Syllabus: Economics 211, Second Half

This module of the PhD economics core covers basic building-block subjects in aggregate applied economics: consumption, investment, and the labor market.

There will be two major problem sets. There will be a quiz on the last day of class, Wednesday, March 9. The grade will be based half on the quiz and half on the problem sets.

The class will meet Monday and Wednesday in Economics 140 from 10 to 11:50, with a break. The section will meet on Fridays from 9:00 to 10:50 in 420-050.

Class materials will be available from https://coursework.stanford.edu/.

Course schedule:

<table>
<thead>
<tr>
<th>Monday</th>
<th>Wednesday</th>
</tr>
</thead>
<tbody>
<tr>
<td>9-Feb</td>
<td>Consumption</td>
</tr>
<tr>
<td>14-Feb</td>
<td>Consumption</td>
</tr>
<tr>
<td>21-Feb</td>
<td>No class (holiday)</td>
</tr>
<tr>
<td>28-Feb</td>
<td>Investment</td>
</tr>
<tr>
<td>7-Mar</td>
<td>Labor market</td>
</tr>
<tr>
<td>2-Mar</td>
<td>Labor market</td>
</tr>
<tr>
<td>9-Mar</td>
<td>Quiz in class</td>
</tr>
</tbody>
</table>

Problem sets

You are allowed and encouraged to work in groups, but you are required to turn in your own writeup, prepared individually. If anything in the assignment is unclear or you are completely stuck, you can ask Krishna questions by email or during office hours. His answers will try to provide you with the right amount of help to resume your learning process. He won’t give away solutions. He will stop answering questions on a given
assignment 24 hours before it is due. This will encourage you to start in time. Also, there will be no option value to waiting for late tips. Late homework will be graded, but you lose 1/3 of the available points per day that you are late.

Notes and readings
Notes will be posted and distributed in advance of each of the three topics. You are strictly responsible for all material in the notes and lectures. Readings are optional.

Consumption


Attanasio, Orazio "Consumption" *Handbook of Macroeconomics*, vol. 1B, Chapter 11, pp. 741-812 (ScienceDirect)


Abel, Andrew, "Consumption and Investment," Chapter 14 in Benjamin Friedman and Frank Hahn (eds.) *Handbook of Monetary Economics*, North-Holland, 1990 (ScienceDirect)


Investment
Romer, Chapