A(v) : v$ homotopic to $u_0\}$. There exist $\Sigma_1, \ldots, \Sigma_k$ and least area maps $u_i : \Sigma_i \to (M, h)$ such that:*

(1) $\sum_{i=1}^k \operatorname{genus}(\Sigma_i) \leq \operatorname{genus}(\Sigma)$.
(2) $A_0 = \sum_{i=1}^k A(u_i)$.

In general, when trying to take a converging sequence of maps whose area tends to A_0, two types of singularities may occur: neck-pinches and bubbles. Each neck-pinch degenerates to a 1-dimensional segment between two parts of surfaces, and bubbles happen when area accumulates at one point. The following picture is an illustration of this phenomenon.

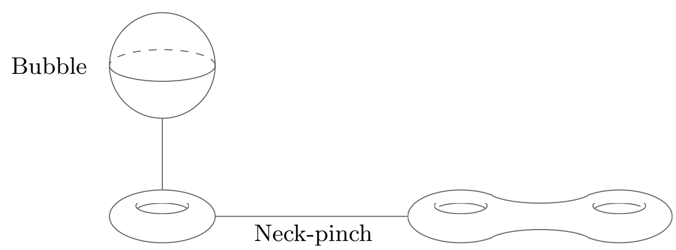

FIGURE 1. Illustration of limit (Σ, u) attaining minimal area. Each bubble is blown up into a sphere, each neck-pinch degenerates to a segment.

However, under some conditions these two types of singularities will not happen. In fact, we have

Corollary 1.9. *If u_0 is incompressible on simple closed curves then one of Σ_i is Σ and all others are genus 0. If further $\pi_2(M) = \{1\}$ then there exists u homotopic to u_0 attaining the least area A_0 (i.e., there is no bubbling).*

Here incompressibility on simple closed curves means: for any nontrivial simple closed curve α in $\pi_1(\Sigma)$, $u_0(\alpha)$ is nontrivial in $\pi_1(M)$.

In most cases harmonic maps are not necessarily conformal, hence not necessarily critical for area functional. However in the case that Σ is the 2-sphere:

Theorem 1.10. *If $u : (S^2, g_S) \to (M, h)$ is harmonic, where g_S is the standard metric on S^2, then u is also conformal, hence minimal.*

Proof. By previous section the Hopf differential $\phi(z)dz^2$ is holomorphic on S^2. That is, $\phi(z)dz^2$ is an entire differential on $\mathbb{C}$ and extends to ∞. Take $\zeta = 1/z$, then near ∞ the Hopf differential is $\phi(1/\zeta)/\zeta^4 d\zeta^2$. Near $\zeta = 0$, $\phi(1/\zeta)/\zeta^4$ is holomorphic. So $\phi(z)z^4$ is an entire function near ∞. Hence $|\phi|$ is bounded by $C/|z|^4$ for every z. By maximum principle we conclude $\phi \equiv 0$. $\square$

Next theorem will be our primary tool for our use.

Theorem 1.11 (Sacks-Uhlenbeck [SU81], Micallef-Moore [MM88])**.** *If $\pi_k(M) \neq \{1\}$ then there exists nonconstant harmonic map $u : S^2 \to M$ and the Morse index of $u \leq k - 2$.*

Remark 1.12. The Morse index is taken with respect to the second variation of the energy functional. In this specific case, the Jacobi operator is: for $V \in \Gamma(u^*TM)$,

$$\mathcal{L}V = \Delta V + \sum_{i=1}^{2} R^M(u_*(e_i), V)u_*(e_i), \quad e_1, e_2 \text{ form an orthonormal basis on } \Sigma.$$

Remark 1.13. Sacks-Uhlenbeck's approach can be (very briefly) sketched as following. For $\alpha \geq 1$, define

$$\mathrm{E}_\alpha(u) = \int_{S^2} \left(1 + \|du\|^2\right)^\alpha da.$$

For $\alpha > 1$, this is a "good" variational problem and they are able to extract converging subsequences of critical points of E_α.

Micallef-Moore further modify E_α to make its critical points non-degenerate, and they proved the modified critical points also converge after passing to a subsequence.

Remark 1.14. We here point out that Colding and Minicozzi have a different approach for minimizers on S^2, and X. Zhou generalized the result to higher genus surfaces.

2. Minimal submanifolds and Bernstein theorem

2.1. **First variation of area functional.** Let $\Sigma^k \subset M^n$ be a submanifold. Denote by D the Levi-Civita connection on M and by h the vector valued second fundamental form

$$h(X, Y) = (D_X Y)^\perp, \quad X, Y \in \Gamma(TM).$$

The vector

$$\vec{H} = \sum_{i=1}^k h(e_i, e_i)$$

is the mean curvature, where $e_1, \dots, e_k$ is an orthonormal basis of tangent vector fields.

Now if X is a vector field on M compactly supported on Σ and F_t is a flow with initial velocity X, consider $\Sigma_t = F_t(\Sigma)$. The variation of area functional can be calculated as following

$$\delta\Sigma(X) = \frac{d}{dt}\bigg|_{t=0} |\Sigma_t| = \int_\Sigma \mathrm{div}_\Sigma\, X d\mu.$$

where $\mathrm{div}_\Sigma(X) = \sum_{i=1}^k \langle D_{e_i} X, e_i\rangle$ and $d\mu$ is the volume measure on Σ.

Decompose X into its tangent and normal components $X = X^T + X^\perp$, we may write $\langle D_{e_i} X, e_i\rangle = \langle D_{e_i} X^T, e_i\rangle + \langle D_{e_i} X^\perp, e_i\rangle$. And the normal component can be further calculated as $\langle D_{e_i} X, e_i\rangle = -\langle X^\perp, (D_{e_i}, e_i)^\perp\rangle$. Therefore

$$\mathrm{div}_\Sigma(X) = \mathrm{div}_\Sigma(X^T) - \langle X, \vec{H}\rangle.$$

And the first variation of area functional is $\delta\Sigma(X) = -\int_\Sigma \langle X, \vec{H}\rangle d\mu$.

Definition 2.1. Call $\Sigma^k \subset M^n$ minimal if $\vec{H} \equiv 0$.

2.2. **Second variation of area functional, Bernstein theorem.** In many cases it's necessary to consider the second variation of area functional. We have

Proposition 2.2. *Assume $\vec{H} \equiv 0$ and $X_p \perp T_p\Sigma$ for every p on Σ, and X is compactly supported on Σ. Then the second variation of area functional is given by*

$$\delta^2\Sigma(X, X) = \int_\Sigma \|D^\perp X\|^2 - \|\langle h, X\rangle\|^2 - \sum_{i=1}^k R^M(e_i, X, e_i, X),$$

with $e_1, \dots, e_k$ being an orthonormal basis on Σ.

Remark 2.3. We split $TM = T\Sigma \oplus N\Sigma$. Then the ambient connection D gives rise to connections on $T\Sigma$ and $N\Sigma$. If $Y \in \Gamma(TM)$ and $X \in \Gamma(NM)$ then we have $D_Y^\perp X = (D_Y X)^\perp$. Then we may rewrite $\|D^\perp X\|^2 = \sum_{i=1}^k \|D^\perp_{e_i} X\|^2$ and $\|\langle h, X\rangle\|^2 = \sum_{i,j}\langle h_{i,j}, X\rangle^2 = \|D^T X\|^2$, and the second variation is given as

$$\delta^2\Sigma(X,X) = \int_\Sigma \|D^\perp X\|^2 - \sum_{i=1}^k R^M(e_i, X, e_i, X) - \|D^T X\|^2.$$

Definition 2.4. Define the Jacobi operator $\mathcal{L}$ on $\Gamma(N\Sigma)$ by

$$\mathcal{L}X = \Delta^\perp X + \sum R^M(e_i, X)e_i + \sum_{i,j}\langle h_{ij}, X\rangle h_{ij}.$$

Then $\mathcal{L}$ is a second order self-adjoint operator on $\Gamma(N\Sigma)$, and $\delta^2\Sigma(X,X) = -\int_\Sigma \langle X, \mathcal{L}X\rangle d\mu$.

We call the number of negative eigenvalues of $\mathcal{L}$ the Morse index of Σ. Σ is called stable if the Morse index is 0, strictly stable if there are also no Jacobi fields.

A famous and important question is to understand the structure of stable minimal surfaces. The first important theorem is given by S. Berstein.

Theorem 2.5 (S. Berstein [Ber27])**.** *Let $\Sigma^2 \subset \mathbb{R}^3$ be a minimal surface and given by a graph $x^3 = u(x^1, x^2)$ defined for all (x^1, x^2). Then Σ is a plane; i.e., u must be a linear function.*

Before proving Bernstein's theorem, we first state some important properties of minimal graphs $\Sigma = \text{graph}(u)$ in $\mathbb{R}^{n+1}$, where $u : \Omega \to \mathbb{R}$ is a C^2 function.

Fact 2.6. *Σ is 2-sided. That is, Σ has a unit normal vector field ν.*

Fact 2.7. *Σ is area minimizing in $\Omega \times \mathbb{R}$.*

The second fact is an easy consequence of calibration theory, which will reappear in later part of the course. We prove this special case here.

Extend ν to a unit vector field in $\Omega \times \mathbb{R}$ by setting $\nu(x, y) = \nu(x, u(x))$. Since ν is parallel in the x^{n+1} direction, we conclude from the minimal surface equation that $\operatorname{div}_{\mathbb{R}^{n+1}} \nu = 0$. In fact, suppose $e_1, \ldots, e_n$ is an orthonormal basis tangent to Σ, $e_{n+1} = \nu$. Then we have

$$\operatorname{div}_{\mathbb{R}^{n+1}} \nu = \sum_{i=1}^{n+1} \langle D_{e_i}\nu, e_i\rangle.$$

Now ν is of unit length, so $\langle D_{e_{n+1}}\nu, e_{n+1}\rangle = 0$. Therefore

$$\operatorname{div}_{\mathbb{R}^{n+1}} \nu = \sum_{i=1}^{n} \langle D_{e_i}\nu, e_i\rangle = -\langle \vec{H}, \nu\rangle = 0.$$

The vector field ν gives a calibration in the region $\Omega \times \mathbb{R}$.

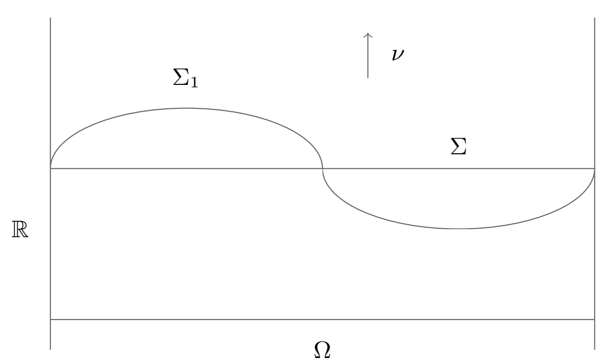

FIGURE 2. Calibration

Suppose $\Sigma_1 \subset \Omega \times \mathbb{R}$ and $\partial\Omega_1 = \partial\Omega$. Denote ν_1 the outer unit normal vector field on Ω_1. Let Ω' be the signed region in $\mathbb{R}^{n+1}$ with $\Sigma - \Sigma_1 = \partial\Omega'$. Then by the divergence theorem,

$$0 = \int_{\Omega'} \operatorname{div}_{\mathbb{R}^{n+1}} \nu = \int_\Sigma \nu \cdot \nu - \int_{\Sigma_1} \nu \cdot \nu_1.$$

So we conclude

$$|\Sigma| = \int_\Sigma \nu \cdot \nu = \int_{\Sigma_1} \nu \cdot \nu_1 \leq |\Sigma_1|, \quad \text{by Cauchy-Schwarz.}$$

Fact 2.8. *If Σ is an entire minimal graph, then*

$$|\Sigma \cap B_R(0)| \leq CR^n, \quad \forall R \geq 1$$

This is an easy consequence of the fact that minimal graphs are area minimizing. Take any $R > 0$. Then Σ divides $\partial B_R(0) = S_R(0)$ into two parts Σ_1, Σ_2. Since Σ is a minimal graph over the domain $S^n_R(0) \subset \mathbb{R}^n$, we have

$$|\Sigma \cap B_R(0)| \leq \min\{|\Sigma_1|, |\Sigma_2|\} \leq CR^n.$$

Now we prove Bernstein's theorem through the following

Theorem 2.9. *Assume $\Sigma \subset \mathbb{R}^3$ is stable, proper, orientable minimal surface with Euclidean area growth. That is, $|\Sigma \cap B_R(0)| \leq CR^2$ for all $R \geq 1$. Then Σ is a plane.*

Proof. Take a normal vector field ν and let $X = \varphi\nu$, $\varphi \in C_c^\infty(\Sigma)$. The stability condition gives

$$0 \leq \delta^2\Sigma(X, X) = \int \|\nabla\varphi\|^2 - \|h\|^2\varphi^2, \quad \text{where } h \text{ is the scalar second fundamental form.}$$

So we know

$$\int_\Sigma \|h\|^2\varphi^2 d\mu \leq \int_\Sigma \|\nabla\varphi\|^2 d\mu, \quad \forall\varphi \operatorname{Lip}_c(\Sigma).$$

We use the logarithmic cut-off trick. Denote $\rho(x) = |x|$, then ρ is a proper function on Σ and $\|\nabla\rho\|^2 \leq \|D\rho\|^2 = 1$. Define

$$\varphi_R(\rho) = \begin{cases} 1 & \text{for } \rho \leq R \\ \frac{\log R^2/\rho}{\log R} & \text{for } R \leq \rho \leq R^2 \\ 0 & \text{for } \rho \geq R^2. \end{cases}$$

Claim: $\int_\Sigma \|\nabla\varphi_R\|^2 \leq C(\log R)^{-1}$.

In fact, we have

$$\int_{\Sigma\cap(B_{R^2}-B_R)} \|\nabla\varphi_R\|^2 \leq \int \frac{\rho^{-2}}{(\log R)^2} = (\log R)^{-2} \int_R^{R^2} r^{-2} \left(\int_{\rho=r} \frac{d\sigma}{\|\nabla\rho\|} \right) dr.$$

The last equality is got by coarea formula. Here again we use coarea formula just for the constant function 1 on $\Sigma \cap B_R(0)$ to get

$$\int_{\rho=r} \frac{d\sigma}{\|\nabla\rho\|} = \frac{d}{dr}|\Sigma \cap B_r(0)|.$$

So

$$\begin{aligned} \int_{\Sigma\cap(B_{R^2}-B_R)} \|\nabla\varphi_R\|^2 &\leq (\log R)^{-2} \left(r^{-2}|\Sigma \cap B_r| \Big|_{r=R}^{r=R^2} + 2\int_R^{R^3} r^{-3}|\Sigma \cap B_r(0)| dr \right) \\ &\leq C_1(\log R)^{-2} + C_2(\log R)^{-1}. \end{aligned}$$

Here we used the area growth of Σ.

Now take R to ∞ we get

$$\int_{\Sigma\cap B_R(0)} \|h\|^2 d\mu \leq \int_{\Sigma\cap B_{R^2}} \|h\|^2\varphi_R^2 d\mu \leq \int_{\Sigma\cap(B_{R^2}-B_R)} \|\nabla\varphi_R\|^2 d\mu \to 0.$$

So $h \equiv 0$ and Σ is a plane. □

Question 2.10. We are curious about possible generalization of Berstein's theorem. The following cases have been of great interest for researchers.

(1) For higher dimensional $\Sigma^n \subset \mathbb{R}^{n+1}$ entire minimal graphs, can we conclude that Σ is affine space? This question has been answered by many authors over many years. The conclusion is true for $n \leq 7$ and false for $n \geq 8$.
(2) Can we get a Berstein type theorem when $\Sigma^n \subset M^{n+1}$ where M is a curved manifold? In some special cases this question can be answered. We'll get back to this question later.
(3) For $\Sigma^2 \subset \mathbb{R}^n$ where $n \geq 4$, can we get a Bernstein type theorem? We'll focus on this direction.

The third question is more complicated than it first appears. The fact is, we can construct a family of area minimizing surfaces in higher dimensional Euclidean spaces. Let $n = 2m$ and $J : \mathbb{R}^n \to \mathbb{R}^n$ being a complex structure, meaning J is orthogonal and $J^2 = -I$. For each fixed J take Σ^2 to be a J-holomorphic curve. Then Σ is area minimizing by a similar calibration argument. In particular, consider

$$\Sigma = \{(z, w) : w = f(z)\},$$

Here f is a J-holomorphic function. Then Σ is an area-minimizing surface in $\mathbb{R}^4$.

3. Bernstein's theorem in higher codimensions

As mentioned in the previous section, Bernstein's theorem in its full generality fails in higher codimensions due to the presence of ***J*-holomorphic curves**, defined as follows.

Definition 3.1. Let $n = 2m$ and let J be an orthogonal complex structure on $\mathbb{R}^n$, i.e. an orthogonal matrix J with $J^2 = -I$. A J-holomorphic curve is a 2-dimensional surface $\Sigma^2 \subseteq \mathbb{R}^n$ such that

$$J(T_x\Sigma) = T_x\Sigma, \text{ for all } x \in \Sigma.$$

Proposition 3.2. *J-holomorphic curves are area-minimizing among orientable competitors.*

Proof. Consider the Kähler form ω, defined by $\omega(X, Y) = JX \cdot Y$. Since J is a constant matrix, we observe that ω is closed. Next we show that ω is a calibrating form that restricts to the area form precisely on J-invariant 2-planes. To see this, take any oriented 2-plane Π in $\mathbb{R}^n$ and let $\{e_1, e_2\}$ be a positive orthonormal basis. By the Schwartz inequality,

$$|\omega(e_1, e_2)| = |Je_1 \cdot e_2| \leq 1.$$

Moreover, $\omega(e_1, e_2) = 1$ if and only if $Je_1 = e_2$, which is equivalent to the J-invariance of Π.

To conclude the proof, let Σ_0 be an oriented surface with $\partial\Sigma_0 = \partial\Sigma$, then we can find a region R with $\Sigma - \Sigma_0 = \partial R$. Then we have

$$\begin{aligned} 0 &= \int_R d\omega = \int_{\partial R} \omega = \int_\Sigma \omega - \int_{\Sigma_0} \omega \\ &= |\Sigma| - \int_{\Sigma_0} \omega \geq |\Sigma| - |\Sigma_0| \end{aligned}$$

and the proof is complete. □

Since the hypotheses of Bernstein's theorem certainly doesn't rule out J-holomorphic curves, Proposition 3.2 shows that Bernstein's theorem is generally false in higher codimensions. The best one could hope for is perhaps the following statement.

Conjecture 3.3. Let $\Sigma^2 \subseteq \mathbb{R}^n$ be a complete stable minimal surface, possibly with some controlled area growth, then there exists $2k \leq n$ and a $2k$-plane $P \subseteq \mathbb{R}^n$ such that Σ is J-holomorphic in P for some complex structure J.

It turns out that even this is false in general. Nonetheless, all hope is not lost as there are some interesting special cases in which Conjecture 3.3 is true. Below we list a few positive results.

(1) When $n = 4$ and Σ is oriented with area growth suitably bounded, the conjecture is true.
(2) If genus$(\Sigma) = 0$ and
$$\int_\Sigma (-K) da < \infty,$$
then the conjecture is true for all n.
(3) If the ambient space is replaced by $\mathbb{T}^n$, then the conjecture is true for $n = 4$.

3.1. **Complexifying the stability operator.** We'll treat the case (1). A key ingredient in the proof is a complexified version of the second variation formula. We first set up some notations before writing down the formula. As before, let (Σ^2, g) be an oriented surface in M^n. Around each point of Σ we can find local isothermal coordinates (x^1, x^2), i.e.

$$g = \lambda^2 \left((dx^1)^2 + (dx^2)^2\right), \text{ where } \lambda^2 = \left|\frac{\partial}{\partial x^1}\right|^2 = \left|\frac{\partial}{\partial x^2}\right|^2$$

Next we write

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right); \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right) \tag{3.1}$$

Now recall that if Σ is minimal and $X \in \Gamma(N\Sigma)$, then the second variation is given by

$$\delta^2\Sigma(X, X) = \int_\Sigma \|D^\perp X\|^2 - \sum_{j=1}^2 R^M(e_j, X, e_j, X) - \|D^T X\|^2 da, \tag{3.2}$$

where $\{e_j\}$ is any orthonormal frame for $T\Sigma$. Now we complexify $T\Sigma$ and $N\Sigma$ and extend the second variation formula to complex vector fields. For $X \in \Gamma(N_\mathbb{C}\Sigma)$, we simply write

$$\delta^2\Sigma(X, \overline{X}) = \int_\Sigma \|D^\perp X\|^2 - \sum_{j=1}^2 R^M(e_j, X, e_j, \overline{X}) - \|D^T X\|^2 da \tag{3.3}$$

Of course now $\|D^\perp X\|^2 = \langle D^\perp X, D^\perp \overline{X}\rangle$ and likewise for $\|D^T X\|^2$. Below we'll use the operators (3.1) to rewrite (3.3). More precisely, we have the following formula.

Proposition 3.4. *Let Σ^2 and M^n be as above and let $X \in \Gamma(N_\mathbb{C}\Sigma)$, then*

$$\delta^2\Sigma(X, \overline{X}) = 4\int_\Sigma \|D^\perp_{\frac{\partial}{\partial \bar{z}}} X\|^2 - R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \bar{z}}, \overline{X}) - \|D^T_{\frac{\partial}{\partial z}} X\|^2 dx^1 \wedge dx^2, \tag{3.4}$$

Remark 3.5. Notice that the integrand

$$\left[\|D^\perp_{\frac{\partial}{\partial \bar{z}}} X\|^2 - R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \bar{z}}, X) - \|D^T_{\frac{\partial}{\partial z}} X\|^2\right] dx^1 \wedge dx^2$$

is conformally invariant. Thus, even though it's written in terms of coordinates, it makes sense globally on Σ.

Proof.

1. We start from (3.3). Introducing isothermal coordinates as above, the area element da becomes $\lambda^2 dx^1 \wedge dx^2$. Plugging this into (3.2) and using the orthonormal frame $e_j = \lambda^{-1}\frac{\partial}{\partial x^j}$, $j = 1, 2$, we find that

$$\delta^2\Sigma(X,\overline{X}) = \int_\Sigma \sum_{j=1}^2 \|D^\perp_{\frac{\partial}{\partial x^j}} X\|^2 - \sum_{j=1}^2 R^M(\frac{\partial}{\partial x^j}, X, \frac{\partial}{\partial x^j}, \overline{X}) - \sum_{j=1}^2 \|D^T_{\frac{\partial}{\partial x^j}} X\|^2 dx^1 \wedge dx^2 \tag{3.5}$$

2. Next we notice that

$$\begin{aligned}\|D^\perp_{\frac{\partial}{\partial z}} X\|^2 + \|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\|^2 &= \frac{1}{4}\langle D^\perp_{\frac{\partial}{\partial x^1}} X - iD^\perp_{\frac{\partial}{\partial x^2}} X,\ D^\perp_{\frac{\partial}{\partial x^1}} X + iD^\perp_{\frac{\partial}{\partial x^2}} X\rangle \\ &\quad + \frac{1}{4}\langle D^\perp_{\frac{\partial}{\partial x^1}} X + iD^\perp_{\frac{\partial}{\partial x^2}} X,\ D^\perp_{\frac{\partial}{\partial x^1}} X - iD^\perp_{\frac{\partial}{\partial x^2}} X\rangle \\ &= \frac{1}{2}\left(\|D^\perp_{\frac{\partial}{\partial x^1}} X\|^2 + \|D^\perp_{\frac{\partial}{\partial x^2}} X\|^2\right)\end{aligned}$$

Likewise, we also have

$$\|D^T_{\frac{\partial}{\partial z}} X\|^2 + \|D^T_{\frac{\partial}{\partial \overline{z}}} X\|^2 = \frac{1}{2}\left(\|D^T_{\frac{\partial}{\partial x^1}} X\|^2 + \|D^T_{\frac{\partial}{\partial x^2}} X\|^2\right)$$

and

$$\begin{aligned}&R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \overline{z}}, \overline{X}) + R^M(\frac{\partial}{\partial \overline{z}}, X, \frac{\partial}{\partial z}, \overline{X}) \\ =&\frac{1}{2}\left(R^M(\frac{\partial}{\partial x^1}, X, \frac{\partial}{\partial x^1}, \overline{X}) + R^M(\frac{\partial}{\partial x^2}, X, \frac{\partial}{\partial x^2}, \overline{X})\right)\end{aligned}$$

Plugging these into (3.5), we obtain

$$\begin{aligned}\delta^2\Sigma(X,\overline{X}) = 2\int_\Sigma &\|D^\perp_{\frac{\partial}{\partial z}} X\|^2 + \|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\|^2 - R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \overline{z}}, \overline{X}) \\ &+ R^M(\frac{\partial}{\partial \overline{z}}, X, \frac{\partial}{\partial z}, \overline{X}) - \|D^T_{\frac{\partial}{\partial z}} X\|^2 - \|D^T_{\frac{\partial}{\partial \overline{z}}} X\|^2 dx^1 \wedge dx^2\end{aligned} \tag{3.6}$$

3. Take the term $\int_\Sigma \|D^\perp_{\frac{\partial}{\partial z}} X\|^2 dx^1 \wedge dx^2$. We want to integrate by parts to write it in terms of $\int_\Sigma \|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\|^2 dx^1 \wedge dx^2$, a curvature term and some other stuff. To do so, we observe

$$\begin{aligned}\|D^\perp_{\frac{\partial}{\partial z}} X\|^2 &= \|D_{\frac{\partial}{\partial z}} X\|^2 - \|D^T_{\frac{\partial}{\partial z}} X\|^2 \\ &= \langle D_{\frac{\partial}{\partial z}} X,\ D_{\frac{\partial}{\partial \overline{z}}} \overline{X}\rangle - \|D^T_{\frac{\partial}{\partial z}} X\|^2 \\ &= \frac{\partial}{\partial \overline{z}}\langle D_{\frac{\partial}{\partial z}} X,\ \overline{X}\rangle - \langle D_{\frac{\partial}{\partial \overline{z}}} D_{\frac{\partial}{\partial z}} X,\ \overline{X}\rangle - \|D^T_{\frac{\partial}{\partial z}} X\|^2 \\ &= \frac{\partial}{\partial \overline{z}}\langle D_{\frac{\partial}{\partial z}} X,\ \overline{X}\rangle - \langle D_{\frac{\partial}{\partial z}} D_{\frac{\partial}{\partial \overline{z}}} X,\ \overline{X}\rangle - R^M(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, X, \overline{X}) - \|D^T_{\frac{\partial}{\partial z}} X\|^2 \\ &= \frac{\partial}{\partial \overline{z}}\langle D_{\frac{\partial}{\partial z}} X,\ \overline{X}\rangle - \frac{\partial}{\partial z}\langle D_{\frac{\partial}{\partial \overline{z}}} X,\ \overline{X}\rangle + \langle D_{\frac{\partial}{\partial \overline{z}}} X,\ D_{\frac{\partial}{\partial z}} \overline{X}\rangle - R^M(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, X, \overline{X}) - \|D^T_{\frac{\partial}{\partial z}} X\|^2 \\ &= \frac{\partial}{\partial \overline{z}}\langle D_{\frac{\partial}{\partial z}} X,\ \overline{X}\rangle - \frac{\partial}{\partial z}\langle D_{\frac{\partial}{\partial \overline{z}}} X,\ \overline{X}\rangle + \|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\| + \|D^T_{\frac{\partial}{\partial \overline{z}}} X\|^2 - R^M(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, X, \overline{X}) - \|D^T_{\frac{\partial}{\partial z}} X\|^2\end{aligned} \tag{3.7}$$

Integrating over Σ, using the fact that X has compact support and plugging into (3.6), we get

$$\delta^2\Sigma(X,\overline{X}) = 2\int_\Sigma 2\|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\|^2 - R^M(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, X, \overline{X}) - R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \overline{z}}, \overline{X})$$

$$- R^M(\frac{\partial}{\partial \overline{z}}, X, \frac{\partial}{\partial z}, \overline{X}) - 2\|D^T_{\frac{\partial}{\partial z}} X\|^2 dx^1 \wedge dx^2 \tag{3.8}$$

By the first Bianchi identity,

$$R^M(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, X, \overline{X}) + R^M(\frac{\partial}{\partial \overline{z}}, X, \frac{\partial}{\partial z}, \overline{X}) = -R^M(X, \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}, \overline{X}) = R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \overline{z}}, \overline{X}).$$

Therefore from (3.8) we get

$$\begin{aligned}\delta^2\Sigma(X, \overline{X}) &= 2\int_\Sigma 2\|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\|^2 - 2R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \overline{z}}, \overline{X}) - 2\|D^T_{\frac{\partial}{\partial z}} X\|^2 dx^1 \wedge dx^2 \\ &= 4\int_\Sigma \|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\|^2 - R^M(\frac{\partial}{\partial z}, X, \frac{\partial}{\partial \overline{z}}, \overline{X}) - \|D^T_{\frac{\partial}{\partial z}} X\|^2 dx^1 \wedge dx^2\end{aligned}$$

as stated. The proof is now complete.

□

In the case where the ambient manifold is $\mathbb{R}^n$, (3.4) simplifies and we have the following beautiful stability criterion.

Corollary 3.6. *Suppose $\Sigma^2 \subseteq \mathbb{R}^n$ is a stable oriented minimal surface, then*

$$\int_\Sigma \|D^T_{\frac{\partial}{\partial z}} X\|^2 dx^1 \wedge dx^2 \leq \int_\Sigma \|D^\perp_{\frac{\partial}{\partial \overline{z}}} X\|^2 dx^1 \wedge dx^2, \text{ for all } X \in \Gamma(N_{\mathbb{C}}\Sigma) \tag{3.9}$$

Proof. Each section $X \in \Gamma(N_{\mathbb{C}}\Sigma)$ can be written as $X = X_1 + iX_2$, where X_1, X_2 are sections of the real normal bundle $N\Sigma$. Then we have

$$\delta^2\Sigma(X, \overline{X}) = \delta^2\Sigma(X_1, X_1) + \delta^2(X_2, X_2) \geq 0,$$

where the last inequality is true by stability. The corollary now follows from Proposition 3.4. □

3.2. **Stable minimal surfaces in $\mathbb{R}^4$ and T^4.** Let's come back to complete oriented stable minimal surfaces in $\mathbb{R}^4$. Recall that our goal is to construct an orthogonal complex structure J on $\mathbb{R}^4$ with respect to which Σ is holomorphic. We introduce some notations before describing the construction. We will roughly be following [Mic84].

For clarity, below we suppose Σ is the image of an isometric stable minimal immersion $F : M^2 \to \mathbb{R}^4$, where M^2 is a complete oriented surface. Let $E \simeq M \times \mathbb{R}^4$ denote the pullback of $T\mathbb{R}^4$ and its metric structure via F. Then we can view TM as a sub-bundle of E and use the metric to define the orthogonal complement bundle, which we denote by NM. Since M is oriented, the pullback metric induces a complex structure J^T on M. Also, still by orientability, we can define a complex structure $J^\perp$ on NM by rotation by 90° in the clockwise or counterclockwise direction (notice that we have a choice here). We then define $J : M \to \mathrm{Hom}(E)$ as follows: for each $p \in M$, given a vector $v \in E_p$, we define,

$$J_p(v) = J^T_p(v^T) + J^\perp_p(v^\perp),$$

where v^T and $v^\perp$ denote the orthogonal projections of v onto T_pM and N_pM, respectively.

The triviality of E allows us to view J as a map from M to $\mathrm{Hom}(\mathbb{R}^4)$. What we want to demonstrate now is that J is constant, so that $J : M \to \mathrm{Hom}(\mathbb{R}^4)$ extends as a complex structure on all of $\mathbb{R}^4$. To see this, we first complexify E, TM and NM and extend J^T and $J^\perp$ to be complex linear maps. Then J^T gives rise to a splitting

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

Likewise, $N_{\mathbb{C}}M$ splits as $N^{1,0}M \oplus N^{0,1}M$. We denote $N^{1,0}M$ by V; then $N^{0,1}M = \overline{V}$. With these notations, we form the following sub-bundle of $E_{\mathbb{C}} \simeq M \times \mathbb{C}^4$:

$$W = T^{1,0}M \oplus V.$$

For each $p \in M$, the fiber W_p is a subspace of $\mathbb{C}^4$. The constancy of J is then translated into the constancy of W.

Proposition 3.7. *If $\Gamma(W)$ is closed under the usual directional derivatives in $\mathbb{C}^4$, then W_p is independent of p.*

Proof. Take $w \in W_p \subseteq \mathbb{C}^4$ and extend it as a constant vector field on M. Note that since

$$\mathbb{C}^4 = W_p \oplus \overline{W}_p,$$

for each $q \in M$ we can decompose $w_q = w_q^{1,0} + w_q^{0,1}$, with $w_q^{1,0} \in W_q$ and $w_q^{0,1} \in \overline{W}_q$. We will show that $w^{0,1}$ is constantly zero. To see this, observe that since w is a constant vector field, letting ∂ denote a directional derivative, we have

$$\begin{aligned} 0 = \partial w &= \partial w^{1,0} + \partial w^{0,1} \\ &= (\partial w)^{1,0} + (\partial w)^{0,1} \end{aligned} \tag{3.10}$$

where the last equality follows from the assumption that $\Gamma(W)$ is closed under differentiation. Since $W_p \oplus \overline{W}_p$ is a direct sum, (3.10) immediately implies that both $w^{1,0}$ and $w^{0,1}$ are constant. In particular, since $w_p^{0,1} = 0$, we see that $w^{0,1}$ is constantly zero. $\square$

To check that $\Gamma(W)$ is closed under differentiation, we will use the following proposition.

Proposition 3.8. *Let $F_{zz}^{\perp}$ denote the projection of F_{zz} onto $N_{\mathbb{C}}M$ and let $F_{zz}^{1,0}$, $F_{zz}^{0,1}$ be the projection of $F_{zz}^{\perp}$ onto V, $\overline{V}$, respectively. If $F_{zz}^{0,1} = 0$ then $\Gamma(W)$ is closed under differentiation.*

Proof. Recall that F is minimal. Introducing isothermal coordinates, F is also conformal. Thus F is harmonic and we have

$$F_z \cdot F_z = 0 \text{ (Conformality)} \tag{3.11}$$

$$F_{z\overline{z}} = 0 \text{ (Harmonicity)} \tag{3.12}$$

Next take a local positive orthonormal frame $\{e_3, e_4\}$ of NM such that

$$J^{\perp}(e_3) = e_4; \;\; J^{\perp}(e_4) = -e_3,$$

and let $\varepsilon = \frac{1}{\sqrt{2}}(e_3 - ie_4)$. Then $V = \operatorname{span}_{\mathbb{C}}(\varepsilon)$ and $W = \operatorname{span}_{\mathbb{C}}(\varepsilon,\ F_z)$.

Now let $s \in \Gamma(W)$ and write

$$s = a(z)F_z + b(z)\varepsilon.$$

To save notations, below we simply write $X \simeq Y$ if $X \equiv Y \bmod W$. Now we compute

$$\frac{\partial}{\partial z} s \simeq aF_{zz} + b\frac{\partial}{\partial z}\varepsilon, \tag{3.13}$$

and expand the two terms on the right using the basis $\{F_z, F_{\overline{z}}, \varepsilon, \overline{\varepsilon}\}$. The first term becomes

$$\begin{aligned} F_{zz} &= \frac{F_{zz} \cdot F_{\overline{z}}}{|F_z|^2} F_z + \frac{F_{zz} \cdot F_z}{|F_z|^2} F_{\overline{z}} + (F_{zz} \cdot \overline{\varepsilon})\varepsilon + (F_{zz} \cdot \varepsilon)\overline{\varepsilon} \\ &\simeq \frac{F_{zz} \cdot F_z}{|F_z|^2} F_{\overline{z}} + (F_{zz} \cdot \varepsilon)\overline{\varepsilon} = \frac{F_{zz} \cdot F_z}{|F_z|^2} F_{\overline{z}} + (F_{zz}^{0,1} \cdot \varepsilon)\overline{\varepsilon}, \end{aligned}$$

where we used the fact that $F_{zz}^{1,0} \cdot \varepsilon = 0$ in the last equality. Now by (3.11), the first term above vanishes. Using the assumption $F_{zz}^{0,1} = 0$, we see that the second term vanishes as well. Thus $F_{zz} \simeq 0$; that is, $F_{zz} \in V$.

Next we look at the second term in (3.13). Then we have

$$\frac{\partial}{\partial z}\varepsilon \simeq \frac{\frac{\partial}{\partial z}\varepsilon \cdot F_z}{|F_z|^2} F_{\overline{z}} + (\frac{\partial}{\partial z}\varepsilon \cdot \varepsilon)\overline{\varepsilon}$$

$$
\begin{aligned}
&= \frac{\frac{\partial}{\partial z}\varepsilon \cdot F_z}{|F_z|^2} F_{\overline{z}} \ (\varepsilon \text{ had unit length}) \\
&= -\frac{\varepsilon \cdot F_{zz}}{|F_z|^2} F_{\overline{z}} \ (\text{integrate by parts in the first term}) \\
&= -\frac{\varepsilon \cdot F_{zz}^{0,1}}{|F_z|^2} F_{\overline{z}} \ (F_{zz}^{1,0} \cdot \varepsilon = 0) \\
&= 0 \ (\text{by assumption}) \ .
\end{aligned}
$$

Thus for each section s of W, we've shown that $\frac{\partial}{\partial z} s \simeq 0$. Similarly we can show that $\frac{\partial}{\partial \overline{z}} s \simeq 0$. Thus $\Gamma(W)$ is preserved by differentiation. □

To verify the assumption of Proposition 3.8, we suppose in addition that M is **parabolic**.

Definition 3.9. Given a Riemannnian surface M, we say that M is parabolic if every positive superharmonic function on M is constant.

Below we give some examples of parabolic manifolds.

Example 3.10.

(1) *The complex plane $\mathbb{C}$ is parabolic. On the other hand, the unit disk $\mathbb{D} \subset \mathbb{C}$ is not parabolic.*
(2) *Any compact Riemann surface with finitely many punctures is parabolic.*
(3) *If M is a complete surface with $|M \cap B_R| \leq CR^2$ for R large, then M is parabolic.*

Proof. Suppose $u > 0$ is a positive superharmonic function on M. Letting $w = \log u$, we have

$$
\begin{aligned}
\Delta w &= \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} \leq \frac{\Delta u}{u} - |\nabla w|^2 \\
&\leq -|\nabla w|^2 \ (\text{since } \Delta u \leq 0) \ .
\end{aligned} \tag{3.14}
$$

Next we test the inequality (3.14) against φ^2, where φ is any test function $\varphi \in C^1_C(M)$, getting

$$
\begin{aligned}
\int_M \varphi^2 |\nabla w|^2 d\,\text{vol} \leq -\int_M \varphi^2 \nabla w d\,\text{vol} &= 2\int_M \phi \langle \nabla\varphi, \nabla w\rangle d\,\text{vol} \\
&\leq \frac{1}{2}\int_M \varphi^2 |\nabla w|^2 d\,\text{vol} + 2\int_M |\nabla\varphi|^2 d\,\text{vol}
\end{aligned}
$$

Hence we get

$$
\int_M \varphi^2 |\nabla w|^2 d\,\text{vol} \leq 4\int_M |\nabla\varphi|^2 d\,\text{vol} \, .
$$

Applying the logarithmic cut-off trick as in the proof of the Bernstein theorem in the last section, we conclude that w, and thus u, is constant. □

(4) *If $\Sigma^2 \subseteq \mathbb{R}^n$ is an entire minimal graph, then Σ with the induced metric is parabolic.*

Proof. We will prove that Σ is conformally equivalent to $\mathbb{C}$. By the uniformization theorem, we know that Σ is conformally equivalent either to $\mathbb{C}$ or to $\mathbb{D}$. Assume by contradiction that the latter holds and let $F : \mathbb{D} \to \Sigma$ be a biholomorphic map. Since Σ is isometrically and minimally embedded, F is harmonic as a map of $\mathbb{D}$ into $\mathbb{R}^n$. Modifying F by an automorphism of $\mathbb{D}$ is necessary, we may assume that $F(0) = (0, 0, u(0, 0))$.

Next denote $\tilde{F}(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2))$. By the previous paragraph, $\tilde{F}$ is a harmonic diffeomorphism from $\mathbb{D}$ to $(\mathbb{R}^2, h)$, where h is obtained by pulling back the induced metric on Σ via $(x_1, x_2) \mapsto (x_1, x_2, u(x_1, x_2))$.

Recall that in complex coordinates, the Jacobian of $\tilde{F}$ can be written as

$$J(\tilde{F}) = |\tilde{F}_z|^2 - |\tilde{F}_{\bar{z}}|^2, \tag{3.15}$$

which is everywhere strictly positive since $\tilde{F}$ is a diffeomorphism. This implies that $|\tilde{F}_z|$ is everywhere non-zero, so we can define a metric $\tilde{g}$ on $\mathbb{D}$ by

$$\tilde{g} = |\tilde{F}_z|^2|dz|^2.$$

Now since $\tilde{F}$ is harmonic, we have $F_{\bar{z}z} = 0$ and hence $\Delta|\tilde{F}_z|^2 = 0$, which means that $(\mathbb{D}, \tilde{g})$ is flat (Gauss curvature zero).

Using (3.15) again, we see that $|d\tilde{F}|$ is dominated by $|\tilde{F}_z|$ and hence

$$\tilde{F}^*(h) \leq c\tilde{g},$$

where c is a dimensional constant. Now for an arbitrary $R > 0$, we can choose r such that

$$\mathrm{dist}_{\tilde{F}^*(h)}(0, \partial\mathbb{D}_r) = \mathrm{dist}_h(0, \partial(F(\mathbb{D}_r))) \geq R.$$

Combining this with the previous inequality, we get

$$c\,\mathrm{dist}_{\tilde{g}}(0, \partial\mathbb{D}_r) \geq R \tag{3.16}$$

Next we take the coordinate function x on $\mathbb{D}$. Then

$$\begin{cases} \Delta_{\tilde{g}} x &= 0 \\ |x| &\leq 1 \end{cases}$$

so by the harmonic function estimates in [?] and the definition of $\tilde{g}$, we have

$$\frac{1}{|F_z|^2} = |\nabla_{\tilde{g}} x|^2(0) \leq \frac{c}{\mathrm{dist}_{\tilde{g}}(0, \partial\mathbb{D}_r)^2} \leq \frac{c}{R^2}.$$

This in turn gives us

$$|d\tilde{F}|^2(0) \geq |\tilde{F}_z|^2(0) \geq cR^2.$$

Since R is arbitrary, we obtain a contradiction, so Σ is conformally equivalent to $\mathbb{C}$ and hence parabolic. $\square$

After this little digression into parabolic manifolds we return to our problem and give the precise statement of the main result of this section.

Theorem 3.11. *Assume $F : M^2 \to \mathbb{R}^4$ is an oriented, stable, parabolic, complete minimal surface. Then F is J-holomorphic for some orthogonal complex structure J on $\mathbb{R}^4$.*

Proof. By Proposition 3.8, the proof reduces to showing that $F^{0,1}_{zz}$ vanishes. We will demonstrate this by plugging special test functions into the stability inequality (3.4) and using parabolicity. To that end we consider a test function of the form fs, where f is a real-valued smooth function with compact support on M, and $s \in \Gamma(N_{\mathbb{C}}M)$. Then we have

$$\delta^2\Sigma(fs, f\bar{s}) = \int_M |f_z|^2|s|^2 - f^2(\mathrm{Re}\langle \bar{s}, D_{z\bar{z}}s\rangle) - f^2|(\partial_z^T s)|^2 dx^1 \wedge dx^2, \tag{3.17}$$

where we're using D to denote $D^{\perp}$. To derive this formula, we recall that by (3.4) we have

$$\delta^2\Sigma(fs, f\bar{s}) = \int_M |D_{\bar{z}}(fs)|^2 - |\partial_z^T(fs)|^2 dx^1 \wedge dx^2 \tag{3.18}$$

To handle the first term we compute

$$\int |D_{\bar{z}}(fs)|^2 = \int D_{\bar{z}}(fs) \cdot D_z(f\bar{s}) = \int (f_{\bar{z}}s + fD_{\bar{z}}s) \cdot (f_z\bar{s} + fD_z\bar{s})$$
$$= \int |f_z|^2|s|^2 + f^2|D_{\bar{z}}s|^2 + ff_zD_{\bar{z}}s \cdot \bar{s} + (\text{complex conjugate of the previous term})$$

$$= \int |f_z|^2|s|^2 + f^2|D_{\overline{z}}s|^2 + 2\mathrm{Re}(ff_zD_{\overline{z}}s\cdot\overline{s}) \tag{3.19}$$

Now notice that

$$\begin{aligned}\int ff_zD_{\overline{z}}s\cdot\overline{s} &= \frac{1}{2}\int (f^2)_zD_{\overline{z}}s\cdot\overline{s}\\ &= -\frac{1}{2}\int f^2D_zD_{\overline{z}}s\cdot\overline{s} - \frac{1}{2}\int f^2|D_{\overline{z}}s|^2\end{aligned}$$

Plugging this back into (3.19), we get

$$\int |D_{\overline{z}}(fs)|^2 = \int |f_z|^2|s|^2 - f^2\mathrm{Re}(D_{z\overline{z}}s\cdot\overline{s}) \tag{3.20}$$

For the second term in (3.18), we notice that

$$\partial_z(fs) = f_zs + f\partial_zs.$$

Since s is a normal section, projection onto TM kills the first term and we're left with

$$\partial_z^T(fs) = f(\partial_z^Ts) \tag{3.21}$$

(3.17) now follows by plugging (3.20) and (3.21) back into (3.18). To proceed, we take a vector $a\in\mathbb{C}^4$ and denote by $a^{1,0}(p)$ its projection onto V_p. Applying (3.17) with $a^{1,0}$ in place of s and using the stability of M in $\mathbb{R}^4$, the result we get is the following

$$\int_M f^2q(a)dA \leq \int_M |\nabla f|^2|a^{1,0}|^2dA \leq |a|\int_M |\nabla f|^2dA, \tag{3.22}$$

where q is the following expression:

$$q(a) = \frac{-2}{|F_z|^4}\mathrm{Re}\left\{(F_{zz}^{1,0}\cdot a)(F_{\overline{zz}}^{1,0}\cdot\overline{a})\right\}. \tag{3.23}$$

Take an orthonormal basks $\{a_1,\dots,a_4\}$ of $\mathbb{C}^4$, denote $q(a_j)$ by q_j and sum over j, we obtain

$$\sum_{j=1}^4 q_j = \frac{-2}{|F_z|^4}\mathrm{Re}\left\{F_{zz}^{1,0}\cdot F_{\overline{zz}}^{1,0}\right\} = 0 \tag{3.24}$$

Now by [FCS80], the inequality (3.22) with q_j in place of $q(a)$ implies the existence of a positive function u_j on M solving

$$\Delta u_j + q_ju_j = 0 \tag{3.25}$$

Letting $w_j = \log u_j$, an easy calculation shows that

$$\Delta w_j = -q_j - |\nabla w_j|^2.$$

Thus we get

$$\begin{aligned}\int_M (q_j + |\nabla w_j|^2)f^2 &= \int_M(-\Delta w_j)f^2 = 2\int_M f\nabla f\cdot\nabla w_j\\ &\leq \frac{1}{2}\int_M f^2|\nabla w_j|^2 + 2\int_M|\nabla f|^2,\end{aligned}$$

and therefore

$$\int_M (q_j + \frac{1}{2}|\nabla w_j|^2)f^2 \leq 2\int_M |\nabla f|^2$$

Summing over j and using (3.24), we deduce that

$$\frac{1}{2}\int_M\sum_{j=1}^4|\nabla w_j|^2f^2 \leq 8\int_M|\nabla f|^2.$$

Again by [FCS80], we get a positive function v such that

$$8\Delta v + \frac{1}{2}(\sum_{j=1}^{4} |\nabla w_j|^2) v = 0$$
$$\Rightarrow \Delta v \leq 0;$$

that is v is a positive superharmonic function. By the parabolicity of M, v must be a (nonzero) constant. Looking back at the PDE satisfied by v, we immediately deduce that $\sum_{j=1}^{4} |\nabla w_j|^2 = 0$, so each w_j, and hence each u_j, is constant. By (3.25), we see that each q_j is zero. Since the a_j's form a basis for $\mathbb{C}^4$, we conclude that

$$q(a) = \frac{-2}{|F_z|^4} \operatorname{Re}\left\{ (F_{zz}^{1,0} \cdot a)(F_{\overline{z}\overline{z}}^{1,0} \cdot \overline{a}) \right\} = 0, \text{ for all } a \in \mathbb{C}^4. \tag{3.26}$$

Now at a point p where $F_{\overline{z}\overline{z}}^{\perp}(p) \neq 0$, we can let $a = \frac{F_{\overline{z}\overline{z}}^{\perp}}{|F_{\overline{z}\overline{z}}^{\perp}|}$ and plug it into (3.26). Then we get

$$|F_{zz}^{1,0}(p)||F_{zz}^{0,1}(p)| = 0. \tag{3.27}$$

Thus at each $p \in M$, one of $F_{zz}^{1,0}(p)$ and $F_{zz}^{0,1}(p)$ must vanish. The fact that F is conformal and harmonic implies that $F_{zz}^{1,0} dz^2$ and $F_{zz}^{0,1} dz^2$ are holomorphic quadratic differentials with values in V and $\overline{V}$, respectively. Hence we conclude, by unique continuation, that either $F_{zz}^{1,0}$ or $F_{zz}^{0,1}$ vanishes identically. In the latter case, the proof is complete by invoking Proposition 3.8. In the former case we simply change the complex structure $J^{\perp}$ on NM. (Recall that we had a choice when constructing $J^{\perp}$. See the remarks before Proposition 3.7.) □

More or less the same argument establishes the same theorem in the compact setting of ambient flat 4-tori instead of $\mathbb{R}^4$.

Theorem 3.12. *Assume $F : M^2 \to T^4$ is an oriented, stable, compact minimal surface and that T^4 is a flat torus. Then F is J-holomorphic for some orthogonal complex structure J on T^4.*

Proof sketch. By arguing as in 3.11 we get $\lambda_0(\Delta + q_j) \geq 0$ on M for all $j \in \{1, 2, 3, 4\}$. Let $u_j = e^{w_j} > 0$ be the lowest eigenfunction so that, as before,

$$\int_M (q_j + \frac{1}{2}|\nabla w_j|^2) f^2 \leq 2 \int_M |\nabla f|^2 \quad \text{ for } j \in \{1, 2, 3, 4\}.$$

Summing over j and recalling the definition of the q_j we conclude

$$\frac{1}{2} \int_M \sum_j |\nabla w_j|^2 f^2 \leq 8 \int_M |\nabla f|^2.$$

Picking $f = 1$ (since M compact) we see that each w_j is constant, so each u_j is constant, so $q_j = \lambda_0(\Delta + q_j)$ is constant. Since the q_j sum to zero they must then all be zero and the result follows like before. □

In the proof of Theorem 3.11 we made use of identity (3.23) in (3.22). Let's justify that now:

Claim 3.13. *We can rewrite*

$$\frac{2}{|F_z|^2} \left[\frac{|a^{1,0} \cdot F_{zz}|^2}{|F_z|^2} + \operatorname{Re}\left(\overline{a^{1,0}} \cdot D_z D_{\overline{z}} a^{1,0} \right) \right] = -\frac{2}{|F_z|^4} \operatorname{Re}\left((F_{zz}^{1,0} \cdot a)(F_{\overline{z}\overline{z}}^{1,0} \cdot \overline{a}) \right)$$

where $a \in \mathbb{C}^4$, $|a| = 1$.

Proof of claim. Recall that $\varepsilon = \frac{1}{\sqrt{2}}(e_3 - ie_4)$ is such that $\{\varepsilon, \overline{\varepsilon}\}$ forms an orthonormal frame for $N_{\mathbb{C}}M = V \oplus \overline{V}$. Note that $\varepsilon \cdot \varepsilon = \overline{\varepsilon} \cdot \overline{\varepsilon} = 0$ and $\varepsilon \cdot \overline{\varepsilon} = 1$. By differentiating $a^{1,0} = (a \cdot \overline{\varepsilon})\varepsilon$ once and using the product rule,

$$\begin{aligned}
D_{\overline{z}}a^{1,0} &= \partial_{\overline{z}}(a \cdot \overline{\varepsilon})\varepsilon + (a \cdot \overline{\varepsilon})D_{\overline{z}}\varepsilon \\
&= (a \cdot (\partial_{\overline{z}}\overline{\varepsilon})^T)\varepsilon + (a \cdot D_{\overline{z}}\overline{\varepsilon})\varepsilon + (a \cdot \overline{\varepsilon})\big[(D_{\overline{z}}\varepsilon \cdot \varepsilon)\overline{\varepsilon} + (D_{\overline{z}}\varepsilon \cdot \overline{\varepsilon})\varepsilon\big] \\
&= (a \cdot (\partial_{\overline{z}}\overline{\varepsilon})^T)\varepsilon + \big[a \cdot \big((D_{\overline{z}}\overline{\varepsilon} \cdot \overline{\varepsilon})\varepsilon + (D_{\overline{z}}\overline{\varepsilon} \cdot \varepsilon)\overline{\varepsilon}\big)\big]\varepsilon + (a \cdot \overline{\varepsilon})\big[(D_{\overline{z}}\varepsilon \cdot \varepsilon)\overline{\varepsilon} + (D_{\overline{z}}\varepsilon \cdot \overline{\varepsilon})\varepsilon\big] \\
&= (a \cdot (\partial_{\overline{z}}\overline{\varepsilon})^T)\varepsilon + (a \cdot \overline{\varepsilon})(D_{\overline{z}}\overline{\varepsilon} \cdot \varepsilon)\varepsilon + (a \cdot \overline{\varepsilon})(D_{\overline{z}}\varepsilon \cdot \overline{\varepsilon})\varepsilon, \\
&\qquad\qquad \text{because } D_{\overline{z}}\overline{\varepsilon} \cdot \overline{\varepsilon} = D_{\overline{z}}\varepsilon \cdot \varepsilon = 0, \text{ as } \overline{\varepsilon} \cdot \overline{\varepsilon} = \varepsilon \cdot \varepsilon = 0 \\
&= (a \cdot (\partial_{\overline{z}}\overline{\varepsilon})^T)\varepsilon, \\
&\qquad\qquad \text{because } \varepsilon \cdot \overline{\varepsilon} = 1 \\
&= \big[a \cdot \big((\partial_{\overline{z}}\overline{\varepsilon} \cdot (F_z/|F_z|^2))F_{\overline{z}} + (\partial_{\overline{z}}\overline{\varepsilon} \cdot (F_{\overline{z}}/|F_z|^2))F_z\big)\big]\varepsilon \\
&= -\left(a \cdot \frac{F_z}{|F_z|^2}\right)(\overline{\varepsilon} \cdot F_{\overline{z}\overline{z}})\,\varepsilon. \qquad (3.28)
\end{aligned}$$

where the last equality follows from the product rule and minimality, $F_{z\overline{z}} = 0$. We will differentiate again in z, but before doing so, first we observe that

$$\begin{aligned}
\partial_z\left(\frac{F_z}{|F_z|^2}\right) \cdot F_z = 0 &\quad \text{by conformality, } F_z \cdot F_z = 0, \text{ and} \\
\partial_z\left(\frac{F_z}{|F_z|^2}\right) \cdot F_{\overline{z}} = 0 &\quad \text{by minimality, } F_{z\overline{z}} = 0.
\end{aligned}$$

Consequently, $\partial_z(F_z/|F_z|^2)$ is purely normal and thus

$$\partial_z\left(\frac{F_z}{|F_z|^2}\right) = \frac{F_{zz}^{\perp}}{|F_z|^2}.$$

Plugging this into (3.28) and exploiting similar cancelations among the derivatives of ε, $\overline{\varepsilon}$, we get

$$D_z D_{\overline{z}} a^{1,0} = -\left(a \cdot \frac{F_{zz}^{\perp}}{|F_z|^2}\right)(\overline{\varepsilon} \cdot F_{\overline{z}\overline{z}})\,\varepsilon = -\left(a \cdot \frac{F_{zz}^{\perp}}{|F_z|^2}\right)F_{\overline{z}\overline{z}}^{1,0}$$

and

$$\overline{a^{1,0}} \cdot D_z D_{\overline{z}} a^{1,0} = -\left(a \cdot \frac{F_{zz}^{\perp}}{|F_z|^2}\right)(\overline{a^{1,0}} \cdot F_{\overline{z}\overline{z}}^{1,0}) = -\left(a \cdot \frac{F_{zz}^{\perp}}{|F_z|^2}\right)(\overline{a} \cdot F_{\overline{z}\overline{z}}^{1,0})$$

by replacing $\overline{a^{1,0}}$ with $\overline{a}$ in the dot product with $F_{\overline{z}\overline{z}}^{1,0}$. By replacing $F_{zz}^{\perp} = F_{zz}^{1,0} + F_{zz}^{0,1}$ and then using $\overline{F_{zz}^{0,1}} = F_{\overline{z}\overline{z}}^{1,0}$ we get

$$\begin{aligned}
\overline{a^{1,0}} \cdot D_z D_{\overline{z}} a^{1,0} &= -\frac{a \cdot F_{zz}^{1,0}}{|F_z|^2}(\overline{a} \cdot F_{\overline{z}\overline{z}}^{1,0}) - \frac{a \cdot F_{zz}^{0,1}}{|F_z|^2}(\overline{a} \cdot F_{\overline{z}\overline{z}}^{1,0}) \\
&= -\frac{a \cdot F_{zz}^{1,0}}{|F_z|^2}(\overline{a} \cdot F_{\overline{z}\overline{z}}^{1,0}) - \frac{|a \cdot F_{zz}^{0,1}|^2}{|F_z|^2} \\
&= -\frac{a \cdot F_{zz}^{1,0}}{|F_z|^2}(\overline{a} \cdot F_{\overline{z}\overline{z}}^{1,0}) - \frac{|a^{1,0} \cdot F_{zz}|^2}{|F_z|^2}.
\end{aligned}$$

From this we conclude

$$\operatorname{Re}\left(\overline{a^{1,0}}\cdot D_z D_{\overline{z}} a^{1,0} + \frac{|a^{1,0}\cdot F_{zz}|^2}{|F_z|^2}\right) = -\frac{1}{|F_z|^2}\operatorname{Re}\left((a\cdot F_{zz}^{1,0})(\overline{a}\cdot F_{\overline{zz}}^{1,0})\right)$$

which gives the required result. □

3.3. **Stable minimal genus-0 surfaces in $\mathbb{R}^n$, $n \geq 5$.** We now try to see what we can prove when $\mathbb{R}^4$ (or T^4) is replaced by $\mathbb{R}^n$, $n \geq 5$. We will show that:

Theorem 3.14. *Let $F : M^2 \to \mathbb{R}^n$, $n \geq 5$, with M complete, oriented, stable, genus 0, and finite total curvature. Then there exists an affine subspace $A^{2k} \subset \mathbb{R}^n$ such that F is J-holomorphic for some J.*

Remark 3.15. The requirement of finite total curvature might appear to be too strong but in fact it isn't. One can check using Gauss-Bonnet that, provided F is proper,

$$\text{quadratic area growth, } |\chi(M)| < \infty \Leftrightarrow \text{finite total curvature.}$$

We will appeal to a theorem by Chern and Osserman [CO67]:

Theorem 3.16 ([CO67])**.** *Suppose $M^2 \subset \mathbb{R}^n$ is a complete orientable minimal surface with finite total curvature, i.e.,*

$$\int_M (-K)\, dA < \infty.$$

Then M is conformally equivalent to a punctured compact surface $\hat{M}$ and the Gauss map extends through the punctures, meaning $p \mapsto T_pM$, N_pM extend smoothly to $\hat{M}$.

We will also need the following consequence of the stability inequality

$$2\int_M \left[f^2 \frac{|(\partial_z s)^T|^2}{|F_z|^2} + \frac{f^2}{|F_z|^2}\operatorname{Re}\left(\overline{s}\cdot D_z D_{\overline{z}} s\right)\right] dA \leq \int_M \int |\nabla f|^2\, |s|^2\, dA, \tag{3.29}$$

for all $f \in C_c^\infty(M)$.

Lemma 3.17. *Suppose M is complete, oriented, stable, parabolic, and that s is a bounded section of $N_{\mathbb{C}}M$ with $D_{\overline{z}} s = 0$. Then $(\partial_z s)^T = 0$.*

Proof. From (3.29) with $D_{\overline{z}} s = 0$ and $|s|$ bounded we conclude that

$$\int_M f^2 \frac{|(\partial_z s)^T|^2}{|F_z|^2}\, dA \leq c\int_M |\nabla f|^2\, dA$$

for all f, so by elliptic theory there exists $u > 0$ with

$$\Delta u + \frac{|(\partial_z s)^T|^2}{c|F_z|^2} u = 0.$$

By parabolicity u needs to be constant, and therefore $(\partial_z s)^T = 0$. □

Proof of Theorem 3.14. By invoking the Chern-Osserman theorem we can construct a complex $(n-2)$-plane bundle $E \to \hat{M}$ extending $N_{\mathbb{C}}M$, where $\hat{M} \approx S^2$ in view of our genus 0 assumption, and we can also extend the connection D from before to a connection on E.

By [KM58] and the fact that $\dim M = 2$ it follows that E is a holomorphic vector bundle; i.e., for all $p \in S^2$ there exists a local basis $s_1, \ldots, s_{n-2}$ of E which is holomorphic ($D_{\overline{z}} s_j = 0$). By [Gro57], the holomorphic vector bundle E necessarily decomposes as a direct sum

$$E = (L_1 \oplus \cdots \oplus L_p) \oplus (L_{p+1} \oplus \cdots \oplus L_r) \oplus (L_{r+1} \oplus \cdots \oplus L_{n-2})$$

of complex line bundles order so that:

(1) $L_1, \ldots, L_p$ have $c_1(L) > 0$,

(2) $L_{p+1}, \ldots, L_r$ have $c_1(L) = 0$, and
(3) $L_{r+1}, \ldots, L_{n-2}$ have $c_1(L) < 0$.

Roughly speaking, we will show that if there are no flat bundles then F is going to be J-holomorphic; conversely, flat bundles will correspond to direction of vanishing of the second fundamental form and will help determine the affine space A^{2k} from the statement of the theorem.

Seeing as to how E was initially constructed as a complexification of a real bundle, the real pairing $E_x \times E_x \to \mathbb{R}$, $(s_1, s_2) \mapsto s_1 \cdot s_2$, gives rise to a holomorphic isomorphism $E \cong E^*$. According to this isomorphism the signs of the first Chern classes flip and therefore our original decomposition has to have as many positive line bundles as it does negative ones; namely, $p = n - 2 - r$.

By definition of $c_1(\cdot)$, the bundles $L_1, \ldots L_r$ (whose first Chern class is non-negative) all admit nontrivial global holomorphic sections $s_1, \ldots, s_{n-2}$. Since S^2 is compact, these sections are additionally bounded. By Lemma 3.17 above, $(\partial_z s_j)^T = 0$ for all $j \in \{1, \ldots, r\}$.

There are two cases to consider. First, suppose that all L_i have $c_1 = 0$. Then $s_1, \ldots, s_{n-2}$ is a global basis of holomorphic sections which we have showed satisfy $(\partial_z s_j)^T = 0$ and therefore the second fundamental form of M vanishes:

$$s = \sum_j a_j s_j \Rightarrow -\frac{(s \cdot F_{\overline{z}z}^{\perp})F_{\overline{z}}}{|F_z|^2} = \frac{\partial_z s \cdot F_{\overline{z}}}{|F_z|^2} = (\partial_z s)^T = 0.$$

Therefore M is totally geodesic and we're done.

Now suppose that $p > 0$, $n - 2 - r = p > 0$. For convenience we set up the following table of index notation:

$$\begin{aligned} 1 \le \mu, \nu &\le p, & r + 1 \le a, b &\le n - 2, \\ 1 \le i, j &\le r, & p + 1 \le A, B &\le n - 2. \end{aligned}$$

In other words, indices μ, ν run over positive line bundles, i, j run over non-negative line bundles, and so on. We list some properties of $s_1, \ldots, s_{n-2}$ that we will need.

(1) $s_\mu \cdot s_j = 0$, since
$$\partial_{\overline{z}}(s_\mu \cdot s_j) = D_{\overline{z}} s_\mu \cdot s_j + s_\mu \cdot D_{\overline{z}} s_j = 0$$
because we know that our sections are holomorphic. Therefore $s_\mu \cdot s_j$ is a holomorphic function on S^2, thus constant. However, the section s_μ belongs to a positive line bundle and necessarily vanishes somewhere. The claim follows. As a consequence, we get:

$$\operatorname{span}\{L_1, \ldots, L_r\}^\perp = \operatorname{span}\{\overline{L}_1, \ldots, \overline{L}_p\}. \tag{3.30}$$

(2) $\partial_z s_j \cdot s_k = 0$, since
$$\begin{aligned} \partial_{\overline{z}}(\partial_z s_j \cdot s_k) &= \partial_{\overline{z}} \partial_z s_j \cdot s_k + \partial_z s_j \cdot \partial_{\overline{z}} s_k \\ &= \partial_z(\partial_{\overline{z}} s_j \cdot s_k) - \partial_{\overline{z}} s_j \cdot \partial_z s_k + \partial_z s_j \cdot \partial_{\overline{z}} s_k. \end{aligned}$$

Since s_j is holomorphic, $\partial_{\overline{z}} s_j$ is purely tangential so the first term drops out by orthogonality. Next, s_k is a bounded holomorphic section so by Lemma 3.17 (which relies on stability), $\partial_z s_k$ is purely normal, the second term drops out by orthogonality. The same goes for the third term. Therefore the expression above vanishes, so $\partial_z s_j \cdot s_k dz$ is a holomorphic 1-form. The claim follows since Riemann-Roch forces such a differential to vanish identically. From this it follows that $\partial_z s_j \in \operatorname{span}\{L_1, \ldots, L_p\}$, and since $(\partial_z s_j \cdot s_k)^T = 0$ by stability (Lemma 3.17), we get

$$\partial_z s_j, D_z s_j \in \operatorname{span}\{L_1, \ldots, L_p\}. \tag{3.31}$$

Now we check the following

Claim 3.18. *The bundle $\xi = L_1 \oplus \cdots \oplus L_r \oplus (T_{\mathbb{C}}M)^{1,0}$ is parallel.*

Proof of claim. By Proposition 3.7 we need to check that ∂_z, $\partial_{\overline{z}}$ map $\Gamma(\xi)$ into itself. By linearity this amounts to showing

$$\partial_z s_j, \partial_{\overline{z}} s_j, \partial_z F_z, \partial_{\overline{z}} F_z \in \Gamma(\xi).$$

By minimality $\partial_{\overline{z}} F_z \in \Gamma(\xi)$ is clear, while $\partial_z s_j \in \Gamma(\xi)$ is just (3.31). For the other two cases we compute

$$\begin{aligned}
\partial_{\overline{z}} s_j &= (\partial_{\overline{z}} s_j)^T \qquad \text{because } D_{\overline{z}} s_j = 0 \text{ by holomorphicity} \\
&= \left(\partial_{\overline{z}} s_j \cdot \frac{F_z}{|F_z|^2}\right) F_{\overline{z}} + \left(\partial_{\overline{z}} s_j \cdot \frac{F_{\overline{z}}}{|F_z|^2}\right) F_z \\
&= \left(\partial_{\overline{z}} s_j \cdot \frac{F_{\overline{z}}}{|F_z|^2}\right) F_z,
\end{aligned}$$

the last equality following from minimality, $F_{z\overline{z}} = 0$, and therefore $\partial_{\overline{z}} s_j \in \Gamma(\xi)$. Likewise we find

$$\begin{aligned}
\partial_z F_z &= \left(\partial_z F_z \cdot \frac{F_z}{|F_z|^2}\right) F_{\overline{z}} + \left(\partial_z F_z \cdot \frac{F_{\overline{z}}}{|F_z|^2}\right) F_z + (\partial_z F_z)^\perp \\
&= \left(\partial_z F_z \cdot \frac{F_{\overline{z}}}{|F_z|^2}\right) F_z + (\partial_z F_z)^\perp,
\end{aligned}$$

seeing as to how the first term drops out in view of conformality, $F_z \cdot F_z = 0$. Now we observe $\partial_z F_z \cdot s_k = -F_z \cdot \partial_z s_k = 0$ by stability, and we conclude $\partial_z F_z \in \Gamma(\xi)$. This completes the proof of the claim. $\square$

Next we check the following

Claim 3.19. *We have* $\dim(\xi \cap \overline{\xi}) = r - p$.

Proof of claim. Recall that $\xi = L_1 \oplus \cdots \oplus L_r \oplus (T_{\mathbb{C}} M)^{1,0}$. For brevity write $V = L_1 \oplus \cdots \oplus L_r$, so that $\xi = V \oplus (T_{\mathbb{C}} M)^{1,0}$. From (3.30) we see that $V^\perp \subset V$, so $\operatorname{span}\{\xi, \overline{\xi}\} = \mathbb{C}^n$. Observe that

$$n = \dim \mathbb{C}^n = \dim \operatorname{span}\{\xi, \overline{\xi}\} = 2r + 2 - \dim(\xi \cap \overline{\xi})$$

which gives $\dim(\xi \cap \overline{\xi}) = 2r + 2 - n$. From the decomposition of E into line bundles by Grothendieck's theorem we further have $p + r = n - 2 \Leftrightarrow r = n - 2 - p$. Combining these two relations we conclude

$$\dim(\xi \cap \overline{\xi}) = 2(n - 2 - p) + 2 - n = n - 2 - 2p = r - p$$

which is the required result. $\square$

The proof of the theorem is now completed via the following sequence of steps:

(1) Since ξ is parallel, let's write $\xi = M \times \Lambda$ for a complex $(r+1)$-dimensional vector space Λ. Notice that the complex $(r-p)$-dimensional vector space $T = \Lambda \cap \overline{\Lambda}$ is (by definition) preserved by complex conjugation and therefore the complexification $W \otimes_{\mathbb{R}} \mathbb{C}$ of a real $(r-p)$-dimensional vector space W.

(2) Seeing as to how $(T_{\mathbb{C}} M)^{1,0}$ is manifestly not preserved by conjugation we get that $M \times W$ is a parallel subbundle of the real normal bundle NM, or in other words, that $\Sigma = F(M)$ is a subset of an affine subspace $P \subset \mathbb{R}^n$ perpendicular to W, the dimension of which is evidently $\dim \mathbb{R}^n - \dim W = n - (r - p) = n - r + p = 2p + 2$. That is, we have constructed an affine subspace $P^{2p+2} \subset \mathbb{R}^n$ that contains the surface Σ.

(3) From the decomposition $N_{\mathbb{C}} M = (M \times T) \oplus (M \times T)^\perp$, the $\perp$ being taken within $N_{\mathbb{C}} M$ of course, we characterize $(M \times T)^\perp$ as the complexified normal bundle of Σ viewed as a surface within P^{2p+2}.

(4) From (3.30) we find that $(M \times T)^{\perp} \subset L_1 \oplus \cdots \oplus L_p \oplus \overline{L}_1 \oplus \cdots \oplus \overline{L}_p$ and, by dimension counting, this inclusion is actually an exact equality. Namely,

$$(M \times T)^{\perp} = L_1 \oplus \cdots L_p \oplus \overline{L}_1 \oplus \cdots \oplus \overline{L}_p.$$

(5) By restricting to the context $\Sigma^2 \subset P^{2p+2}$ and the parallel nature of ξ we find that there exists a constant almost complex structure on P^{2p+2} with respect to which Σ^2 is J-holomorphic as in the proof of Theorem 3.11.

□

4. Positive isotropic curvature

We'll see that a number of the techniques developed in the previous section will extend to non-flat ambient spaces and thereby give important geometric consequences. Instead of studying the second variation operator for area, however, we will study the second variation operator for energy:

$$\mathrm{E}(F) = \int_{\Sigma} |dF|_h^2 \, dA_h$$

where $F : \Sigma^2 \to (M^n, g)$. For the purposes of computing the energy integral, the Riemann surface Σ^2 is thought of as being a Riemannian manifold (Σ^2, h), though the Dirichlet energy is conformally invariant as we have seen before.

By a computation similar to that for second variation of area, we find:

Proposition 4.1. *If $X \in \Gamma(F^*(TM))$ and F is a critical point for the energy functional, then*

$$\frac{1}{2}\delta^2 E(X, X) = \int_{\Sigma} |\nabla X|^2 - \sum_{i=1}^{2} R(e_i, X, e_i, X) \, dA_h.$$

Remark 4.2. This is reminiscent of the formula for the second variation of energy on geodesics $\gamma \subset M$,

$$\frac{1}{2}\delta^2 E(X, X) = \int_{\gamma} |\nabla_{\gamma'} X|^2 - R(\gamma', X, \gamma', X) \, ds.$$

We will complexify the (ambient) tangent bundle and the stability operator like we did before. For $X \in \Gamma(F^*(T_{\mathbb{C}}M))$ of the form $X = X_1 + iX_2$, we define

$$\frac{1}{2}\delta^2 E(X, \overline{X}) = \int_{\Sigma} \langle \nabla X, \nabla \overline{X} \rangle - \sum_{i=1}^{2} R(e_i, X, e_i, \overline{X}) \, dA_h,$$

and by arguing as in Proposition 3.4 we get:

Proposition 4.3. *If $X \in \Gamma(F^*(T_{\mathbb{C}}M))$ and F is a critical point for the energy functional, then in complex coordinates $z = x + iy$ we have*

$$\frac{1}{8}\delta^2 E(X, \overline{X}) = \int_{\Sigma} |\nabla_{\overline{z}} X|^2 - R(\partial_z, X, \partial_{\overline{z}}, \overline{X}) \, dx \, dy.$$

Remark 4.4. In general variations of energy and area behave differently. Critical points of the prior are harmonic maps, and critical points of the latter are minimal surfaces. (Recall that we've seen that these coincide on a round S^2.) The second variation of energy and the second variation of area behave differently, too. The stability operator for energy is easier to work with since it has one less term in it but is also coarser–for example, every harmonic map into flat space is clearly stable.

It is important to be able to understand the effect of curvature on stabiliity. In the context of the area functional, we know that positive curvature gives rise to instability. Likewise, we can force instability in Proposition 4.3 provided we can construct global holomorphic sections X and that the complex sectional curvatures $R(\partial_z, X, \partial_{\overline{z}}, \overline{X})$ are positive. This section aims to pursue these ideas further.

Let's set up our notation. Recall that M^n is a real Riemannian manifold with real metric $\langle \cdot, \cdot \rangle$. We complexify $T_{\mathbb{C}} M = TM \otimes_{\mathbb{R}} \mathbb{C}$ and extend $\langle \cdot, \cdot \rangle$ to $T_{\mathbb{C}} M$, mimicking the extension of the dot product $X \cdot Y$ on $\mathbb{R}^n$ to a dot product on $\mathbb{C}^n$. Namely, for $X = X_1 + iX_2 \in T_{\mathbb{C}} M$ we set

$$\langle X, X \rangle = \langle X_1, X_1 \rangle - \langle X_2, X_2 \rangle + 2i \langle X_1, X_2 \rangle.$$

Notice that this is *not* a Hermitian metric, just symmetric and bilinear over $\mathbb{C}$.

Definition 4.5. A vector $X \in T_{\mathbb{C}} M$ is called **isotropic** if $\langle X, X \rangle = 0$; i.e., if $|X_1| = |X_2|$ and $\langle X_1, X_2 \rangle = 0$. A plane $\Pi^2 \subset T_{\mathbb{C}} M$ is isotropic if every $X \in \Pi$ is isotropic.

Example 4.6. *If F is conformal, then $F_z = dF(\partial_z)$ is isotropic. We made extended use of this fact in the previous section.*

Lemma 4.7. *If $\Pi^2 \subset T_{\mathbb{C}} M$ is isotropic then there exist real vectors $e_1, e_2, e_3, e_4 \in TM$, orthonormal with respect to the real metric, such that*

$$\Pi^2 = \operatorname{span}\{e_1 + ie_2, e_3 + ie_4\}.$$

Proof. The pairing $(X, Y) = \langle X, \overline{Y} \rangle$ is Hermitian, and by Gram-Schmidt over $\mathbb{C}$ we may arrange for a basis X, Y of Π^2 to be such that $(X, X) = (Y, Y) = 1$ and $(X, Y) = 0$. Write $X = \frac{1}{\sqrt{2}}(e_1 + ie_2)$, $Y = \frac{1}{\sqrt{2}}(e_3 + ie_4)$. We make the following observations:

(1) The isotropy of X and Y and the fact that $\langle X, \overline{X} \rangle = \langle Y, \overline{Y} \rangle = 1$ together give
$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = \langle e_4, e_4 \rangle = 1,$$
$$\text{and } \langle e_1, e_2 \rangle = \langle e_3, e_4 \rangle = 0.$$
(2) The isotropy of $X + Y = \frac{1}{\sqrt{2}}(e_1 + e_3 + i(e_2 + e_4))$ gives
$$\langle e_1 + e_3, e_1 + e_3 \rangle = \langle e_2 + e_4, e_2 + e_4 \rangle \Leftrightarrow \langle e_1, e_3 \rangle = \langle e_2, e_4 \rangle,$$
$$\text{and } 0 = \langle e_1 + e_3, e_2 + e_4 \rangle = \langle e_1, e_4 \rangle + \langle e_2, e_3 \rangle.$$
(3) The complex orthogonality $\langle X, \overline{Y} \rangle = 0$ gives
$$0 = \langle e_1 + ie_2, e_3 - ie_4 \rangle = \langle e_1, e_3 \rangle + \langle e_2, e_4 \rangle + i[\langle e_2, e_3 \rangle - \langle e_1, e_4 \rangle].$$

These facts put together show that e_1, e_2, e_3, e_4 are real orthonormal vectors. ☐

Definition 4.8. A (real) Riemannian manifold (M^n, g) is called PIC (short of positive isotropic curvature, or originally positive curvature on totally isotropic 2-planes) if every isotropic 2-plane $\Pi^2 \subset T_{\mathbb{C}} M$ and every complex orthonormal basis X, Y for Π satisfy $R(X, Y, \overline{X}, \overline{Y}) > 0$.

Remark 4.9. Just for the sake of comparison, we recall that a Riemannian manifold is said to have positive (sectional) curvature if $R(X, Y, X, Y) > 0$ for every real orthonormal basis X, Y of every real 2-plane $\Pi^2 \subset TM$.

We make the following observations regarding the definition of PIC:

(1) PIC manifolds are *not* necessarily Ricci positive. In particular, round products $S^1 \times S^{n-1}$ are always PIC but not Ricci positive.
(2) We can perturb the spherical metrics above in such a way that $S^1 \times S^{n-1}$ is still PIC and yet has negative Ricci curvature somewhere.
(3) PIC manifolds are always scalar positive.

(4) All 2- and 3-manifolds are vacuously PIC, because they have no isotropic complex 2-planes, since isotropic subspaces can be checked to take up no more than half the total dimension of their ambient vector space.

There are a number of interesting PIC manifolds:

Theorem 4.10 ([MM88]). *The following manifolds are PIC:*

(1) (M^n, g) *with positive curvature operator* $\mathscr{R} : \Lambda^2 TM \to \Lambda^2 TM$*; i.e.,* $\langle \mathscr{R}(\xi), \xi \rangle > 0$ *for all* $\xi \in \Lambda^2 T_pM \setminus \{0\}$.[1] *In fact, it's enough for* $\mathscr{R}$ *to be (2,2)-positive, i.e. the positivity condition be met for 2-vectors* ξ *with tensor-rank at most 2.*
(2) (M^n, g) *with positive pointwise strictly* $\frac{1}{4}$*-pinched curvature; i.e., there exists a continuous* $\kappa : M \to (0, \infty)$ *such that* $\frac{1}{4}\kappa(p) < K_\Pi \leq \kappa(p)$ *for all* $\Pi^2 \subset T_pM$.

The following theorem by Micallef and Wang shows that the class of PIC manifolds is rich enough to support connected sums.

Theorem 4.11 ([MW93]). *If* (M_1^n, g_1), (M_2^n, g_2) *have isotropic curvatures bounded from below by a positive constant (e.g., if they are compact and PIC), then* $M_1^n \# M_2^n$ *supports a PIC metric.*

We proceed by proving the fact that a manifold is PIC if the curvature operator is positive definite or it's 1/4-pinched.

Proof. By previous lemma any isotropic plane Π is spanned by vectors X, Y with

$$X = \frac{1}{2}e_1 + ie_2, \quad Y = \frac{1}{2}(e_3 + ie_4).$$

Then the complexified curvature is given by

$$R(X, Y, \bar{X}, \bar{Y}) = \mathscr{R}(X \wedge Y, \bar{X} \wedge \bar{Y}) = \frac{1}{4}\mathscr{R}((e_1 + ie_2) \wedge (e_3 + ie_4), (e_1 - ie_2) \wedge (e_3 - ie_4)).$$

So

$$K(\Pi) = \frac{1}{4}\left[\mathscr{R}(e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4) + \mathscr{R}(e_1 \wedge e_4 + e_2 \wedge e_3, e_1 \wedge e_4 + e_2 \wedge e_3)\right] > 0,$$

if the curvature operator $\mathscr{R}$ is positive definite. We also see that it suffices to require the curvature operator is positive operator on the sum of 2 wedges, which is called $(2, 2)$ positive by Michallef-Moore.

We next prove pointwise strict 1/4-pinching condition implies PIC.

We fix a point on the manifold and let $e_1, \ldots, e_4$ be 4 orthonormal vectors in the tangent space. Further expanding the above equation, we have

$$\begin{aligned} K(\Pi) &= \frac{1}{4}\left[K_{13} + K_{24} - 2R(e_2, e_4, e_1, e_3) + K_{23} + K_{14} + 2R(e_1, e_4, e_2, e_3)\right] \\ &= \frac{1}{4}(K_{13} + K_{24} + K_{14} + K_{23} - 2R_{1234}). \end{aligned}$$

Here we've used the first Bianchi identity.

We conclude the proof by the following property of 1/4-pinched manifold.

Proposition 4.12. *If* $p \in M$ *and* $k(p) > 0$ *such that* $\frac{1}{4}k(p) < K(\Pi) \leq k(p)$ *for all two-plane* $\Pi \subset T_pM$*, then* $|R_{1234}| < \frac{1}{2}k(p)$.

The proof is straightforward consequence of the following two identities. Let u, v, w, x be 4 orthonormal vectors in T_pM.

$$(i) \quad 4R(u, v, w, v) = R(u + w, v, u + w, v) - R(u - w, v, u - w, v)$$

[1] Recall that if $\xi = u \wedge v$ is a simple 2-vector, with u, v orthonormal, then $\langle \mathscr{R}(u \wedge v), u \wedge v \rangle = R(u, v, u, v) = K_{\mathrm{span}\{u,v\}}$.

$$\begin{aligned}(ii) \quad 6R(u,v,w,x) = &R(u,v+x,w,v+x) - R(u,v-x,u,v-x)\\ &- R(v,u+x,w,u+x) + R(v,u-x,w,u-x).\end{aligned}$$

From (i) we know $4|R(u,v,w,v)|$ is bounded above by $2k(p) - 2\cdot\frac{1}{4}k(p) = \frac{3}{2}k(p)$, so $|R(u,v,w,v)| < \frac{3}{8}k(p)$. From (ii) we conclude $6|R(u,v,w,x)|$ is bounded by $4\cdot 2\cdot\frac{3}{8}k(p)$, so $|R(u,v,w,x)| < \frac{1}{2}k(p)$. □

4.1. **High homotopy groups of PIC manifolds.** We are now ready to state a beautiful theorem of Micallef-Moore on high homotopy groups of PIC manifolds. The proof of this result relies on the complexified second variation of energy functional.

Theorem 4.13 ([MM88])**.** *If M^n is compact and PIC then $\pi_2(M) = \ldots = \pi_{[\frac{n}{2}]}(M) = \{0\}$.*

Corollary 4.14. *If M^n is a compact, simply-connected PIC manifold then M is homeomorphic to S^n.*

Proof. From theorem we see that $\pi_1(M) = \ldots = \pi_{[\frac{n}{2}]}(M) = \{0\}$. By Hurwicz theorem we know the corresponding homology groups with real coefficients $H_1(M) = \ldots = H_{[\frac{n}{2}]}(M) = \{0\}$. By Poincar'e duality we conclude that every $H_j(M)$ with $0 < j < n$ is trivial. So M^n is homotopic sphere. By validity of Poincare conjecture M^n is homeomorphic to S^n. □

Proof. The proof of this theorem contains two parts.

(1) Existence theorem. If $\pi_k(M) \neq \{0\}$ then there exists a nonconstant harmonic map $F : S^2 \to M^n$ with Morse index of F is at most $k-2$.
(2) Index estimate. If M^n is PIC and $F : S^2 \to M$ is a nonconstant harmonic map, then the Morse index of F is at least $[\frac{n}{2}] - 1$.

Combing these two facts, if M^n is PIC, $k \geq 2$ and $\pi_k(M) \neq \{0\}$, then we find a nonconstant harmonic map $F : S^2 \to M$ with $\text{index}(F) \leq k-2$. On the other hand, since M is PIC we know $\text{index}(F) \geq [\frac{n}{2}] - 1$. This gives $k - 2 \geq [\frac{n}{2}] - 1$, which means $k \geq [\frac{n}{2}] + 1$.

We quote the existence part from chapter 1 of our notes. Now we focus on index estimate.

Suppose $F : S^2 \to M$ is a nonconstant harmonic map. Then the complexified second variation of energy functional is given by

$$\frac{1}{8}\delta^2 E(X,\bar{X}) = I(X,\bar{X}) = \int_{S^2} \left[|\nabla_{\bar{z}} X|^2 - R(F_z, X, F_{\bar{z}}, \bar{X})\right] dxdy.$$

Where $X \in \Gamma(F^*(N_{\mathbb{C}}M))$.

Now the index form is real, and the complexified index form is the Hermitian extension of its real form, so we have

$$\text{index} \geq \min\{\dim_{\mathbb{C}} V : V \subset \Gamma(F^*(N_{\mathbb{C}}M)), I < 0 \text{ on } V\}.$$

Note that whenever we have a holomorphic isotropic section X of the pullback bundle, naturally we have $I(X,\bar{X}) < 0$. The proof is done by constructing a large family of holomorphic isotropic sections.

Claim 4.15. *There exists a subspace $W \subset \Gamma(F^*(N_{\mathbb{C}}M))$, such that $\forall X \in W$, $\nabla_{\bar{z}} X = 0, \langle X, X\rangle = 0$ and $\dim(W) \geq [\frac{n}{2}]$.*

Clearly F_z is in W by harmonicity of F. So the compliment of F_z in W gives a $[\frac{n}{2}]-1$ dimensional subspace of holomorphic isotropic sections as the theorem infers.

Denote $E = F^*(N_{\mathbb{C}}M)$. As before, E is a Hermitian bundle over S^2, so the extended connection is automatically holomorphic. Again by [Gro57], E splits into line bundles, listed in decreasing order of first Chern class:

$$E = (L_1 \oplus \cdots \oplus L_p) \oplus (L_{p+1} \oplus \cdots \oplus L_r) \oplus (L_{r+1} \oplus \cdots \oplus L_{n-2}).$$

Where $L_1, \ldots, L_p$ have positive first Chern class, $L_{p+1}, \ldots, L_r$ have zero first Chern class, and $L_{r+1}, \ldots, L_{n-2}$ have negative first Chern class. Choose a section $s_j \in \Gamma(L_j)$ for $j = 1, \ldots, p$. This is always possible since the first Chern class is positive. Also we know that s_j must vanish somewhere. The complex linear pairing $\langle s_j, s_k \rangle$ for $1 \leq j, k \leq p$ is then a holomorphic function on S^2 and vanishes somewhere, so $\langle s_j, s_k \rangle = 0$ everywhere on S^2. Hence we conclude

$$\mathscr{P} = \operatorname{span}\{s_1, \ldots, s_p\}$$

is a totally isotropic p-dimensional subspace of $\Gamma(E)$.

Now consider $F = L_{p+1} \oplus \ldots \oplus L_r$. On each L_q, $p+1 \leq q \leq r$, there is also a section s_q. But now $c_1(L_q) = 0$ so s_q does not necessarily vanish somewhere. However, the complex linearly extended pairing $\langle \cdot, \cdot \rangle$ defines an isomorphism $E \to E^*$. In this isomorphism, line bundles with positive and negative first Chern class map to one another, hence F maps to itself. That's to say, $\langle \cdot, \cdot \rangle$ defines a **non-degenerate** bilinear form $F \to F$. Hence we are able to take an orthonormal basis $s_{p+1}, \ldots, s_r$ of sections of F such that $\langle s_q, s_t \rangle = \delta_{qt}$. Then the following $[\frac{n}{2}] - p$ sections

$$e_1 = s_{p+1} + i s_{p+2}, \quad e_2 = s_{p+3} + i s_{p+4}, \ldots$$

is a totally isotropic collection of sections. Denote $F_0 = \operatorname{span}\{e_1, \ldots, e_{[\frac{n}{2}]-p}\}$.

Let $W = \mathscr{P} \oplus F_0$. Then we claim W is a totally isotropic subspace. Indeed, any section s_j of positive line bundle and s_q of zero line bundle must satisfy $\langle s_j, s_q \rangle = 0$, since the pairing gives a holomorphic function on S^2 that vanishes somewhere. Furthermore, $\dim_{\mathbb{C}}(W) = \dim_{\mathbb{C}}(\mathscr{P}) + \dim_{\mathbb{C}}(F_0) = p + [\frac{n}{2}] - p = [\frac{n}{2}]$, as desired. This concludes the proof of the claim. □

Remark 4.16. This theorem puts essential obstruction on high homotopy groups for a manifold to carry a PIC metric. However, the fundamental group of a PIC manifold remains open. In fact, we know $S^1 \times S^3$, equipped with product metric, is strict PIC ($R_{1234} = 0$ for any orthonormal vectors $e_1, \ldots, e_4$). And by [MW93], the connected sum of two PIC manifolds supports a PIC metric. For example, we have $\pi_1((S^1 \times S^3) \# (S^1 \times S^3)) = F_2$, the free group with two generators. This observation leads to the following conjecture on the fundamental group of PIC manifolds.

Conjecture 4.17. *M is a PIC manifold. Then the fundamental group of M is virtually free. That is, there exists $F \subset \pi_1(M)$ with F being free and finite index.*

It is known this conjecture in PIC manifold is related to a more geometric statement of PIC manifold. The following geometric property implies the above conjecture.

Conjecture 4.18. *If M is κ-PIC, that is, for any isotropic plane Π, $K(\Pi) \geq \kappa > 0$, and $\Sigma^2 \subset M$ is a stable minimal disk, then for every $p \in \Sigma$, we have $d(p, \partial\Sigma) \leq c/\sqrt{\kappa}$.*

4.2. Fundamental group of PIC manifolds.

Previously we've shown all high homotopy groups of a PIC manifold must vanish. The fundamental group of a PIC manifold turns out to be quite different.

In low dimensional cases, all 2 or 3 dimensional manifolds are PIC since there is no isotropic plane. 4-dimensional PIC manifolds have been completely classified by Hamilton and Chen-Zhu using Ricci flow, since the PIC condition is preserved under Ricci flow. The following result is the best known about the fundamental group of a high dimensional PIC manifold till today, with a proof that also comes from a variational approach.

Theorem 4.19 ([Fra03])**.** *Suppose $n \geq 5$ and M^n is a PIC manifold. Then there is no free Abelian subgroup of $\pi_1(M)$ of rank greater than 1.*

Ideally from a variational point of view one may try the following type of argument. Assume $\mathbb{Z} \oplus \mathbb{Z} \in \pi_1(M)$, then there exists conformal minimal branched immersion $u : T^2 \to M$ such that $u_* : \pi_1(T^2) \to \mathbb{Z} \oplus \mathbb{Z}$ isomorphically. The question is, can we have stable tori in a PIC manifold?

Unfortunately the answer is yes. For example one may take the Cartesian product $S^1 \times S^n/\mathbb{Z}_p$, where $S^n/\mathbb{Z}_p$ is the lens space with positive constant curvature. Clearly this gives a PIC manifold. Then we may choose a shortest non-contractible geodesic γ in $S^3/\mathbb{Z}_p$ and $S^1 \times \gamma$ will be a stable tori.

Instead we are going to prove that if we take a sufficiently high degree cover of a stable tori, it becomes unstable. Before proceeding to proof, we first recall that the complexified second variation for energy functional implies for stable tori $F : \Sigma \to M$, we have

$$\int_\Sigma R(F_z, s, F_{\bar{z}}, \bar{s})dxdy \leq \int_\Sigma |\nabla_{\bar{z}} s|^2 dxdy, \quad \forall s \in \Gamma(E), E = F^*(N\Sigma \otimes \mathbb{C}).$$

If further s is isotropic, then by PIC condition we'll have

$$\kappa \int_\Sigma |s|^2 dA \leq \int_\Sigma |\nabla_{\bar{z}} s|^2 dA.$$

We argue as following that when Σ is 'large' enough this cannot happen.

Proof. The proof proceeds in a few steps. Step 1: For any $k\mathbb{Z} \oplus k\mathbb{Z} \subset \pi_1(M)$ there is a branched minimal immersion Σ_k representing $k\mathbb{Z} \oplus k\mathbb{Z}$.

Step 2: Suppose for now that for every $\epsilon > 0$, there is a sufficiently large k and a smooth map $f : \Sigma_k \to S^2$ satisfying $\deg f = 1$ and $|df| < \epsilon$.

We now use this ϵ-contracting map f to construct 'almost' holomorphic sections of the bundle E.

Definition 4.20. Let $\epsilon > 0$. A section $s \in \Gamma(E)$ is called ϵ-holomorphic if $\int_\Sigma |\nabla_{\bar{z}} s|^2 dA < \epsilon \int_\Sigma |s|^2 dA$.

A immediate consequence from the second variation formula is, if $\epsilon < \kappa$ then any ϵ-holomorphic isotropic section s must vanish.

Now for the holomorphic bundle E over T^2, since the complex linearly expanded pairing $(\cdot,\cdot)$ gives an isomorphism $E \to E^*$, we have $c_1(E) = 0$. Note that we are unable to obtain a section of E since Riemann-Roch theorem only guarantees a section when $c_1(E) > 0$. However if ξ is a line bundle over T^2 with $c_1(\xi) = 2$ then $c_1(\xi \otimes E) > 0$, hence we are able to get a bundle of $\xi \otimes E$.

Take a line bundle L over S^2 with $c_1(L) = 2$, and let $\xi = f^*(L)$. Then $c_1(\xi) = \deg(f)c_1(L) = 2$. Extend complex pairing $(\cdot,\cdot)$ to $(\xi\otimes E)\times(\xi\otimes E) \to \xi\otimes\xi$, denoted also by $(\cdot,\cdot)$, by $(t_1\otimes s_1, t_2\otimes s_2) = (s_1, s_2)t_1 \otimes t_2$. Also let $H(\xi \otimes E)$ be the space of holomorphic sections of $\xi \otimes E$. Then by Riemann-Roch,

$$\dim H(\xi \otimes E) \geq c_1(\det(\xi \otimes E)) = (n-2)c_1(\xi) + c_1(E) = 2n - 4.$$

Step 3: We are ready to find a holomorphic isotropic section of $\xi \otimes E$. By Riemann-Roch, if σ is a holomorphic section such that $(\sigma,\sigma) = 0$ at more than $2c_1(\xi) = 4$ points, then (σ,σ) is identically 0. Define, $H_x = \{\sigma \in H(\xi \otimes E) : (\sigma,\sigma)_x = 0\}$. Note that $\dim_{\mathbb{C}}(H(\xi \otimes E)) = d \geq 2n - 4$. Take 5 arbitrary points $x_1, \ldots, x_5$ on T^2. We want to argue the intersection $\cap_{j=1}^5 H_j$ is nonempty. Now each H_j is defined by a homogeneous degree 2 polynomial on $H(\xi \otimes E) \approx \mathbb{C}^d$, it can be viewed as a $(d-2)$ dimensional hypersurface in $\mathbb{C}P^{d-2}$. By intersection formula, we have

$$\dim \left(\cap_{j=1}^5\right) \geq d - 6 \geq 2n - 10 \geq 0, \quad \text{if } n \geq 5.$$

So there exists $\sigma \in \cap_{j=1}^5 H_j$. That means, $(\sigma,\sigma) \equiv 0$ in $\xi \otimes E$.

Step 4: From σ obtained above we construct almost holomorphic isotropic section s of E.

Notice if τ^* is a section of the due bundle ξ^* then $\tau^*(\sigma) = s$ is a section of E. Of course τ^* is not holomorphic and neither is s, but if we are able to construct τ^* through pull back by f of a section on L then by ϵ-contractibility of f the derivative of $s = \tau^*(\sigma)$ will be sufficiently small.

We look at the bundle L over S^2. Let U_+, U_- be small open neighborhoods of the south and north poles, and S_+, S_- be the southern and northern hemisphere. By contractibility of disk the bundle L^* over U_+, U_- is trivial. Take t_1^* be a trivialization of L^* in $S^2 - U_-$ such that $|t_1^*| = 1$ pointwisely on S_+. Then use cut-off function to extend t_1^* identically 0 on U_-. Similarly define t_2^*. Then we can find sections $t_1^*, t_2^* \in \Gamma(L^*)$ such that $1 \leq |t_1^*| + |t_2^*| \leq 2$ everywhere on S^2. Define, using the ϵ-contracting map f, $\tau_j = f^*(t_j^*)$, $j = 1, 2$. Then by the chain rule

$$|\nabla \tau_j^*| = |\nabla(t_j^* \circ f)| = |(\nabla f) \circ (\nabla t_j^*)| \leq C\epsilon.$$

Let $s_j = \tau_j^*(\sigma)$, $j = 1, 2$. Then we have

$$\begin{aligned} |s_1|^2 + |s_2|^2 &= |\tau_1^*(\sigma)|^2 + |\tau_2^*(\sigma)|^2 \\ &= (|\tau_1^*|^2 + |\tau_2^*|^2)|\sigma|^2 \\ &\geq |\sigma|^2. \end{aligned}$$

Therefore either $\int_\Sigma |s_1|^2 dA \geq \frac{1}{2}\int_\Sigma |\sigma|^2 dA$ or $\int_\Sigma |s_1|^2 dA \geq \frac{1}{2}\int_\Sigma |\sigma|^2 dA$ is true. We therefore get a section $s = s_1$ or s_2 with

$$\int_\Sigma |s|^2 dA \leq C\epsilon \int_\Sigma |\sigma|^2 dA \leq 2C \int_\Sigma |s|^2,$$

which concludes the proof. □

The same method shows

Theorem 4.21. *If Σ is a stable incompressible torus in κ-PIC manifold then a sufficiently high degree covering of Σ is unstable.*

Finally we prove the existence of the ϵ-contracting map f.

Theorem 4.22. *Given $u : T^2 \to M$ with $u_*(\pi_1(T^2)) = k\mathbb{Z} \oplus k\mathbb{Z}$. Then there exists $f : (T^2, u * g) \to S^2$ with $\deg(f) = 1$ and $|df| < \epsilon$ if k is sufficiently large.*

Proof. For each k, denote by Σ_k the preimage of u. Recall the systole of Σ_k is defined by the number

$$\mathscr{L} = \min\{L(\gamma) : \gamma \text{ is a noncontractible closed geodesic in } \Sigma_k\}.$$

Since M is compact it is routine to check that for k large enough the surface Σ_k has large systole, say, larger than L.

Look at the universal cover $\tilde{\Sigma}$ of Σ. Since $\tilde{\Sigma}$ is noncompact with compact quotient Σ, there is a geodesic line $r : \mathbb{R} \to \tilde{\Sigma}$. Choose T very large, $T >> L$ and define $D_1 : \tilde{\Sigma} \to \mathbb{R}$ by

$$D_1(x) = d(x, r(T)) - T.$$

And define $D_2(x)$ to be the signed distance function to r such that D_2 attains positive on one component of $\tilde{\Sigma} - r$ and negative on the other.

On the square region

$$\Omega = \{|D_1| < \frac{L}{4}, |D_2| < \frac{L}{4}\},$$

define $f : \Omega \to [-\frac{L}{4}, \frac{L}{4}] \times [-\frac{L}{4}, \frac{L}{4}]$ by $f = (D_1, D_2)$. Clearly f is a Lipschitz function, and with proper choice of T $|df| < 2$. Then the boundary of Ω is mapped into the boundary of the square of length $L/2$ in $\mathbb{R}^2$. Also $r(0)$ is the only point mapped to 0 in $\mathbb{R}^2$. Hence f is a local diffeomorphism, and in particular, degree 1 map from a neighborhood of $r(0)$ in Ω to one component of $\mathbb{R}^2 - f(\partial\Omega)$.

We then smooth f out to get a map $\tilde{f}$ from $\Omega \to B_{L/5}(0) \subset \mathbb{R}^2$, and compose $\tilde{f}$ with a contracting map which takes $B_{L/5}(0)$ to $B_\pi(0)$. Finally, glue $B_{L/5}(0)$ to a punctured sphere $S^2 - q$ and map the every point in the fundamental domain of Σ in $\tilde{\Sigma}$ to q, we obtain the desired map. □

5. Positive scalar curvature

5.1. Positive curvature obstructing stability.

Theorem 5.1 (J. Simons, [Sim68]). *There are no stable minimal submanifolds (of any codimension) in the round (S^n, g_0).*

Proof sketch. The idea is to think of S^n as being the unit sphere in $\mathbb{R}^{n+1}$ and then use the ambient Killing vector fields that represent isometries. If V_i, $i = 1, \ldots, n+1$, represents those ambient Killing fields and $X_i = (V_i)^t$ are their projections to the sphere, then one can show that on any minimal $\Sigma^k \subset S^n$ we have

$$\sum_{i=1}^{n+1} \delta^2\Sigma(X_i, X_i) < 0$$

and therefore there can be no stable minimal surfaces. □

This result was later improved to work under much weaker regularity assumptions by Lawson and Simons.

Theorem 5.2 (Lawson-Simons, [LS73]). *There are no stable stationary integral currents, mod p currents, or varifolds in the round sphere (S^n, g_0).*

Remark 5.3. Currents and varifolds don't come equipped with a normal bundle, so variations have to be considered in the ambient space and therefore stability is interpreted as the lack of ambient flows that decrease mass.

Li and Yau showed in [LY82] that, in the case $k = 2$, the flow ϕ_t generated by one of the vector fields X_i actually satisfies $|\phi_t(\Sigma)| < |\Sigma|$, $t \neq 0$, provided Σ is not entirely contained in any equator of S^n. Consequently, $\operatorname{index}(\Sigma^2) \geq n + 1$. El Soufi and Ilias handled the higher dimensional case in [ESI92].

There is a conjecture that aims to generalized these stability obstructions to 1/4-pinched manifolds.

Conjecture 5.4 (Lawson-Simons conjecture). Let (S^n, g) be 1/4-pinched; i.e., $1/4 < K \leq 1$. Then there exists no stable minimal $\Sigma^k \subset S^n$.

This is known to hold true in the following cases:

(1) when $\Sigma \approx S^2$, by Micallef-Moore, and
(2) when Σ is a hypersurface (i.e., codimension 1) in S^n (as we remark in the proposition below),

but is otherwise open, even when Σ is a general two dimensional surface.

Proposition 5.5. *There are no stable two-sided closed hypersurfaces in a manifold (M^n, g) with positive Ricci curvature.*

Proof. The second variation formula for a two-sided $\Sigma^{n-1} \subset M^n$ says that if ν is a unit normal field to Σ, and we vary Σ along the direction $X = \varphi\nu$, then

$$\delta^2\Sigma(\varphi, \varphi) = \int_\Sigma |\nabla\varphi|^2 - (\operatorname{Ric}(\nu, \nu) + |A|^2)\varphi^2 \, d\mu.$$

Therefore if $\operatorname{Ric} > 0$ and $\varphi \equiv 1$, we have

$$\delta^2\Sigma(1, 1) = \int_\Sigma -(\operatorname{Ric}(\nu, \nu), |A|^2) \, d\mu < 0,$$

so Σ is unstable. □

Notice that from this we get the following:

Corollary 5.6. *If (M^n, g) is compact,* $\mathrm{Ric} > 0$*, then* $H_{n-1}(M, \mathbb{Z}) = 0$.

Remark 5.7. This uses a hard result [**?**] on minimizing volume in homology classes. One should view this as a sort of counterpart of Bochner's theorem on the triviality of 1-dimensional cohomology of closed manifolds with positive Ricci curvature.

The next thing to wonder about, instead of stability, is the geometry or topology of index 1 hypersurfaces. In particular, from min-max theory we know that every Riemannian manifold of positive curvature has at least one index 1 hypersurface.

Question 5.8. Suppose $\Sigma^{n-1} \subset M^n$, $K > 0$, and that $\mathrm{index}(\Sigma) = 1$. Can we bound the first Betti number $b_1(\Sigma) \leq c(n)$?

This is known to hold true, for instance, in the following setting:

Theorem 5.9. *If (M^3, g) has positive Ricci curvature, then any $\Sigma^2 \subset M^3$ with index 1 has* $\mathrm{genus}(\Sigma) \leq 3$.

Remark 5.10. Conjecturally the genus bound $\mathrm{genus}(\Sigma) \leq 3$ can be improved to $\mathrm{genus}(\Sigma) \leq 2$, as the study of Heegard splittings of 3-manifolds suggests.

It is remarkable that positive scalar curvature alone is enough to give very interesting stability results, this being a consequence of the fact that on hypersurfaces the stability operator can be written purely in terms of scalar curvature.

Proposition 5.11. *If Σ is minimal then* $\mathrm{Ric}(\nu,\nu) + |A|^2 = \frac{1}{2}(R_M - R_\Sigma + |A|^2)$.

Proof. Choose an orthonormal basis $e_1, \dots, e_{n-1}, e_n = \nu$. Then

$$\begin{aligned} \mathrm{Ric}(\nu,\nu) - \frac{1}{2}R_M &= \sum_{i=1}^{n-1} R_{inni} - \frac{1}{2}\sum_{a,b=1}^{n} R_{abba} = -\frac{1}{2}\sum_{a,b=1}^{n-1} R_{abba} \\ \text{(Gauss equation)} &= -\frac{1}{2}\sum_{a,b=1}^{n-1}\left[R^\Sigma_{abba} - h_{aa}h_{bb} + h_{ab}^2\right] \\ &= -\frac{1}{2}R_\Sigma + \frac{1}{2}H^2 - \frac{1}{2}|A|^2. \end{aligned}$$

So indeed when $H = 0$, $\mathrm{Ric}(\nu,\nu) + |A|^2 = \frac{1}{2}R_M - \frac{1}{2}R_\Sigma + \frac{1}{2}|A|^2$. □

As a result of this proposition, we can rewrite our stability operator as

$$\delta^2\Sigma(\varphi,\varphi) = \int_\Sigma |\nabla\varphi|^2 - \frac{1}{2}(R_M - R_\Sigma + |A|^2)\varphi^2,$$

in which case we find that

$$\text{Stability} \Leftrightarrow \lambda(-\Delta - \frac{1}{2}(R_M - R_\Sigma + |A|^2)) \geq 0.$$

5.2. **Bonnet-type theorem.**

Proposition 5.12. *If $R_M \geq \kappa > 0$, then $\lambda_1(-\Delta_\Sigma + \frac{1}{2}R_\Sigma) \geq \kappa/2$ for every stable Σ.*

Proof. By stability,

$$\begin{aligned} \int_\Sigma |\nabla\varphi|^2 &\geq \int_\Sigma \frac{1}{2}(R_M - R_\Sigma + |A|^2)\varphi^2 \\ &\geq \frac{\kappa}{2}\int_\Sigma \varphi^2 - \frac{1}{2}\int_\Sigma R_\Sigma\varphi^2 \end{aligned}$$

so

$$\int_\Sigma |\nabla\varphi|^2 + \frac{1}{2}R_\Sigma\varphi^2 \geq \frac{\kappa}{2}\int_\Sigma \varphi^2$$

for all $\varphi \in C_c^\infty(\Sigma)$, and the result follows. □

Corollary 5.13. *When $n = 3$, Σ is a 2-dimensional surface with $R_\Sigma = 2K_\Sigma$, so the conclusion above can be rewritten as $\lambda_1(-\Delta_\Sigma + K_\Sigma) \geq \kappa/2$.*

We recall Bonnet's theorem:

Theorem 5.14 (Bonnet's theorem)**.** *Let (Σ^2, g) be such that $K_\Sigma \geq \kappa > 0$. Then the length of any stable geodesic is $\leq \pi/\sqrt{\kappa}$. Consequently,*

1. *If Σ is complete, then it is also compact with* diam $\leq \pi/\sqrt{\kappa}$.
2. *If Σ has boundary, then* $\operatorname{dist}(p, \partial\Sigma) \leq \pi/\sqrt{\kappa}$ *for all $p \in \Sigma$.*

Our goal is to show that this extends beyond surfaces with positive Gauss curvature, to surfaces that satisfy the eigenvalue positivity condition $\lambda(-\Delta + K) \geq \kappa/2$.

Theorem 5.15. *If (Σ^2, g) satisfies $\lambda(-\Delta_\Sigma + K_\Sigma) \geq \kappa/2 > 0$, then*

1. *if Σ is complete, it must have* diam $\leq 2\pi/\sqrt{\kappa}$, *and*
2. *if Σ has boundary, then* $\operatorname{dist}(p, \partial\Sigma) \leq 2\pi/\sqrt{\kappa}$ *for all $p \in \Sigma$.*

Remark 5.16. We can no longer estimate the lengths of stable geodesics. Instead we will construct a new functional on curves, and study stable critical points of that and provide upper bounds on the (original) length. This will clearly bound the lengths of the optimal (original) geodesics.

Proof. Let $u > 0$ be the first eigenfunction of $-\Delta_\Sigma + K_\Sigma$, so that $-\Delta_\Sigma u + K_\Sigma u = \lambda u$ with $\lambda \geq \kappa/2$.

We construct a compact 3-manifold $M^3 = \Sigma \times \mathbb{S}^1$ and endow it with a warped product metric $g + u^2\, dt^2$. By a calculation, we see that the scalar curvature $\widetilde{R}$ of M^3 satisfies

$$\widetilde{R} = 2K_\Sigma - 2\frac{\Delta_\Sigma u}{u} = 2\frac{-\Delta_\Sigma u + K_\Sigma u}{u} \geq \kappa.$$

Let $s \mapsto \gamma(s)$ be any curve in Σ parametrized by arclength, $s \in [0, \ell]$. Note that $\gamma \times \mathbb{S}^1$ is a surface in M^3, whose area is

$$\operatorname{area}(\gamma \times \mathbb{S}^1) = \int_0^\ell u(\gamma(s))\, ds \triangleq L_u(\gamma).$$

The functional L_u will be our new functional on curves of Σ. The result will follow once we establish the following

Claim 5.17. *Stable curves for L_u satisfy the Bonnet-type property $\ell \leq 2\pi/\sqrt{\kappa}$.*

The stability inequality applied to the surface $\gamma \times \mathbb{S}^1 \subset M^3$ yields

$$\int_{\gamma\times\mathbb{S}^1} \left[\frac{1}{2}(\widetilde{R} + |\widetilde{A}|^2) - \widetilde{K}\right]\varphi^2\, u\, dt\, ds \leq \int_{\gamma\times\mathbb{S}^1} |\widetilde{\nabla}\varphi|^2\, u\, dt\, ds.$$

We restrict to $\mathbb{S}^1$-invariant variations $\varphi = \varphi(s)$. This way $|\widetilde{\nabla}\varphi|^2 = (\varphi')^2$ and the t-integrals drop out. Estimating $\widetilde{R} \geq \kappa$ and $|\widetilde{A}| \geq 0$ we get:

$$\frac{\kappa}{2}\int_0^\ell \varphi^2\, u\, ds + \int_0^\ell \frac{u''}{u}\,\varphi^2\, u\, ds \leq \int_0^\ell (\varphi')^2\, u\, ds$$

for all φ with $\varphi(0) = \varphi(\ell) = 0$. Since we don't actually know what u is, our goal is to choose φ that makes the u dependence disappear. To that end, we choose $\varphi = u^{-1/2}\psi$ for some ψ with $\psi(0) = \psi(\ell) = 0$. The stability inequality becomes

$$\frac{\kappa}{2}\int_0^\ell \psi^2 + \int_0^\ell \frac{u''}{u}\psi^2 \leq \int_0^\ell \Big[-\frac{1}{2}u^{-3/2}u'\psi + u^{-1/2}\psi'\Big]^2 u.$$

Estimating the entire (non-negative) right hand side by its double, and expanding the square

$$\frac{\kappa}{2}\int_0^\ell \psi^2 + \int_0^\ell \frac{u''}{u}\psi^2 \leq 2\int_0^\ell \frac{1}{4}u^{-2}(u')^2\psi^2 + (\psi')^2 - \psi\psi' u^{-1}u'$$

$$(\text{use } \psi\psi' = \frac{1}{2}(\psi^2)') = \int_0^\ell \frac{1}{2}\Big(\frac{u'}{u}\Big)^2\psi^2 + 2(\psi')^2 + \psi^2\Big(\frac{u''}{u} - \frac{u'}{u}\Big)$$

and canceling the u'' terms on the left and right hand sides, we conclude that

$$\frac{\kappa}{2}\int_0^\ell \psi^2 \leq 2\int_0^\ell (\psi')^2$$

and therefore that $\lambda_1(d^2/dt^2) \geq \kappa/4$ on the interval $(0,\ell)$. But we know that $\lambda_1(d^2/dt^2) = \pi^2/\ell^2$, and the result follows. □

5.3. **Some obstructions to positive scalar curvature.** As an immediate corollary of the Bonnet-type theorem we proved we get:

Corollary 5.18. *Let (M^3, g) have $R_M > 0$. Then any closed stable minimal surface $\Sigma^2 \subset M^3$ is necessarily diffeomorphic to $\mathbb{S}^2$ or $\mathbb{RP}^2$.*

Proof. The universal cover $\widetilde{\Sigma}$ of Σ is also a stable minimal immersion, and by the Bonnet-type theorem it is in fact compact. By the classification of surfaces, $\widetilde{\Sigma} \approx \mathbb{S}^2$ and the claim follows. □

Theorem 5.19. *If M^3 is compact and carries a metric g_0 of non-positive sectional curvature, then it cannot carry any metric of positive scalar curvature.*

Proof. We argue by contradiction and assume that g were a metric of positive scalar curvature on M^3. By compactness, $R_g \geq \kappa$ and there exists $C > 0$ so that $C^{-1}g \leq g_0 \leq Cg$.

By Cartan-Hadamard, the universal covering space $(\widetilde{M}, g_0)$ of (M, g_0) is diffeomorphic to $\mathbb{R}^3$ via the exponential map, and $g_0 \geq \delta$. Therefore $\delta \leq Cg$ on $\widetilde{M} \approx \mathbb{R}^3$. This upper bound of the flat metric δ in terms of the positive scalar curvature metric g will yield the contradiction.

Consider the x^1x^2 plane in $\mathbb{R}^3$, and the x^3 axis which cuts it orthogonally. For $R > 0$, let C_R be the circle of radius R on the x^1x^2 plane. Any point q on the x^3 axis satisfies $d_\delta(q, C_R) \geq R$, and therefore $d_g(q, C_R) \geq R/C$.

Solve the Plateau problem with metric g and boundary C_R. For topological reasons, the g-area minimizing disk Σ^2 will intersect the x^3 axis at some point q. By the previous estimate, $d_g(q, C_R) \geq R/C \to \infty$ as $R \to \infty$. However, our Bonnet-type theorem places a uniform upper bound on $d_g(q, C_R)$ in terms of κ. This contradicts the existence of g. □

In fact the following stronger result is also true provided the proof above is appropriately generalized.

Theorem 5.20. *Let (M^3, g) be closed and with sectional curvature $K \leq 0$. If M_1^3 is closed and there exists a map $f : M_1 \to M$ of nonzero degree, then M_1 has no metric of positive scalar curvature.*

Corollary 5.21. *If M^3 is as above and M_0^3 is any closed manifold, then $M\#M_0$ cannot carry a metric of positive scalar curvature.*

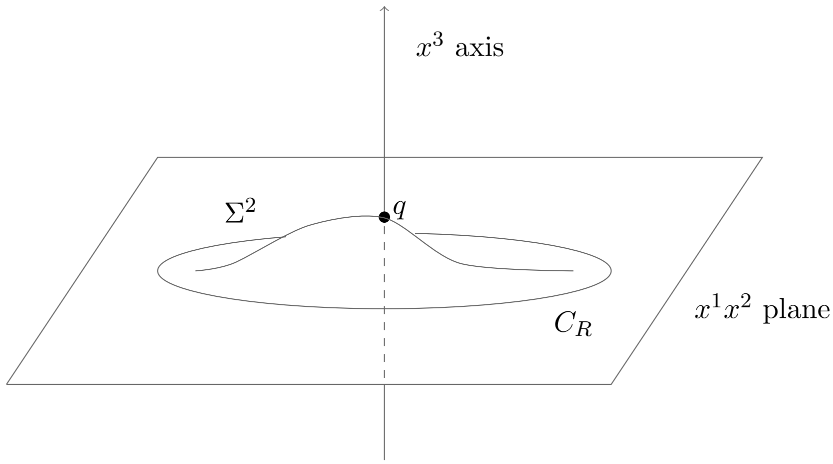

FIGURE 3. Solving the Plateau problem, $\partial\Sigma^2 = C_R$.

5.4. **Asymptotically flat manifolds and ADM mass.** It turns out that the tools we've developed for positive scalar curvature in the compact setting can be used to understand certain complete noncompact manifolds with nonnegative scalar curvature–namely, those which look approximately Euclidean outside large compact sets.

Definition 5.22. We say that (M^n, g) is asymptotically flat of order $p > \frac{n-2}{2}$ if[2] $R_g = O(|x|^{-n-\alpha})$ for some $\alpha > 0$, and there exists a compact $K \subset M$ such that $M \setminus K \approx \mathbb{R}^3 \setminus B_1$ so that under the induced Euclidean coordinates $x^1, \ldots, x^n$ on $M \setminus K$, the metric g satisfies the fall-off conditions $g_{ij} = \delta_{ij} + \alpha_{ij}$ with

$$|\alpha_{ij}(x)| + |x|\,|\partial\alpha_{ij}(x)| + |x|^2\,|\partial\partial\alpha_{ij}(x)| \leq c\,|x|^{-p}$$

for some $c > 0$.

Definition 5.23. For any manifold we define the ADM mass to be

$$m = \frac{1}{c(n)} \lim_{r\to\infty} \int_{S_r} (g_{ij,j} - g_{jj,i})\nu^i d\sigma$$

where S_r represents the coordinate radius r sphere, $\nu^i = \frac{x^i}{|x|}$ is the Euclidean outward unit normal, and $d\sigma$ the Euclidean volume element of S_r. The normalization constant $c(n)$ above is chosen so that

$$m\left(\left(1 + \frac{m_s}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta\right) = m_s.$$

For example, $c(3) = 16\pi$.

Example 5.24 (Schwarzschild metrics)**.** *The metrics*

$$g_m = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta$$

we're using to normalize our ADM mass definition are called (Riemannian) mass m Schwarzschild metrics and correspond to static black hole solutions of the Einstein equations. The parameter m reflects their "total mass." Notice that for $m = 0$ we get flat Euclidean space, whereas for $m < 0$ the metrics we get are incomplete with a nonremovable singularity at $x = 0$.

[2]The most general requirement for scalar curvature is $R_g \in L^1$, but we adopt this stronger decay assumption for expository convenience.

Proposition 5.25. *All complete, rotationally symmetric and scalar flat metrics are conformally flat and asymptotically flat:*

$$g = u^{\frac{4}{n-2}}\delta$$

with

$$u = u(r) = 1 + \frac{m}{2r^{n-2}}$$

for some constant $m \geq 0$. In other words, Schwarzschild metrics (for $m \in \mathbb{R}$) are the only *rotationally symmetric and scalar-flat manifolds (without any asymptotic assumptions).*

Proof. Rotational symmetry is characterized by metrics on $\mathbb{R}^n \setminus \{0\}$ (or $\mathbb{R}^n$) given by $g = dr^2 + f(r)^2 g_{\mathbb{S}^{n-1}}$. Notice that by changing coordinates it follows that every such g is conformally flat, i.e.

$$g = u^{\frac{4}{n-2}}\delta$$

for some radial function $u = u(r) > 0$. Scalar flatness translates to $\Delta u = 0$ with respect to the flat metric, and since u is radial we have

$$\Delta u = 0 \Leftrightarrow \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) = 0$$

$$\Leftrightarrow \frac{\partial u}{\partial r} = \frac{b}{r^{n-1}} \Leftrightarrow u = a + \frac{b}{r^{n-2}}.$$

We may rescale so that $a = 1$, and therefore

$$u(r) = 1 + \frac{m}{2r^{n-2}}$$

for some $m \in \mathbb{R}$, thus recovering the various Schwarzschild metrics. □

It's not clear that the definition of ADM mass actually makes sense because it's not clear that the limit even exists. This is so because the integrand $g_{ij,j} - g_{jj,i}$ is only of order $|x|^{-p-1}$, and $p + 1 > n/2$ does not guarantee existence of the limit. Nevertheless, it is true that ADM mass is well defined.

Lemma 5.26. *The ADM mass m is well defined on asymptotically flat manifolds (M^n, g) of order $p > \frac{n-2}{2}$.*

Proof of lemma. On the asymptotic flat chart we have

$$R = g_{ij,ij} - g_{jj,ii} + O((g-\delta)\partial^2 g) + O((\partial g)^2).$$

Notice that the error terms are both of order $|x|^{-2p-2}$ with $2p + 2 > n$, so integrable. The scalar curvature is also assumed to be integrable, so

$$\int_{M\setminus K} |g_{ij,ji} - g_{jj,ii}|\, d\mu < \infty.$$

By the divergence theorem we have that for all $r_1 < r_2$ sufficiently large

$$\int_{S_{r_2}} (g_{ij,j} - g_{jj,i})\nu^i d\sigma - \int_{S_{r_1}} (g_{ij,j} - g_{jj,i})\nu^i d\sigma = \int_{B_{r_2}\setminus B_{r_1}} \operatorname{div}(g_{ij,i} - g_{jj,i})\, dx = \int_{B_{r_2}\setminus B_{r_1}} (g_{ij,ji} - g_{jj,ii})\, dx$$

Therefore we the limit as $r_2 \to \infty$ will exist by the dominated convergence theorem since we know that $g_{ij,ji} - g_{jj,ii} \in L^1(d\mu)$ and we can exchange $d\mu$ with dx integration.

Finally we need to check the formula for scalar curvature. By computing in coordinates we know that

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\ell}(g_{\ell i,j} + g_{j\ell,i} - g_{ij,\ell}) = \frac{1}{2}(g_{ki,j} + g_{kj,i} - g_{ij,k}) + O((g-\delta)\partial g),$$

and since $R^i{}_{jk\ell} = \partial\Gamma - \partial\Gamma + \Gamma^2 + \Gamma^2$,

$$R = \Gamma^k_{ii,k} - \Gamma^k_{ki,i} + O((g-\delta)\partial^2 g)$$

$$
\begin{aligned}
&= g_{ik,ki} - \frac{1}{2} g_{ii,kk} - \frac{1}{2} g_{kk,ii} + O((g-\delta)\partial^2 g) + O((\partial g)^2) \\
&= g_{ij,ji} - g_{ii,jj} + O((g-\delta)\partial^2 g) + O((\partial g)^2)
\end{aligned}
$$

as claimed. □

5.5. **Positive mass theorem.** Now, while we've just shown that ADM mass is well-defined, it's far from clear that it represents a non-negative quantity[3]. The positive mass theorem validates this assertion:

Theorem 5.27 (Positive mass theorem). *If (M^n, g) is complete and asymptotically flat with $R_g \geq 0$ then $m \geq 0$, and $m = 0$ if and only if $(M^n, g) \cong (\mathbb{R}^n, \delta)$.*

In this section we proceed to check this theorem in the base case $n = 3$, though the steps that hold true for all dimensions are performed in full generality. We present the proof as a sequence of reductions. The plan is:

(1) Reduce to simpler asymptotic behavior at infinity, replacing our asymptotically flat manifold with a so-called asymptotically conformally flat manifold that is also scalar-flat.
(2) Arguing by contradiction, reduce $m < 0$ to a manifold that is precisely Euclidean outside a compact set.
(3) Argue that the manifold obtained above can be turned into a nonflat manifold $\approx T^3 \# M^3_1$ with non-negative scalar curvature, obtaining a contradiction and thus finishing the proof.

Throughout this proof we will make extensive use of the conformal Laplacian,

$$L_g u = \Delta_g u - c_n R_g u$$

where $c_n = \frac{n-2}{4(n-1)}$ (not to be confused with the ADM mass-normalizing factor $c(n)$). The significance of the conformal Laplacian is that under a conformal change of metric $u^{\frac{4}{n-2}} g$ we have

$$R(u^{\frac{4}{n-2}}) = -c_n^{-1} u^{-\frac{n+2}{n-2}} L_g u. \tag{5.1}$$

Step 1. Simplification of asymptotics.

Definition 5.28. We say that a manifold (M^n, g) is asymptotically conformally flat if outside a compact set we have $g = u^{\frac{4}{n-2}} \delta$ with $u \to 1$ at ∞. If such a manifold is additionally scalar flat, then by (5.1) it follows that $\Delta u = 0$.

Remark 5.29. If u is as above, then since it is harmonic on Euclidean space we know how to expand it into spherical harmonics near infinity. The expansion is

$$u(x) = 1 + \frac{A}{2|x|^{n-2}} + O(|x|^{1-n}).$$

Proposition 5.30. *If u is as above, then $m(u^{\frac{4}{n-2}} \delta) = A$.*

Proof. This is just a calculation. □

Theorem 5.31. *Let (M^n, g) be asymptotically flat with $R_g \geq 0$. Then for every $\varepsilon > 0$ there exists a metric $\widehat{g}$ that is asymptotically conformally flat, scalar flat, and with $m_{\widehat{g}} \leq m_g + \varepsilon$.*

Proof. First we check the following:

Claim 5.32. *Without loss of generality we may assume that $R_g \equiv 0$.*

[3] ADM mass can be negative: Schwarzschild metrics with $m < 0$ certainly have negative mass. To get non-negativity we ought to impose particular conditions on the boundary of M^3, or assume that there is no boundary. We do the latter.

Proof of claim. If $R_g > 0$ but is not identically zero, then we may solve the equation $L_g u = \Delta_g u - c(n) R_g u = 0$, $u > 0$, $u \to 1$ at infinity. (Some further analysis is required to show that this can be done.) The solution $u(x)$ of this PDE can be expanded as

$$u(x) = 1 + \frac{a}{2|x|^{n-2}} + O_2(|x|^{1-n})$$

for some $a \in \mathbb{R}$. For $r > 0$ large we have

$$\begin{aligned} 0 < c(n) \int_{B_r} R_g u \, dV_g &= \int_{B_r} \Delta_g u \, dV_g = \int_{\partial B_r} \frac{\partial u}{\partial \nu} \, d\sigma_g \\ &= \int_{\partial B_r} \left[-\frac{n-2}{2} \frac{a}{|x|^{n-1}} + O_1(|x|^{-n}) \right] d\sigma_g = -\frac{n-2}{2} |S^{n-1}| a \end{aligned}$$

and therefore $a < 0$. Setting $\widetilde{g} = u^{\frac{4}{n-2}} g$ yields $R_{\widetilde{g}} \equiv 0$ and $m_{\widetilde{g}} = m_g + a < m_g$. ☐

Now fix $\sigma > 0$ large and take a cut-off function $\chi(r)$ such that $\chi(r) = 1$ for $r \le \sigma$, $\chi(r) = 0$ for $r \ge 2\sigma$, $0 \le \chi'(r) \le c/\sigma$ for $\sigma \le r \le 2\sigma$, $|\chi''(r)| \le c/\sigma^2$ for $\sigma \le r \le 2\sigma$. Set

$$\widetilde{g} = \chi g + (1 - \chi) \delta$$

where δ is Euclidean. This metric agrees with g in $|x| \le \sigma$ and is Euclidean on $|x| \ge 2\sigma$, so asymptotically flat. Note that $R_{\widetilde{g}} \equiv 0$ except in the annulus $\sigma \le |x| \le 2\sigma$, where $|R_{\widetilde{g}}| = O(|x|^{-p-2})$. Even though $R_{\widetilde{g}}$ may not be non-negative, one can show with some analysis that we can uniquely solve

$$L_{\widetilde{g}} u = \Delta_{\widetilde{g}} u + c(n) R_{\widetilde{g}} u = 0, \quad u > 0, \quad u \to 1 \text{ at infinity}$$

provided $\int_M |R_{\widetilde{g}}|^{\frac{n}{2}} \, dV_{\widetilde{g}}$ is small enough. (This comes from the Sobolev inequality.) This is indeed the case here since

$$|R_{\widetilde{g}}| \le c|x|^{-p-2} \Rightarrow |R_{\widetilde{g}}|^{\frac{n}{2}} \le c' |x|^{\frac{n}{2}(-p-2)}$$

and $\frac{n}{2}(p+2) > \frac{n(n+2)}{4} > n$ for $n \ge 3$, and we can therefore make the integral be sufficiently small by sending the (still free) parameter $\sigma \to \infty$.

Having solved this equation, we may set $\widehat{g} = u^{\frac{4}{n-2}} \widetilde{g}$, which is globally scalar flat, and asymptotically conformally flat since $\widetilde{g} = \delta$ for $|x| \ge 2\sigma$. Furthermore, as $\sigma \to \infty$ we get $m_{\widehat{g}} \to m_g$. ☐

Why is this reduction desirable? If we can prove the positive mass theorem for these special asymptotics, then we can prove the general case by arguing that if a metric g with $m_g < 0$ were to exist, we could find a nearby metric $\widehat{g}$ with the simplified asymptotics which also satisfies $m_{\widehat{g}} < 0$, a contradiction.

Step 2. An observation of Lohkamp and reduction to a compact problem.

Having reduced to asymptotically conformally flat asymptotics and zero scalar curvature, we know that $g = u^{\frac{4}{n-2}} \delta$ for some harmonic u. By expanding u along spherical harmonics near infinity we have

$$u(x) = 1 + \frac{m}{2|x|^{n-2}} + O_2(|x|^{1-n}),$$

where m is the ADM mass of the manifold because of Proposition 5.30.

Theorem 5.33. *If in these asymptotics we have $m < 0$, then outside large compact sets we can perturb g to another metric which has non-negative scalar curvature and is exactly Euclidean near infinity.*

Proof. Recall that $\Delta u = 0$. We first make the observation that if ψ is any positive smooth concave function, then $\Delta \psi(u) = \psi'(u) \Delta u + \psi''(u) |\nabla u|^2 \le 0$, so $R(\psi(u)^{\frac{4}{n-2}} \delta_{ij}) \ge 0$.

Now since we're assuming $m < 0$, we know that for $|x|$ large, $u(x)$ approaches 1 from below. Construct a smooth concave function $\psi : \mathbb{R} \to \mathbb{R}$ such that ψ is the identity on $(-\infty, 1 - \varepsilon)$, increasing, and $\psi \equiv 1 - \varepsilon/2$ on $(1-\varepsilon, \infty)$. Then perturb g outside a large compact set to $\widehat{g}_{ij} = \psi(u)^{\frac{4}{n-2}}\delta_{ij}$. Evidently, $\widehat{g} = g$ in a compact region, $\widehat{g}$ is Euclidean near infinity, and $R_{\widehat{g}} \geq 0$. □

Step 3. Obtaining a $T^n \# M_1^n$ with $R \geq 0$.

Having reduced to a manifold (M^n, g) with $R \geq 0$ and which is Euclidean outside a compact set, let C be a sufficiently large coordinate cube such that $M \setminus C \approx \mathbb{R}^n \setminus B_1$. Then construct a new manifold (M', g') by periodically patching copies of C together. This manifold clearly has a free, properly discontinuous $\mathbb{Z}^n$ action, and therefore we can further construct

$$(M'', g'') \cong (M', g')/\mathbb{Z}^n$$

to be a compact manifold with non-negative scalar curvature, which is topologically $T^n \# M_1^n$. Since curvature is local in nature, the scalar curvature descends to this manifold and therefore we will have produced a scalar non-negative $T^n \# M_1^n$.

Now we need to specialize to $n = 3$. Our goal is to further reduce to the setting of Corollary 5.21. This requires some general understanding of the variational theory of scalar curvature. To that end, we need the following:

Definition 5.34. The total scalar curvature functional (also known as the Einstein-Hilbert functional) is

$$\mathcal{R}(g) = \int_M R_g \, dV_g.$$

Theorem 5.35. *If M^n carries no metrics of positive scalar curvature, then either we can perturb any g with $R_g \geq 0$ to have positive scalar curvature, or g satisfies $\mathrm{Ric}_g \equiv 0$*

Proof. Let g_0 have $R_{g_0} \geq 0$. There are two cases to consider.

The first case is the one in which $R_{g_0} > 0$ at some point. Let $u > 0$ be the first eigenfunction of L_{g_0} and $\lambda \in \mathbb{R}$ the corresponding eigenvalue. Since $R_{g_0} \geq 0$, but does not vanish identically, it follows that $\lambda = \lambda(L_{g_0}) \geq 0$. In fact, $\lambda > 0$ because otherwise the first eigenfunction $u > 0$ of L_{g_0} would be superharmonic on M^n, so constant, so R_{g_0} would be constant, so zero–which is false. Therefore $\lambda > 0$, so $R(u^{\frac{4}{n-2}} g_0) = -c_n^{-1} u^{-\frac{n+2}{n-2}}(-\lambda)u > 0$. This is also impossible, since we're assuming that M^n cannot carry positive scalar curvature.

The second case to consider is that of $R_{g_0} \equiv 0$. We need to check that $\mathrm{Ric}_{g_0} \equiv 0$. Consider an arbitrary variation $g_t = g_0 + th$, and set $\lambda(t) = \lambda_0(-L_{g_t})$. We know that $\lambda(t)$ is smooth, $\lambda(t) \leq 0$, and $\lambda(0) = 0$. So $\lambda'(0) = 0$.

For every t near 0, let u_t be the first eigenfunction of L_{g_t} normalized to have $\int_M u_t^2 \, dV_{g_t} = V(g_0)$, so that $u_0 = 1$. Then

$$V(g_0)\lambda(t) = \int_M |\nabla u_t|^2 + c(n) R_{g_t} u_t^2 \, dV_{g_t}$$

and differentiating in t at $t = 0$,

$$\begin{aligned} 0 = V(g_0)\lambda'(0) &= c(n) \int_M \left[\frac{d}{dt} R_{g_t}\right]_{t=0} dV_{g_0} \\ &= c(n) \int_M (g^{ij})^{\cdot} R_{ij} + g^{ij} \dot{R}_{ij} \, dV_{g_0} \\ &= c(n) \int_M \langle -h, \mathrm{Ric}_{g_0} \rangle \, dV_{g_0} \end{aligned}$$

where the second term has dropped off because it is a boundary term and $\partial M = \emptyset$. Therefore, $\langle -h, \mathrm{Ric}_{g_0} \rangle = 0$ for all perturbations h, so $\mathrm{Ric}_{g_0} = 0$.

Alternatively, the second case can also be seen to hold true by Ricci flow. If $R_{g_0} \geq 0$, then running Ricci flow for a short time would deform a non-Ricci flat g_0 to positive scalar curvature, which is impossible, so g_0 is Ricci flat. □

Remark 5.36. There do exist nontrivial Ricci-flat metrics on manifolds which don't admit any positive scalar curvature metrics: K3 manifolds are such an example. Furthermore, Calabi-Yau manifolds are examples in 6 dimensions of Ricci-flat metrics which are not perturbable to positive scalar curvature metrics, although the background manifold does carry such metrics.

This theorem is closely related with the following trichotomy theorem:

Theorem 5.37 (Trichotomy theorem). *Let (M^n, g), $n \geq 3$, and $[g] = \{e^{2u} g : u \in C^\infty(M^n)\}$. Then:*

1. *$[g]$ contains a metric with $R > 0$ if and only if $\lambda(-L) > 0$.*
2. *$[g]$ contains a metric with $R \equiv 0$ if and only if $\lambda(-L) = 0$.*
3. *$[g]$ contains a metric with $R < 0$ if and only if $\lambda(-L) < 0$.*

Now we return to the positive mass theorem. By applying Theorem 5.35 in combination with Corollary 5.21 when $n = 3$, we conclude that the compact manifold $T^3 \# M_1^3$ we have constructed (by assuming $m < 0$, by way of contradiction) must in fact be Ricci flat. But of course Ricci flat metrics are flat when $n = 3$, so our original metric must have been flat, which contradicts that $m < 0$.

Remark 5.38 (Higher dimensions). There are two parts of the proof that don't immediately generalize to arbitrary dimensions as stated: the fact that $T^n \# M_1^n$ doesn't carry scalar positive metrics (which we only checked for $n = 3$), and the fact that g being Ricci flat implies that the original background manifold is flat (this is trivial when $n = 3$).

In fact, only the first of those statements is nontrivial. The second statement follows from the general fact that every asymptotically flat and Ricci flat manifold (M^n, g) is necessarily Euclidean as one can see, for example, by volume comparison on large balls.

The second statement is known to hold true up to $n \leq 7$ via an inductive argument that reduces to the 3-dimensional case, and by a theorem of Smale ([**?**]) and some additional modifications can likely be adapted to $n = 8$ as well. For larger n there are hard technical obstacles in the minimal surface proof, but it has been shown by Witten ([**?**], [**?**]) through the use of spinors that the positive mass theorem holds true for all (M^n, g) in all dimensions, provided M^n admits a spin structure.

5.6. **Rigidity case.** We have yet to prove the rigidity case of the positive mass theorem. It follows from the following observation that works in all dimensions.

Theorem 5.39 (Rigidity case of PMT). *Assume $m \geq 0$ for every asymptotically flat (M^n, g) with $R \geq 0$. If (M^n, g) is then asymptotically flat*

Proof. If $R > 0$ at some point then we can solve $Lu = 0$, $u \to 1$ at ∞, and show that

$$m(u^{\frac{4}{n-2}} g) - m(g) = -c(n) \lim_{r \to \infty} \int_{S_r} \frac{\partial u}{\partial \nu} \, d\sigma = -\int_M \Delta_g u \, dV_g = m(g) - \int_M R_g \, dV_g < 0.$$

We have already shown that $m(u^{\frac{4}{n-2}} g) \geq 0$, so $m(g) > 0$, a contradiction.

So we can assume that $R \equiv 0$. Let h be a C_c^∞ symmetric $(0, 2)$-tensor, and $g_t = g + th$. The metrics g_t are still asymptotically flat, and in fact coincide with g at infinity. Of course, these metrics don't satisfy $R \geq 0$ anymore. For t small, let $u_t > 0$ be such that $Lu_t = 0$, $u_t \to 1$ at infinity (which we can indeed solve, like we did in Claim 5.32). Then define $\widehat{g}_t = u_t^{\frac{4}{n-2}} g_t$, which satisfy $R(\widehat{g}_t) \equiv 0$. Observe, further, that $u_0 \equiv 1$ and $\widehat{g}_0 = g$.

Define $\widehat{m}(t) = m(\widehat{g}_t)$, and observe that

$$\widehat{m}(t) = -c(n) \lim_{r \to \infty} \int_{S_r} \frac{\partial u}{\partial \nu} \, d\sigma_t = -c(n) \int_M \Delta_{g_t} u_t \, dV_t = -c(n) \int_M R_{g_t} u_t \, dV_t.$$

This is differentiable in t and we know by assumption that $\widehat{m}(t) \geq 0$, with $\widehat{m}(0) = m(g_0) = 0$, so $\widehat{m}'(0) = 0$. We can also compute $\widehat{m}'(0)$ by differentiation, which gives

$$0 = \widehat{m}'(0) = -c(n) \int_M \langle h, \mathrm{Ric}_g \rangle \, dV_g.$$

Recall that h were an arbitrary symmetric $(0, 2)$-tensor, and conclude that g is Ricci flat, thus flat by remark 5.38. □

6. Calibrated geometry

6.1. **Definitions and examples.**

Definition 6.1. A smooth p-form φ on a Riemannian manifold (M, g) is called a **calibrating form** or simply a **calibration** if it is closed (i.e. $d\varphi = 0$) and satisfies

$$\varphi_x(\xi) \leq 1, \forall\, x \in M \text{ and } \forall\, \xi \in G_p(T_x M), \tag{6.1}$$

where we denote by $G_p(V)$ the set of all simple, unit length p-vectors in $\wedge^p V$, which can be identified with the set of oriented p-planes in V.

Definition 6.2. Given a calibration φ and $x \in M$, the **contact set** at x is defined to be

$$\{\xi \in G_p(T_x M) | \varphi_x(\xi) = 1\}. \tag{6.2}$$

Definition 6.3. Given a calibration φ, a p-dimensional submanifold Σ^p is called $\varphi-$**calibrated**, or just **calibrated**, if

$$\varphi_x(T_x \Sigma) = 1, \ \forall\, x \in \Sigma. \tag{6.3}$$

Remark 6.4. Note that a calibrated submanifold is automatically oriented because φ restricts to a volume form.

One of the most important properties of calibrated submanifolds is that they minimize area in their relative homology class. More precisely, we have the following theorem.

Theorem 6.5. *Suppose Σ is φ-calibrated and let Σ_0 be another oriented p-submanifold with $\partial\Sigma_0 = \partial\Sigma$. Then $|\Sigma| \leq |\Sigma_0|$.*

Proof. By assumption we can write $\Sigma - \Sigma_0 = \partial R^{p+1}$. Then by Stokes' theorem and the closedness of φ, we get

$$\begin{aligned} 0 = \int_R d\varphi &= \int_{\Sigma - \Sigma_0} \varphi \\ &= \int_\Sigma \varphi - \int_{\Sigma_0} \varphi. \end{aligned}$$

Therefore

$$|\Sigma| = \int_\Sigma \varphi = \int_{\Sigma_0} \varphi \leq |\Sigma_0|.$$

The first equality is due to the fact that Σ is calibrated, while the third equality follows from (6.1). □

Before we go on let's give some examples of calibrations and calibrated submanifolds.

Example 6.6.

(1) *Let $\Omega \in M$ be an open subset and let r be a smooth function with $|\nabla r|_g = 1$. Then the integral curves of ∇r are length-minimizing.*

Proof. Consider the 1-form dr, which is certainly closed. Moreover, for all $x \in \Omega$ and each unit vector $v \in T_x M$,

$$dr(v) = g(\nabla r, v) \leq 1,$$

with equality holding if and only if $v = \nabla$. This shows that dr is a calibration and the integral curves of ∇r are calibrated. □

(2) *Suppose u is a solution to the minimal surface equation in $\Omega \in \mathbb{R}^n$ and let $\Sigma \in \mathbb{R}^{n+1}$ denote the graph of u. Then Σ is calibrated in $\Omega \times \mathbb{R}$.*

Proof. Take an orthonormal frame $e_1, \ldots, e_{n+1}$ with $e_1, \ldots, e_n$ tangent to Σ and e_{n+1} normal. Let $w^1, \ldots, w^{n+1}$ be the dual co-frame. Let $\varphi = w^1 \wedge \cdots \wedge w^n$ and extend φ to $\Omega \times \mathbb{R}$ so that it does not depend on x^{n+1}. The minimal surface equation implies that φ is closed. Moreover, (6.1) is satisfied since φ is a wedge product of 1-forms dual to unit vectors. Finally, Σ is φ-calibrated because φ restricts to the volume form on Σ by construction. □

(3) *More generally, if we have a foliation of a Riemannian manifold M^n by hypersurfaces, and if at each $x \in M$ we define φ_x to be the volume form of the leaf containing x, then φ is closed if and only if each leaf is a minimal hypersurface.*
(4) *Consider a Kähler manifold (M^{2n}, g, J) and recall that the Kähler form is defined by $\omega(X, Y) = g(JX, Y)$. The simplest example of a Kähler manifold is $\mathbb{R}^{2n}$ with the standard complex structure J, i.e.*

$$J(\frac{\partial}{\partial x^j}) = \frac{\partial}{\partial y^j};\; J(\frac{\partial}{\partial y^j}) = -\frac{\partial}{\partial x^j}. \tag{6.4}$$

Letting

$$dz^j = dx^j + idy^j,\; d\bar{z}^j = dx^j - idy^j, \tag{6.5}$$

we find that the Kähler form in this case is

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.$$

Now let's return to the general situation and fix $p \leq n$. Define a $2p$-form on M^{2n} by $\varphi = \frac{1}{p!}\omega^p$. Then φ is a calibration and the calibrated submanifolds are exactly the complex submanifolds.

Proof. First note that since M is Kähler, $d\omega = 0$ and hence φ is closed. Next we verify (6.1). More precisely, for any $x \in M$ we claim that

$$\frac{1}{p!}w^p(\xi) \leq 1, \forall\, \xi \in G_{2p}(T_x M) \text{ (Wirtinger's inequality)} \tag{6.6}$$

with equality holding if and only if ξ represents a complex p-plane (J-invariant oriented real $2p$-plane). The case $p = 1$ is already handled in Proposition 3.2, so we can assume that $p \geq 2$. For $\xi \in G_{2p}(T_x M)$, let E denote the oriented $2p$-plane represented by ξ. Since $\omega(X, Y) = g(JX, Y)$ we can view ω as a non-degenerate skew-symmetric bilinear form on E. By linear algebra we can find a positive basis $e_1, f_1, \ldots, e_p, f_p$ of E such that

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0,\; \omega(e_i, f_j) = \lambda_i \delta_{ij}$$

Furthermore, up to re-ordering we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ and that $\lambda_{p-1} \geq 0$. Then, with respect to the basis $\{e_1, f_1, \ldots, e_p, f_p\}$, ω is represented by the matrix

$$\begin{bmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$$

In other words, letting $\{\theta^i\}$ and $\{\tau^i\}$ be dual to $\{e_i\}$ and $\{f_i\}$ respectively, we have

$$\omega = \sum_{i=1}^{p} \lambda_i \theta^i \wedge \tau^i.$$

Hence, noting that 2-forms commute, we deduce that

$$\varphi = \frac{1}{p!} w^p = (\lambda_1 \lambda_2 \cdots \lambda_p) \theta^1 \wedge \tau^1 \wedge \cdots \wedge \theta^p \wedge \tau^p = (\lambda_1 \lambda_2 \cdots \lambda_p) \xi^\flat \tag{6.7}$$

Now notice that since e_i, f_i have unit length,

$$\lambda_i = \omega(e_i, f_i) = g(Je_i, f_i) \leq 1$$

Hence by (6.7), we see that

$$\varphi(\xi) \leq 1,$$

with equality holding if and only if $\lambda_1 = \cdots = \lambda_p = 1$, in which case we have

$$g(Je_i, f_i) = \omega(e_i, f_i) = 1.$$

Thus by the equality case of Cauchy-Schwarz inequality, we conclude that $Je_i = f_i$ and hence $J\xi = \xi$, i.e. ξ represents a complex p-plane. □

Remark 6.7.

(1) Concerning example (3), it is worth noting that if we have a foliation $\mathcal{F}$ of an n-manifold M by p-dimensional minimal submanifolds where $n - p \geq 2$, then in general the p-form φ constructed by the same procedure will not be closed. Nonetheless, if we have a $p+1$-submanifold N that is tangent to $\mathcal{F}$, i.e. $\mathcal{F}$ restricts to a foliation of N, then $\varphi|_N$ is closed.
(2) An interesting question is that given a p-dimensional oriented foliation of a compact n-manifold M, can we equip M with a Riemannian metric so that the leaves of the foliation become minimal submanifolds? In this direction, Dennis Sullivan proved that the answer is yes if and only if the foliation is "homologically taut". See [**?**] for details.

Remark 6.8. Example (4) shows that any complex submanifold of a Kähler manifold minimizes volume and is thus stable minimal. The theorem of Micallef [Mic84] that we presented earlier shows that the converse is true in some cases. Below we list two more results in this direction.

Theorem 6.9. *(Lawson-Simons,* [LS73]*) Any stable minimal submanifold Σ^{2p} of $\mathbb{CP}^n$ is $\pm$ holomorphic.*

Theorem 6.10. *(Siu-Yau,* [**?**]*) Every stable minimal S^2 in a compact Kähler manifold with positive bisectional curvature is $\pm$ holomorphic.*

6.2. **The Special Lagrangian Calibration.** We now introduce another calibration of great geometric interest; namely the special Lagrangian calibtration. We take $(\mathbb{R}^{2n}, J)$, which has the structure of an n-dimensional vector space over $\mathbb{C}$, as our ambient space, although the calibration makes sense under more general settings.

Before we go on, let us recall some facts from complex linear algebra. As in the previous section, on $(\mathbb{R}^{2n}, J)$ we introduce coordinates so that (6.4) holds and define $dz^j, d\overline{z}^j$ as in (6.5). An $\mathbb{R}$-linear map of $\mathbb{R}^{2n}$ to itself is said to be **complex** if it commutes with J, in which case it becomes a $\mathbb{C}$-linear map of $(\mathbb{R}^{2n}, J) \simeq \mathbb{C}^n$. Thus, a complex linear map A can be represented either as a $(2n \times 2n)$-real matrix of an $(n \times n)$-complex matrix. If we let $\det A$ and $\det_{\mathbb{C}} A$ denote the determinants of the two matrices, respectively, then we have the following relation whose proof will be given after the next paragraph.

Lemma 6.11. $\det A = |\det_{\mathbb{C}} A|^2$

The **unitary** group, denoted $U(n)$, is defined to be

$$\{A \in O(2n) | AJ = JA\} \tag{6.8}$$

Elements in $U(n)$ have the property that they preserve both the inner-product on $\mathbb{R}^{2n}$ and the Kähler form. Moreover, by Lemma 6.11, for all $A \in U(n)$ we have $|\det_{\mathbb{C}} A| = 1$ and we define the **special unitary** group as

$$SU(n) = \{A \in U(n) | \det_{\mathbb{C}} A = 1\}. \tag{6.9}$$

Proof of Lemma 6.11. With respect to the $\mathbb{R}$-basis $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \cdots, \frac{\partial}{\partial y^n}\}$ of $\mathbb{R}^{2n}$, we have

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Thus, since A commutes with J, it must have the following form with respect to the same basis as above:

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

It is then not hard to see that the complex $(n \times n)$-matrix representing A is exactly $\alpha + i\beta$. Performing some column operations, we find that

$$\begin{aligned} \det A = \det \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} &= \det \begin{bmatrix} \alpha + i\beta & \beta \\ -\beta + i\alpha & \alpha \end{bmatrix} = \det \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix} \\ &= \det(\alpha + i\beta) \det(\alpha - i\beta) = |\det(\alpha + i\beta)|^2 = |\mathrm{det}_{\mathbb{C}} A|^2 \end{aligned}$$

□

Definition 6.12. Given $\xi \in G_n(\mathbb{R}^{2n})$, let L be the oriented n-plane it represents. L is called a **Lagrangian** plane if

$$\xi = A(\frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}), \text{ for some } A \in U(n). \tag{6.10}$$

ξ is called a **special Lagrangian** plane if (6.10) holds for some $A \in SU(n)$. A submanifold $\Sigma^n \subseteq \mathbb{R}^{2n}$ is said to be a Lagrangian (special Lagrangian, resp.) submanifold if $T_x\Sigma$ is a Lagrangian (special Lagrangian, resp.) plane for each $x \in \Sigma$.

For the discussions to follow it will be useful to know some alternative characterisations of Langrangian planes. Recall that the Kähler form ω on $(\mathbb{R}^{2n}, J)$ is given by $\omega = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j = \sum_{j=1}^{n} dx^j \wedge dy^j$.

Lemma 6.13. *Let ξ and L be as in Definition 6.12. The following are equivalent:*

(a) $\omega(X,Y)=0,\ \forall,\ X,Y\in L$
(b) *The 1-form* $\sum_{j=1}^{n} y^j dx^j$ *is closed on* L.
(c) $J(L)=L^{\perp}$.
(d) $\xi = A(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^n})$ *for some* $A\in U(n)$.
(e) $|dz(u_1\wedge\cdots\wedge u_n)|=1$, *where* $\{u_1,\cdots,u_n\}$ *is an orthonormal basis of* L.
In the case where L *is the graph of a linear function* $f:\Omega\to\mathbb{R}^n$, *with* Ω *simply-connected, the above are equivalent to*
(f) $f=\nabla u$, *with* u *quadratic.*

Proof.

(1) $((a)\Leftrightarrow(b))$ Simply note that $d(\sum y^j dx^j)=-\omega$.
(2) $((a)\Leftrightarrow(c))$ Since $\omega(X,Y)=\langle JX,Y\rangle$, (a) holds if and only if $JX\perp Y$, $\forall X,Y\in L$, which is exactly (c).
(3) $((d)\Rightarrow(a))$ Condition (d) implies that $\{A(\frac{\partial}{\partial x^j})\}_{j=1}^n$ is an orthonormal basis for L. Moreover, since $A\in O(2n)$ and commutes with J, we have that

$$\omega(A(\frac{\partial}{\partial x^j}),A(\frac{\partial}{\partial x^k}))=\omega(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k})=\langle\frac{\partial}{\partial y^j},\frac{\partial}{\partial x^k}\rangle=0,$$

for all $1\le j,k\le n$. Thus (a) holds.
(4) $((c)\Rightarrow(d))$ Let $\{u_1,u_2,\cdots,u_n\}$ be an orthonormal basis for L. Then by assumption $\{u_1,u_2,\cdots,u_n,Ju_1,\cdots,Ju_n\}$ is an orthonormal basis for $\mathbb{R}^{2n}$. This implies that there exists $A\in U(n)$ such that $A(\frac{\partial}{\partial x^j})=u_j$. Clearly $\xi=A(\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^n})$.
(5) $((e)\Leftrightarrow(d)))$ Let $\{u_1,\cdots,u_n\}$ be an orthonormal basis for L. Then there is a unique complex linear map A with $A(\frac{\partial}{\partial x^j})=u_j$, i.e. $\pm\xi=u_1\wedge\cdots\wedge u_n=A(\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^n})$. Since

$$\begin{aligned}A(\frac{\partial}{\partial x^k})=u_k&=dx^j(u_k)\frac{\partial}{\partial x^j}+dy^j(u_k)J\frac{\partial}{\partial x^j}\\&=(dx^j+idy^j)(u_k)\frac{\partial}{\partial x^j}=dz^j(u_k)\frac{\partial}{\partial x^j},\end{aligned}$$

the complex matrix representation of A is given by $[dz^j(u_k)]$. Thus,

$$dz(u_1\wedge\cdots\wedge u_n)=\det[dz^j(u_k)]=\det_{\mathbb{C}}A \tag{6.11}$$

On the other hand, since $A(\frac{\partial}{\partial y^j})=Ju_j$, the real matrix representation of A is given by

$$\begin{bmatrix} | & & | & | & & | \\ u_1 & \cdots & u_n & Ju_1 & \cdots & Ju_n \\ | & & | & | & & | \end{bmatrix}.$$

Using lemma 6.11, we see that

$$\begin{aligned}|dz(u_1\wedge\cdots\wedge u_n)|^2&=|\det_{\mathbb{C}}A|^2=\det A\\&\le|u_1|\cdots|u_n||Ju_1|\cdots|Ju_n|=1,\end{aligned} \tag{6.12}$$

with equality holding if and only if $\{u_1,\cdots,Ju_n\}$ is an orthonormal basis for $\mathbb{R}^{2n}$, in which case $A\in U(n)$.
(6) $((b)\Leftrightarrow(f)$ in the graphical case) By assumption, $L=\{(x,f(x))|x\in\Omega\}$ and (b) translates into

$$d(\sum_{j=1}^n f_j dx^j)=0.$$

Since Ω is simply-connected, this is equivalent to the existence of a function u on Ω such that $f_j = \frac{\partial u}{\partial x^j}$, i.e. $f = \nabla u$. Since f is linear, u must be quadratic.

□

Remark 6.14. The proof for $(b) \Leftrightarrow (f)$ in the graphical case also shows that if Σ^n is an oriented n-submanifold of $\mathbb{R}^{2n}$ which is the graph of a function $f : \Omega \to \mathbb{R}^n$ with Ω simpy-connected, then Σ is Lagrangian if and only if $f = \nabla u$ for some $u : \Omega \to \mathbb{R}$.

Now we introduce the special Lagrangian calibration, defined by

$$\varphi = \mathrm{Re}(dz) \tag{6.13}$$

Theorem 6.15. *φ is a calibration and the calibrated submanifolds are exactly the special Lagrangian submanifolds.*

Proof. Apparently φ is closed. Moreover, given $\xi \in G_n(\mathbb{R}^{2n})$, if we assume $\xi = u_1 \wedge \cdots \wedge u_n$ (i.e. $\{u_1, \cdots, u_n\}$ is a positive orthonormal basis of L) and let A be the complex linear map with $A(\frac{\partial}{\partial x^j}) = u_j$, then by (6.11) and (6.12),

$$\varphi(\xi) = \mathrm{Re}\, dz(\xi) = \mathrm{Re} \det_{\mathbb{C}} A \leq |\det_{\mathbb{C}} A| \leq 1,$$

with equality holding if and only if $\det_{\mathbb{C}} A = 1$, which by (6.12) and (6.9) is equivalent to $A \in SU(n)$. Thus we've shown that φ is a calibration and φ restricts to the volume form of ξ if and only if ξ represents a special Lagrangian plane. Recalling definition 6.12, the proof is complete. □

Remark 6.16. Fix $\theta \in \mathbb{R}$, the form

$$\varphi_\theta = \mathrm{Re}(e^{-i\theta} dz)$$

is also a calibrating form. The contact set consists of the planes $\xi = A(\frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n})$ where $A \in U(n)$ with $\det A = e^{i\theta}$. Submanifolds calibrated by φ_θ are called special Lagrangian with **phase** θ.

As mentioned in Remark 6.14, given a graphical Lagrangian submanifold $\Sigma^n = \mathrm{graph}(f)$ over some simply-connected domain Ω, we can find $u : \Omega \to \mathbb{R}$ so that $f = \nabla u$. We now derive the associated PDE satisfied by u in the case Σ is special Lagrangian.

Since Σ is the graph of ∇u, we know that at each $x \in \Sigma$, if we let

$$v_j = \frac{\partial}{\partial x^j} + u_{kj} \frac{\partial}{\partial y^k},$$

then $v_1 \wedge \cdots \wedge v_n$ is a real multiple of the unit simple n-vector ξ representing $T_x\Sigma$. Now by Lemma 6.13 we know that $dz(\xi)$ is a unit complex number for $x \in \Sigma$. Thus by Definition 6.12, Σ is special Lagrangian with respect to some orientation if and only if $dz(\xi) = \pm 1$, or equivalently,

$$\begin{aligned} \mathrm{Im}(dz(\xi)) = 0 &\Leftrightarrow \mathrm{Im}(dz(v_1 \wedge \cdots \wedge v_n)) = 0 \Leftrightarrow \mathrm{Im} \det(dz^j(v_k)) = 0 \\ &\Leftrightarrow \mathrm{Im} \det(I + iD^2 u) = 0 \end{aligned} \tag{6.14}$$

Recalling the following identity

$$\det(I + tB) = 1 + t\,\mathrm{tr}(B) + \cdots + t^k \sigma_k(B) + \cdots + t^n \det(B),$$

where $\sigma_k(B)$ denotes the k-th elementary symmetric polynomial in the eiganvalues of B, (6.14) is equivalent to

$$\sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \sigma_{2k+1}(D^2 u) = 0. \tag{6.15}$$

If we diagonalize D^2u at a point and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues, (6.14) becomes

$$\mathrm{Im}(\prod_{k=1}^{n}(1+i\lambda_k)) = 0 \Leftrightarrow \sum_{k=1}^{n} \tan^{-1}\lambda_k \equiv 0 \bmod \pi$$

6.3. **Varitational Problems for (special) Lagrangian Submanifolds.** First we give an interesting characterization for special Lagrangian submanifolds with some phase θ.

Proposition 6.17. *Let Σ^n be a connected Lagrangian submanifold of $\mathbb{R}^{2n}$. Then Σ is calibrated by φ_θ for some θ if and only if Σ is minimal.*

Proof. By Theorem 6.13, the Lagrangian condition implies that there exists $\theta : \Sigma \to \mathbb{R}/2\pi\mathbb{Z}$ so that

$$dz(T_x\Sigma) = e^{i\theta(x)},\ \forall x \in \Sigma.$$

Locally there exists a lift $\tilde{\theta}$ of θ to an $\mathbb{R}$-valued function, and the following identity holds

$$H = J(\nabla\tilde{\theta}), \tag{6.16}$$

where H denotes the mean curavture vector of Σ. A proof of this will be given at the end. It follows that, by connectedness, $H = 0$ if and only if θ is constant, which is equivalent to Σ being calibrated by some φ_{θ_0}.

Now we prove (6.16). The key fact is that dz is a parallel $(n, 0)$-form on $\mathbb{R}^{2n}$. Since Σ is Lagrangian, by Theorem 6.13 we can choose a local orthonormal frame $e_1, \dots, e_n$ so that $\{e_k, Je_k\}_{k=1}^n$ is an orthonormal basis for $\mathbb{R}^{2n}$. Moreover, we can assume that the tangent components of $\nabla_{e_j}e_k$ vanish at a point. Now for each j, we compute

$$\begin{aligned} e_j(dz(e_1,\dots,e_n)) &= \sum_{k=1}^{n} dz(\dots,\nabla_{e_j}e_k,\dots) \\ &= \sum_{k=1}^{n}(\nabla_{e_j}e_k \cdot Je_k)dz(e_1,\dots,Je_k,\dots,e_n) \\ &= \sum_{k=1}^{n} i(\nabla_{e_j}e_k \cdot Je_k)dz(e_1,\dots,e_k,\dots,e_n) \\ &= i\left(\sum_{k=1}^{n}\nabla_{e_j}e_k \cdot Je_k\right)dz(e_1,\dots,e_n). \end{aligned}$$

Note that by the symmetries of the second fundamental form, the term in parentheses is exactly $H \cdot Je_j$. On the other hand,

$$\begin{aligned} e_j(dz(e_1,\dots,e_n)) = e_j(e^{i\tilde{\theta}}) &= ie_j(\tilde{\theta})dz(e_1,\dots,e_n) \\ &= i(\nabla\tilde{\theta}\cdot e_j)dz(e_1,\dots,e_n). \end{aligned}$$

Comparing the two computations above, we get for each j,

$$H \cdot Je_j = \nabla\tilde{\theta}\cdot e_j \iff H = J(\nabla\tilde{\theta}).$$

□

Remark 6.18. Given a function $h : \Sigma \to \mathbb{R}$, the associated **Hamiltonian vector field** is given by

$$X_h = J\nabla h$$

and has the nice property that $\mathcal{L}_{X_h}\omega = 0$, i.e. ω is preserved by the flow generated by X_h. From this and (6.16), we deduce that the mean curvature vector is locally Hamiltonian and thus the mean curvature flow preserves the Lagrangian condition.

Proposition 6.17 suggests that in order to produce a special Lagrangian submanifold in a homology class, we can try minimizing volume among homologous Lagrangian competitors. More precisely, we consider the following variational problem, which makes sense in any Kähler manifold M^{2n}:

Given a class $[\Sigma_0] \in H^n(M^{2n}, \mathbb{Z})$ with Σ_0 Lagrangian, find $\Sigma \in [\Sigma_0]$ such that

$$|\Sigma| = \inf\{|\Sigma'| | \Sigma' \in [\Sigma_0], \Sigma' \text{ Lagrangian}\}.$$

By Remark 6.18, any sufficiently regular solution to the Lagrangian Plateau problem must be minimal, for if not, then by the remark, the variation generated by $H = J(\nabla\theta)$ would produce a homologous Lagrangian submanifold with less volume.

Now we turn our focus into a more general setting, that is, minimal Lagrangian submanifold of a Kähler manifold.

Suppose (M^{2n}, g, J) is a Kähler manifold, $\Sigma^n \subset M^{2n}$ is a Lagrangian submanifold. Remember that if we take $e_1, \ldots, e_n$ to be a basis of the real tangent space $T_x\Sigma$, then $Je_1, \ldots, Je_n$ is a basis of the normal bundle of Σ. The second fundamental form is given by

$$h_{ij}^k = \langle \nabla_{e_i} e_j, Je_k \rangle.$$

Now by simple calculation,

$$h_{ij}^k = \langle \nabla_{e_i} e_j, Je_k \rangle = -\langle e_j, \nabla_{e_i}(Je_k) \rangle = \langle Je_j, \nabla_{e_i} e_k \rangle = h_{ik}^j.$$

So $h_{ij}^k = h_{kj}^i = h_{ik}^j$.

On Σ, let $\theta_H = (JH)^\#$ be the 1-form dual to JH, where H is the mean curvature vector. Also define a 2-form ρ such that $\rho(X, Y) = \mathrm{Ric}(JX, Y)$. Then we have the following relationship between these forms.

Proposition 6.19. $d(\theta_H) = \frac{1}{2}\rho|_\Sigma$.

Proof. Let $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ be a basis normal at one point, $\eta_1, \ldots, \eta_n$ be dual 1-forms of $e_1, \ldots, e_n$. Then $\theta_H = \sum_i h_{ii}^k \eta_k$. So

$$\begin{aligned}
d\theta_H = \nabla_j h_{ii}^k \eta_j \wedge \eta_k &= \frac{1}{2}(\nabla_j h_{ii}^k - \nabla_k h_{ii}^j)\eta_j \wedge \eta_k \\
&= \frac{1}{2}(\nabla_j h_{ki}^i - \nabla_k h_{ji}^i)\eta_j \wedge \eta_k \\
&= \frac{1}{2}(e_j \langle \nabla_{e_k} e_i, Je_i \rangle - e_k \langle \nabla_{e_j} e_i, Je_i \rangle)\eta_j \wedge \eta_k \\
&= \frac{1}{2}\langle R^M(e_j, e_k)e_i, Je_i \rangle \eta_j \wedge \eta_k \\
&= \frac{1}{2}\mathrm{Ric}(Je_j, e_k)\eta_j \wedge \eta_k \\
&= \frac{1}{2}\rho\bigg|_\Sigma.
\end{aligned}$$

The last step can be done as following: for any two tangent vectors X, Y,

$$\begin{aligned}
\mathrm{Ric}(X, JY) &= \sum R(X, e_i, e_i, JY) + R(X, Je_i, Je_i, JY) \\
&= \sum -R(X, e_i, Je_i, Y) + R(X, Je_i, e_i, Y) \\
&= \sum R(X, Y, e_i, Je_i).
\end{aligned}$$

By first Bianchi identity. ∎

Definition 6.20. We call a Kähler manifold (M^{2n}, g) Kähler-Eistein, if there is some constant c such that $\rho = c\omega$.

Corollary 6.21. *If (M^{2n}, g) is Kähler-Einstein and Σ^n is a Lagrangian submanifold, then $d\theta_H = 0$. In particular, mean curvature flow starting from Σ perserves Lagrangian condition.*

Corollary 6.22. *If (M^{2n}, g) is Kähler-Einstein and Σ^n is regular submanifold, and Σ is stationary for volume among Lagrangian deformation, then $H_\Sigma = 0$. That is, Σ is stationary for volume for all deformations.*

Proof. Consider the mean curvature flow from Σ. By the short time existence for mean curvature flow for regular submanifold, we get a family Σ_t of submanifolds. By the previous corollary each Σ_t is Lagrangian. Therefore from condition we know $|\Sigma_t| \geq |\Sigma|$, so

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| \geq 0.$$

On the other hand,

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| = -\int_\Sigma |H|^2 d\mu.$$

Therefore we conclude $H \equiv 0$ on Σ. □

Definition 6.23. A manifold (M^{2n}, g, J) is called Calabi-Yau, if it's Káhler and there exists a nonzero parallel $(n, 0)$ form.

On a Calabi-Yau manifold we can always write the parallel $(n, 0)$ form as $\alpha = f(z)dz^1 \wedge \ldots \wedge dz^n$ in local holomorphic coordinates, with $\nabla\alpha = 0$. We also normalize α so that $\|\alpha\| = 2^{n/2}$.

Proposition 6.24. *Suppose $\Sigma^n \subset M^{2n}$ is Lagrangian submanifold in a Calabi-Yau manifold. Then $\alpha|_\Sigma = e^{i\beta} d\,\mathrm{Vol}_\Sigma$, where β is a function satisfying $d\beta = \theta_H$.*

Proof. Again take an orthonormal basis $e_1, \ldots, e_n, Je_1, \ldots, Je_n$. Then

$$\begin{aligned}
e_j\left(\alpha(e_1, \ldots, e_n)\right) &= (\nabla_{e_j}\alpha)(e_1, \ldots, e_n) + \sum_{k=1}^n \alpha(e_1, \ldots, \nabla_{e_j} e_k), \ldots, e_n) \\
&= \sum_{k,l=1}^n \alpha(e_1, \ldots, h_{jk}^l Je_l, \ldots, e_n) \\
&= \sum_k i h_{jk}^k \alpha(e_1, \ldots, e_k, \ldots, e_n) \\
&= iH^j \alpha(e_1, \ldots, e_n).
\end{aligned}$$

Suppose locally we have a function β so that $\alpha(e_1, \ldots, e_n) = e^{i\beta}$. Then $e_j(e^{i\beta}) = iH^j e^{i\beta}$ immediately gives $\beta_j = H^j$. □

Now if in addition $H \equiv 0$ on Σ, then β is a constant.

Corollary 6.25. *Suppose M^{2n} is Calabi-Yau manifold, $\Sigma^n \subset M^{2n}$ is a submanifold. Then Σ is Lagrangian and minimal if and only if Σ is calibrated by $\mathrm{Re}(e^{-i\theta_0}\alpha)$ for some θ_0.*

When people study minimal Lagrangian submanifold of Kähler manifolds, there are three classes of Lagrangian manifolds that often come into play.

(1) Hamiltonian stationary submanifolds. $\Sigma \subset M$ is called Hamiltonian stationary, if for any compactly supported function $h \in C_0^\infty(M)$, we have $\delta_{X_h}\Sigma = 0$, where $X_h = J(\nabla h)$ is the Hamiltonian vector field associated to h. If Σ is Hamiltonian stationary, then by first variation formula, we have

$$\delta_{X_h}\Sigma = \int_\Sigma \mathrm{div}_\Sigma(X_h)$$

$$= -\int_\Sigma \langle X_h, H\rangle$$
$$= -\int_\Sigma J(\nabla h), H\rangle$$
$$= \int_\Sigma \langle \nabla h, JH\rangle$$
$$= \int_\Sigma \langle dh, \theta_H\rangle.$$

Therefore $\delta_{X_h}\Sigma = 0$ for all compactly supported smooth h if and only if $\delta\theta_H = 0$.

Note that on a Kähler-Eistein manifold we already $d\theta_H = 0$ for Lagrangian submanifold Σ. So Σ being Hamiltonian stationary is equivalent to θ_H being harmonic.

(2) Lagrangian stationary. We call Σ Lagrangian stationary if it is a critical point for any smooth deformation through Lagrangian submanifolds. That is,

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| = 0$$

if each Σ_t is Lagrangian.

There are two examples to keep in mind. The first example is a unit circle S^1 in $\mathbb{R}^2$. The mean curvature vector H of S^1 is just the position vector, and JH, the vector obtained by rotating the mean curvature vector, is just the unit tangent vector of S^1. From this observation, for any compactly supported smooth function h on S^1, the pairing $\langle \nabla h, JH\rangle = h'(s)$, where $s \in [0, 2\pi]$ is the usual parameter of S^1. However S^1 is not Lagrangian stationary, since any deformation in $\mathbb{R}^2$ is Lagrangian, and S^1 is definitely not a stationary submanifold of $\mathbb{R}^2$.

The second example is the Clifford torus $T^n \subset \mathbb{R}^{2n}$, T^n given by $S^1_{r_1} \times \dots S^1_{r_n}$, each of them embeds into $\mathbb{R}^2$. For a similar reason as before the Clifford torus is Hamiltonian stationary and not Lagrangian stationary. It is also an open question that whether Clifford tori minimize volume among Hamiltonian deformations.

(3) Minimal Lagrangian submanifolds. If Σ is simuteneously a minimal submanifold and Lagrangian, then we call it minimal Lagrangian. We will study minimal Lagrangian submanifold in more detail. Now let us look at the following example of Lagrangian Plateau problem.

Let $\Gamma \subset \mathbb{R}^4$ be a curve. By a fact in symplectic geometry, $\Gamma = \partial\Sigma^2$ for some Lagrangian surface Σ if and only if the following condition holds

$$\int_\Gamma \sum_{j=1}^{2} x^j dy^j = 0.$$

By means of minimal surface theory, there exists a least area orientable surface Σ bounding Γ satisfying that Σ is Lagrangian stationary.

Theorem 6.26 ([McL96])**.** *Given $\Sigma^n \subset M^{2n}$, M is Calabi-Yau manifold. Suppose Σ_0 is a regular special Lagrangian submanifold. Then the moduli space of special Lagrangian submanifold Σ near Σ_0 is a smooth manifold of dimension $b_1(\Sigma_0)$.*

Proof. Consider all surfaces nearby Σ_0 as a graph over Σ_0. That is, suppose

$$\Sigma = \{\exp_x(v(x)) : v \in \Gamma(N\Sigma_0), v \text{ is smooth}\}.$$

Now the condition Σ_0 being special Lagrangian gives us $\omega|_{\Sigma_0} = 0, \mathrm{Im}(\alpha)|_{\Sigma_0} = 0$. Define a map $F : \Gamma(N\Sigma_0) \to \mathscr{E}^2(\Sigma_0) \times \mathscr{E}^n(\Sigma_0)$ by letting

$$F(v) = (\exp(v)^*(\omega), \mathrm{Im}(\exp(v)^*(\alpha)))\,.$$

The linearization at 0 is given by (which will be calculated later)

$$F'(0)(v) = \left(d((Jv)^{\#}), -d*((Jv)^{\#})\right).$$

So $F'(0)$ is surjective onto the Cartesian product of Banach space of exact 2-forms with the Banach space of exact n-forms, choosing proper topology ($W^{1,2}$ topology, for example). Moreover, the null space of $F'(0)$ is precisely the space of harmonic 1-forms. By inverse function theorem and Hodge theory, all nearby special Lagrangian submanifold form a smooth manifold of dimension equal to $b_1(\Sigma_0)$.

To conclude the proof, we only need to calculate the linearization. The calculation proceeds as follows.

$$\left.\frac{d}{dt}\right|_{t=0} F(tv) = (L_v\omega, L_v\mathrm{Im}\alpha).$$

Where L_v is the Lie derivative in v direction. By Cartan formula,

$$L_v\omega = i_v(d\omega) + d(i_v\omega) = d(i_v\omega) = d((Jv)^{\#}),$$

$$L_v(\mathrm{Im}\alpha) = i_v(d(\mathrm{Im}\alpha)) + d(i_v(\mathrm{Im}\alpha)) = \mathrm{Im}(d(i_v\alpha)).$$

Suppose locally we have $v = \sum_{i=1}^n a_i Je_i$, a_i real. Then on Σ_0 we have

$$i_v(dz^1 \wedge \ldots \wedge dz^n) = \sum_{j=1}^n (-1)^{j+1} dz^1 \wedge \ldots \wedge dz^j(v) \wedge \ldots \wedge dz^n.$$

And $dz^j(v) = dz^j(a_j Je_j) = ia_j$.

On the other hand,

$$\theta_v = (Jv)^{\#} = -\sum (-1)^{j+1} a_j dz^1 \wedge \ldots \wedge dz^{j-1} \wedge dz^{j+1} \wedge \ldots \wedge dz^n.$$

We got the desired equality. □

6.4. **Minimizing volume among Lagrangians.** Given the success of the area minimization problem in minimal surface theory, it is reasonable to ask:

Question 6.27. Is there a regularity theory for area minimization if we only consider Lagrangian competitors?

We need to be more precise about what sort of area minimization problem we are trying to solve. There are various possibilities:

(1) **Plateau problem**. Given $\Gamma^{n-1} \subset \mathbb{R}^{2n}$, find an oriented Lagrangian Σ^n with $\partial\Sigma = \Gamma$ and

$$|\Sigma| = \min\{|\Sigma'| : \Sigma' \text{ Lagragian and } \partial\Sigma' = \Gamma\}.$$

(2) **Minimization in homology**. Suppose M^{2n} is compact and that $\alpha \in H_n(M;\mathbb{Z})$ is a homology class that contains at least one Lagrangian submanifold. We wish to find a Lagrangian submanifold $\Sigma \in \alpha$ with

$$|\Sigma| = \min\{|\Sigma'| : \Sigma' \text{ is Lagrangian and belongs to homology class } \alpha\}.$$

Remark 6.28. When $n = 2$ there is an alternative approach to area minimization, similar to the mapping problem for minimal surfaces, which we will address in the next section.

As with minimal surface theory, existence and weak compactness play a key role in the theory of area minimization among Lagrangians. To that end we need to work in a larger class of submanifolds than just smooth submanifolds. We work with:

Definition 6.29. A Lagragian integral current $T = (\Sigma^n, \Theta, \xi)$ is an integral current H^n-a.e. of whose tangent spaces are Lagrangian. Recall that an integral current acts of smooth compactly supported n-forms as

$$\int_T \eta = \int_{\Sigma^n} \langle \eta|_x, \xi|_x \rangle \, \Theta(x) \, dH^n(x),$$

and that its total mass is defined to be

$$\mathbb{M}(T) = \int_{\Sigma^n} \Theta(x) \, dH^n(x).$$

There is a standard weak topology associated with Lagrangian integral currents, the weak-$*$ topology. In particular, $T_i \rightharpoonup T$ if for all compactly supported smooth n-forms η,

$$\int_{T_i} \eta \to \int_T \eta.$$

We note in passing that this is equivalent to the so-called flat convergence.

Lagrangian integral currents do satisfy a compactness theorem, like we wanted.

Theorem 6.30. *Let $\{T_i\}$ be a sequence of Lagrangian integral currents with $\mathbb{M}(T_i) \leq C$. Then there exists a Lagrangian integral current T with $\mathbb{M}(T) \leq C$ such that, after passing to a subsequence, $T_{i'} \rightharpoonup T$. Furthermore, if the T_i all belong to the same homology class α, then T does too. Likewise, if $\partial T_i = \Gamma$ for all i, then $\partial T = \Gamma$ too.*

Remark 6.31. Other than the additional conclusion that the limit is Lagrangian, this follows from the Federer-Fleming theory of integral currents [**?**]. To check that the limit current is Lagrangian (i.e., Lagrangians are closed under weak convergence) we note that it suffices to check

$$\int_T \omega \wedge \eta = 0$$

for every compactly supported smooth $(n-2)$-form η, where ω is the Kähler form. This is an integral condition, and is indeed preserved by weak limits.

A large part of the regularity theory for area minimization in the minimal surface setting relies on the monotonicity formula, which helps establish upper bounds for volumes on all scales. Unfortunately, the general theory of Lagrangian integral currents and Lagrangian minimizers lacks a monotonicity formula.

Proposition 6.32. *The cylinder $\Sigma^2 = S^1(\varepsilon) \times [0,1] \subset \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4 = \{(x^1, y^1, x^2, y^2)\}$ solves the Lagrangian Plateau problem.*

Proof. Let $\widetilde{\Sigma}$ be any other Lagrangian surface with $\partial\widetilde{\Sigma} = \partial\Sigma$ (in the sense of currents). For a.e. $t \in (0,1)$, $\widetilde{\Sigma} \cap \{x^2 \leq t\}$ is a surface with boundary and therefore $\widetilde{\Sigma} \cap \{x^2 = t\}$ consists of finitely many closed curves. By Stokes's theorem and the Lagrangian property,

$$0 = \int_{\widetilde{\Sigma} \cap \{x^2 \leq t\}} \omega = \int_{\widetilde{\Sigma} \cap \{x^2 = t\}} (-y^1 dx^1) - \int_{S^1(\varepsilon) \times \{0\}} (-y^1 dx^1) = \int_{\widetilde{\Sigma} \cap \{x^2 = t\}} (-y^1 dx^1) - \pi\varepsilon^2.$$

By Green's theorem the integral on the right measures $\pm$ the total signed area enclosed by the projections of the curves comprising $\widetilde{\Sigma} \cap \{x^2 = t\}$ onto the $x^1 y^1$ plane. Then by the isoperimetric inequality on $\mathbb{R}^2$,

$$H^1(\widetilde{\Sigma} \cap \{x^2 = t\}) \geq 2\pi\varepsilon, \text{ for a.e. } t \in (0,1).$$

Now by the coarea formula,

$$\text{area}(\widetilde{\Sigma}) \geq \int_0^1 H^1(\widetilde{\Sigma} \cap \{x^2 = t\}) \, dt \geq 2\pi\varepsilon,$$

which is the area of the cylinder Σ. The result follows since Σ is evidently Lagrangian. $\square$

Remark 6.33. We just showed that the cylinder $\Sigma^2 = S^1(\varepsilon) \times [0,1] \subset \mathbb{R}^4$ solves the Lagrangian Plateau problem, but the monotonicity formula evidently doesn't apply.

6.5. **Lagrangian 2D mapping problem.** When $n = 2$ we may attempt to minimize area as with the mapping problem in minimal surface theory:

(1) **(Classical) Plateau problem**. Given $\Gamma \subset \mathbb{R}^{2n}$, minimize $|u(D)|$ over all $u : D \to \mathbb{R}^{2n}$ such that $u(\partial D) = \Gamma$ and $u(D)$ is Lagrangian.
(2) **Minimization in homotopy**. Suppose M^{2n} is compact, Σ_g is a compact surface, and $u_0 : \Sigma \to M^{2n}$ is such that $u_0(\Sigma_g)$ is Lagrangian. Find $u : \Sigma_g \to M^{2n}$ such that $u(\Sigma_g)$ is Lagrangian and
$$|u(\Sigma_g)| = \min\{|u'(\Sigma_g)| \text{ where } u' : \Sigma_g \to M^{2n} \text{ is Lagrangian and homotopic to } u_0\}.$$

Like before, it is convenient to formulate a weak notion of Lagrangian maps.

Definition 6.34. A map $u \in W^{1,2}(\Sigma^2, M^{2n})$ is weakly Lagrangian if $u^*\omega = 0$ a.e. on Σ, where ω is the Kähler form of M^{2n}. Equivalently, u is weakly Lagrangian if
$$\int_\Sigma f u^*\omega = 0$$
for all $f \in C_c^\infty(\Sigma)$.

The following observation is the mapping problem equivalent of Theorem 6.30:

Proposition 6.35. *For every $C > 0$, the set*
$$\{u \in W^{1,2}(\Sigma^2, M^{2n}) \text{ is weakly Lagrangian and } \|u\|_{1,2} \le C\}$$
is closed in the weak topology.

Proof. We present the proof in the case $M^{2n} = \mathbb{R}^{2n}$, though the argument carries through to the general case with some modifications.

Let $u_i \rightharpoonup u$, with $\|u_i\|_{1,2}$, $\|u\|_{1,2} \le C$. On $\mathbb{R}^{2n}$ the Kähler form ω is exact: $\omega = d\eta$. Let $f \in C_c^\infty(\Sigma)$. Then
$$0 = \int_\Sigma f u_i^*\omega = \int_\Sigma f u_i^* d\eta = \int_\Sigma f d(u_i^*\eta) = -\int_\Sigma df \wedge u_i^*\eta.$$
The latter is a linear combination (with smooth and appropriately convergent coefficients) of first derivatives of u_i, and therefore we may take $i \to \infty$ and deduce
$$0 = -\int_\Sigma df \wedge u^*\eta = \int_\Sigma f d(u^*\eta) = \int_\Sigma f u^*\omega.$$
The claim follows since $f \in C_c^\infty(\Sigma)$ was arbitrary. $\square$

In the Lagrangian 2D mapping problem, unlike the previous section, it turns out that we do have a notion of monotonicity and therefore there is a way to work out a regularity theory as in the case of minimal surfaces. This has been carried out in [**?**] when $n = 2$. Putting the issue of existence of minimizers to the side, the regularity theory here goes as follows:

(1) Derive a monotonicity formula that allows for upper volume control on arbitrarily small scales for weakly Lagrangian, weakly conformal maps.
(2) Use this volume control to derive a global Hölder estimate for weakly Lagrangian, weakly conformal maps.
(3) Introduce the concept of tangent cones of weakly Lagrangian, weakly conformal maps and study the relationship between singularities and the cone's flatness or lack thereof.

Putting this all together we get:

Theorem 6.36 (Schoen-Wolfson, [?]). *Let $u : D \to M^4$ be an area minimizing weakly Lagrangian, weakly conformal map. If D' is a smaller disk, there is a finite set $S \subset D'$ such that $u|_{D' \setminus S}$ is a smooth immersion. Every point in S is either a branch point of u or a point at which u has a nonflat tangent cone. The map u is smooth everywhere except across points with nonflat tangent cones, where it is Lipschitz.*

Remark 6.37. The requirement that u be weakly conformal can be arranged alongside the existence theory.

6.6. **Lagrangian cones.** Lagrangian cones come up in the process of blowing up singular points as in the minimal surface theory.

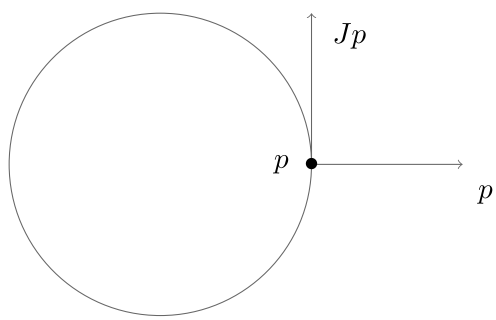

FIGURE 4. Cone link, Legendrian links

Definition 6.38. The link Σ^{m-1} of an m-dimensional cone C in $\mathbb{R}^{2n}$ is defined to be $\Sigma = S^{2n-1} \cap C$. It is said to be Legendrian when $T_p\Sigma \subseteq (Jp)^\perp$ for all $p \in \Sigma$.

From this point on we specialize to $m = 2$. It is not hard to prove that:

Proposition 6.39. *A cone C is Lagrangian if and only if its link Σ is Legendrian.*

Proof. We have that $T_pC = T_p\Sigma \oplus \langle p \rangle$. ($\Rightarrow$) If C is Lagrangian then $J(T_pC) \perp T_pC$, so every $v \in T_p\Sigma$ must be $\perp$ to Jp, so Σ is Legendrian.

($\Leftarrow$) If Σ is Legendrian then $Jp \perp T_p\Sigma$ and therefore $Jp \perp T_pC$. Likewise, since $T_p\Sigma$ is 1-dimensional we evidently have $J(T_p\Sigma) \perp T_p\Sigma$ and, finally, for all $v \in T_p\Sigma$,

$$\langle Jv, p \rangle = \omega(v, p) = -\omega(p, v) = -\langle Jp, v \rangle = 0,$$

so, altogether, $J(T_pC) \perp T_pC$ which shows that C is Lagrangian. $\square$

Corollary 6.40. *C is minimal and Lagrangian if and only if Σ is minimal and Legendrian.*

A singular area minimizing Lagrangian map $u : D \to \mathbb{R}^{2n}$ will give rise to an area minimizing Lagrangian tangent cone $C^2 \subset \mathbb{R}^{2n}$. It is important to understand the structure of these cones for regularity theory and to do so we restrict to $n = 2$. The following theorem of [?] exhausts the list of Hamiltonian stationary 2-cones in $\mathbb{R}^4$:

Theorem 6.41 ([?]). *Links of Hamiltonian stationary cones $C^2 \subset \mathbb{R}^4$ look like*

$$\gamma(s) = \frac{1}{\sqrt{p+q}} \begin{pmatrix} \sqrt{q} e^{is\sqrt{p/q}} \\ i\sqrt{p} e^{-is\sqrt{q/p}} \end{pmatrix},$$

with $0 \leq s \leq 2\pi\sqrt{pq}$ and p, q coprime.

Proof. Let β be the Lagrangian angle, which satisfies $H = J\nabla\beta$ or, equivalently, $H \lrcorner\, \omega = -d\beta$. Note that β is degree 0 homogeneous on C. We claim that $d\beta$ is a harmonic 1-form on Σ.

Let $X = J\nabla h$ be a Hamiltonian vector field. If Σ is Hamiltonian stationary then

$$0 = \int_\Sigma \langle X, H\rangle = \int_\Sigma \langle J\nabla h, H\rangle = -\int_\Sigma \omega(H, \nabla h) = -\int_\Sigma \langle H \lrcorner\, \omega, dh\rangle = -\int_\Sigma h\, \delta(H \lrcorner\, \omega)$$

for all smooth compactly supported h, so $\delta(H \lrcorner\, \omega) = 0$. By a computation (see [?]) we know that $d(H \lrcorner\, \omega) = 0$ on $\mathbb{R}^4$; in particular we have $d(H \lrcorner\, \omega) = 0$ and $\delta(H \lrcorner\, \omega) = 0$ which makes $H \lrcorner\, \omega = -d\beta$ a harmonic 1-form on Σ, as claimed.

In turn this means that β is harmonic and therefore of the form $\beta = 2as$. Writing $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ we find that the condition for it to be a Legendrian curve on the sphere, the complex 2×2 matrix (γ, γ') must be SU(2). Since $\det(\gamma, \gamma') = e^{i\beta}$. The result follows by fairly straightforward algebraic manipulations. □

Remark 6.42. These curves $C_{p,q}$ are not great circles except when $p = q = 1$, and in general they lie on Clifford tori in S^3. If either $p = 1$ or $q = 1$ then the curves are unknotted; otherwise they are knotted.

By studying the second variation formula one obtains:

Theorem 6.43. *If $|p - q| > 1$ then $C_{p,q}$ is strictly unstable for Hamiltonian variations compactly supported on $C \setminus \{0\}$. For $|p - q| = 1$, $C_{p,q}$ is strictly stable.*

By an indirect argument one can further show:

Theorem 6.44. *There exists at least one p for which the Hamiltonian stationary $C_{p,p+1}$ minimizes area among Lagrangian competitors homeomorphic to a disk.*

Remark 6.45. The argument above is indirect, and so it is not known which cones $C_{p,p+1}$ are minimizers.

6.7. **Monotonicity and regularity of minimizers in 2D.** First let's recall the monotonicity formula for stationary submanifolds of Euclidean space:

Theorem 6.46 (Monotonicity for minimal submanifolds)**.** *Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold without boundary. Then*

$$\frac{d}{d\sigma}\Big(\sigma^{-k}|\Sigma \cap B_\sigma|\Big) \geq 0.$$

Proof. This follows from the first variation formula. Let $\mathbf{x}$ denote the position vector field, $r = |\mathbf{x}|$, ζ some smooth cutoff function, and $X = \zeta(r)\mathbf{x}$. Then by stationarity

$$0 = \int_\Sigma \operatorname{div}_\Sigma X\, d\mu = \int_\Sigma k\zeta(r) + r\zeta'(r)|\nabla^T r|^2\, d\mu.$$

The result follows by taking ζ to approximate the indicator function of $[0, \sigma] \subset \mathbb{R}$. □

This monotonicity formula is very important for regularity theory in the minimal surface setting. Ideally, we can find a similar useful and monotone quantity in the Lagrangian area minimization setting.

Monotonicity in this new setting is more difficult. We've seen in a previous section that $S^1(\varepsilon) \times \mathbb{R}$ is Hamiltonian stationary (in fact, minimizing among Lagrangians) and yet it evidently does not satisfy monotonicity. One problem is that we don't have very good choices of (Hamiltonian) deformations. For example if $h(x, y) = h(x^1, x^2, y^1, y^2)$ is a Hamiltonian function with compact support, then its associated Hamiltonian vector field

$$X_h = h_x \frac{\partial}{\partial y} - h_y \frac{\partial}{\partial x}$$

preserves ω and volume. Ideally we can make use of vector fields that represent dilations or other collapsing deformations.

Definition 6.47. Let $u : \Sigma \to \mathbb{R}^4$ be a Lagrangian map and $\gamma : S^1 \to \Sigma$ be a curve into Σ. If η is such that $\omega = d\eta$ on $\mathbb{R}^4$ (e.g., $\eta = xdy - ydx$) then we define

$$\text{period}(\gamma) = \int_\gamma u^*\eta.$$

The Lagrangian map $u : \Sigma \to \mathbb{R}^4$ is said to be exact if all its periods vanish. In that case $u^*\eta$ is an exact 1-form.

Proposition 6.48. *The period of γ does not depend on our particular choice of η and only depends on the homology class $[\gamma] \in H_1(\Sigma, \mathbb{Z})$.*

Proof. Let $\omega = d\eta' = d\eta$. Then $d(\eta - \eta') = 0$, so $\eta - \eta'$ is a closed 1-form, so $\eta - \eta' = d\psi$ on $\mathbb{R}^4$, so

$$\int_\gamma u^*\eta - \int_\gamma u^*\eta' = \int_\gamma u^*(\eta - \eta') = \int_\gamma u^* d\psi = 0.$$

Likewise, if γ' is another closed curve with $[\gamma'] - [\gamma] = 0 \in H_1(\Sigma, \mathbb{Z})$, then $\gamma - \gamma' = \partial\Omega$ and

$$\int_\gamma u^*\eta - \int_{\gamma'} u^*\eta = \int_{\partial\Omega} u^*\eta = \int_\Omega u^* d\eta = 0,$$

since u is a Lagrangian map. □

The following alternative characterization of exactness of Lagrangian maps will be useful:

Proposition 6.49. *Endow $\mathbb{R}^5$ with coordinates $(x, y, \varphi) = (x^1, x^2, y^1, y^2, \varphi)$ and consider the projection $\pi : (x, y, \varphi) \mapsto (x, y)$ onto $\mathbb{R}^4$. Let $\alpha = d\varphi - (xdy - ydx)$ be the contact 1-form on $\mathbb{R}^5$. A Lagrangian map $u : \Sigma \to \mathbb{R}^4$ is exact if and only if it admits a Legendrian lift $\widetilde{u} : \Sigma \to \mathbb{R}^5$, i.e., there exists $\widetilde{u}$ with $\pi \circ \widetilde{u} = u$ and $\widetilde{u}^*\alpha = 0$.*

Proof. If u admits a Legendrian lift $\widetilde{u}$, then we have $0 = \widetilde{u}^*\alpha = \widetilde{u}^* d\varphi - \widetilde{u}^*\eta$, so $u^*\eta = \widetilde{u}^*\eta$ is an exact 1-form on Σ, so $u : \Sigma \to \mathbb{R}^4$ is Lagrangian. Conversely, if $u : \Sigma \to \mathbb{R}^4$ is exact and Lagrangian, then $u^*\eta = d\phi$ for a function ϕ on Σ. The map $\widetilde{u} = (u, \phi)$ is then Legendrian. □

Not every Lagrangian map is exact, but instead they are locally exact in a suitable sense. This is good enough for the purposes of regularity theory.

Definition 6.50. A diffeomorphism F on $\mathbb{R}^5$ is a contact transformation if $F^*\alpha = f\alpha$ for a scalar field f. Note that if $\widetilde{u}$ is a horizontal Legendrian map then so is $F \circ \widetilde{u}$.

There is a large class of contact transformations on $\mathbb{R}^5$. In fact, a computation shows that:

Proposition 6.51. *If $h = h(x, y, \varphi)$ is smooth, then the vector field*

$$X_h = h_x \frac{\partial}{\partial y} - h_y \frac{\partial}{\partial x} - h_\varphi \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + (-2h + (x h_x + y h_y)) \frac{\partial}{\partial \varphi}$$

generates contact diffeomorphisms.

We now vaguely describe the proof of the monotonicity formula.

Auxiliary variables. First we will need to introduce new coordinates. Recall that in the minimal surface case, monotonicity was obtained by plugging $X = \mathbf{x} = \nabla s$, $s = \frac{1}{2}(x^2 + y^2)$ into the first variation formula. To that end, introduce:

(1) We use $s = \frac{1}{2}(x^2 + y^2)$ in this setting, too. Its associated vector field is

$$X_s = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} = J(\nabla s).$$

(2) We will also use the φ coordinate function, whose associated vector field is
$$X_\varphi = -\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = -\nabla s = J(X_s).$$
(3) Write $\widetilde{s} = \sqrt{s^2+\varphi^2}$, which will play the role of the square of the distance function, and define
$$t = \log\widetilde{s} \qquad \text{and} \qquad \theta = \arctan(\varphi/s) \in [-\pi/2, \pi/2].$$
Equivalently, t, θ satisfy $t + i\theta = \log(s + i\varphi)$.

(4) We will be considering Hamiltonian maps in these new t, θ coordinates; i.e., $\eta(t,\theta)$. The divergence of the vector field generated by η is computed to be
$$\begin{aligned}\operatorname{div}_\Sigma X_\eta &= (2\eta_{\theta t} - 2\eta_\theta)|\nabla^T\theta|^2 - 2(\eta_{\theta t}\widetilde{s}^{-1}\cos\theta + \eta_t\widetilde{s}^{-1}\sin\theta) \\ &\quad + (\eta_{tt} - \eta_{\theta\theta} - 2\eta_t)\nabla^T\theta\cdot\nabla^T t.\end{aligned}$$

Goal. We will try to particularly convenient functions η for which the last term drops out. To do this we will arrange that $\eta_{tt} - \eta_{\theta\theta} - 2\eta_t = 0$ by treating it as a wave equation with time variable θ. More specifically, we will solve
$$\begin{aligned}\eta_{tt}(t,\theta) - \eta_{\theta\theta}(t,\theta) - 2\eta_t(t,\theta) &= 0 \quad \text{ on } (-\infty,\infty)\times[-\pi/2,\pi/2] \\ \eta(t,0) &= 0 \\ \eta_\theta(t,0) &= \zeta(t),\end{aligned}$$
for a smooth function ζ which is such that
$$\zeta \text{ is non-increasing, } \zeta \equiv 0 \text{ for } t \geq \log(1/2) \text{ and } \zeta(t) = 1 - 2\lambda e^t \text{ for } t \leq -c,$$
for constants c, $\lambda > 0$ to be determined. If we can solve this wave equation, the divergence of X_η collapses to
$$\operatorname{div}_\Sigma X_\eta = -2G\cdot|\nabla^T\theta|^2 + 4F$$
where $G = \eta_\theta - \eta_{\theta t}$ and $F = -\frac{1}{2}e^{-t}(\eta_{\theta t}\cos\theta - \eta_t\sin\theta)$.

Remark 6.52. In regularity theory we mostly care about the monotonicity formula on very small scales. In our logarithmic coordinates, this translates into the fact that we are only interested in the behavior of η for very negative values of t. This explains why we're diligently prescribing the behavior of ζ for $t \leq -c$ and have it vanish for $t \geq \log(1/2)$. In particular, this construction will yield a Hamiltonian that is supported in B_1.

Solving the wave equation. Because of the finite speed of propagation in wave equations and the explicit description of ζ for $t \leq -c$, we can explicitly compute
$$\eta(t,\theta) = \theta - 2\lambda e^t\sin\theta \text{ for } t \leq -c - \pi/2.$$
In this same region we see that
$$\begin{aligned}G &= 1 - 2\lambda e^t\cos\theta + 2\lambda e^t\cos\theta = 1, \text{ and} \\ F &= -\frac{1}{2}e^{-t}(-2\lambda e^t\cos^2\theta - 2\lambda e^t\sin^2\theta) = \lambda,\end{aligned}$$
and so $\operatorname{div}_\Sigma X_\eta = -2|\nabla^T\theta|^2 + 4$ for $t \leq -c - \pi/2$.

Rescaling the support. It was pointed out in Remark 6.52 that η is a Hamiltonian with support in B_1. The functions
$$\begin{aligned}\eta_a(t,\theta) &= \eta(t - 2\log a, \theta), \\ F_a(t,\theta) &= F(t - 2\log a, \theta), \text{ and} \\ G_a(t,\theta) &= G(t - 2\log a, \theta)\end{aligned}$$

give rise to a Hamiltonian supported in B_a instead. This contact vector field satisfies

$$\operatorname{div}_\Sigma X_{\eta_a} = -2G_a|\nabla^T\theta|^2 + \frac{4}{a^2}F_a.$$

Monotonicity. The monotonicity formula is obtained, as in the minimal surface case, by studying the first variation formula at two different scales a and b. This is done by plugging η_a, η_b into the first variation formula. The following lemma is key in handling the various error terms:

Lemma 6.53. *There exist c, $\lambda > 0$ such that $F \geq 0$, $0 \leq G \leq 1$. Furthermore, there exists $\theta_0 \in (0,1)$ such that $G_a - G_b \geq 0$ for $0 < b \leq \theta_0 a$.*

By putting this all together we finally obtain:

Theorem 6.54 (Density bounds)**.** *The limit*

$$\Theta(p) = \lim_{\sigma\downarrow 0} \frac{1}{\pi\sigma^2}\int_\Sigma F_\sigma \, dA$$

exists and is upper semicontinuous. Furthermore,

$$c_1 \leq \frac{\operatorname{Area}(u(\Sigma)\cap B_\sigma)}{\pi\sigma^2} \leq c_2$$

for small enough $\sigma > 0$.

From this we get:

Theorem 6.55 (Hölder regularity)**.** *Suppose $u : D_1 \to N^4$ is weakly conformal, exact, and Lagrangian stationary. Write $\widetilde{u}$ for the Legendrian lift of u. If there exist r_0, $c > 0$ such that*

$$\operatorname{Area}(\widetilde{u}(D_1)\cap B_r(p)) \leq cr^2$$

for all $p \in N$, $r \leq r_0$, then $\widetilde{u}$ is Hölder continuous in $D_{1/2}$. If $u|_{\partial D_1} : \partial D_1 \to N$ has finite energy then $\widetilde{u}$ is Hölder continuous on all of D_1. There exists $\varepsilon_0 > 0$ such that if $\operatorname{Area}(u(D_1)) \leq \varepsilon_0$, then there is a uniform upper bound on the global Hölder modulus of continuity of u on D_1.

Partial regularity. So far we have shown that our minimizers (should they exist and be exact) satisfy a monotonicity are globally Hölder continuous. By a tilt-excess decay type iteration scheme (similar to the one in Allard's theorem) one can show that being weakly close to a plane gives $C^{1,\alpha}$ bounds and, at this point, this can be improved to C^∞ bounds by adapting the standard argument. Furthermore, the exactness condition can be lifted because all Lagrangian maps are locally exact and regularity is a local result.

Theorem 6.56 (Regularity near almost-flat points)**.** *Let $u : D \to \mathbb{R}^4$ be weakly conformal and area minimizing among Lagrangians in $W^{1,2}(D,\mathbb{R}^4)\cap C^0(\partial D)$ and $u(\partial D) = \Gamma$. If $z \in D$ is such that u is approximately differentiable at z; i.e., there exists an affine map $\ell : \mathbb{R}^2 \to \mathbb{R}^4$ such that*

$$\int_D |u-\ell|^2 + |\nabla u - \nabla \ell|^2 \leq \varepsilon,$$

for ε sufficiently small, then u is C^∞ near z.

Main regularity. We now come to the main regularity theorem. We study minimizing Lagrangian maps $u : D \to N^4$ and separate points $p \in D$ into two categories:

(1) Points at which every tangent cone is flat. These form an open set Ω. In this case the tangent cones are weakly conformal maps of $\mathbb{C}$ into $\mathbb{C}$ and can be shown to be of the form az^n, $a \in \mathbb{C}$. These are the branch points, they are isolated, and u is a smooth immersion on the open set Ω away from the branch points, and is in fact smooth across the branch points (with vanishing differential).

(2) Points at which there exists a nonflat tangent cone. By studying those cones carefully [**?**] show that u is Lipschitz across these singular points, and that these points also form a discrete set.

Putting it altogether we obtain:

Theorem 6.57 (Schoen-Wolfson, [**?**])**.** *Let $u : D \to N^4$ be an area minimizing weakly Lagrangian, weakly conformal map. If D' is a smaller disk, there is a finite set $S \subset D'$ such that $u|_{D' \setminus S}$ is a smooth immersion. Every point in S is either a branch point of u or a point at which u has a nonflat tangent cone. The map u is smooth everywhere except across points with nonflat tangent cones, where it is Lipschitz.*

Remark 6.58. This has been extended to higher codimensions in [**?**].

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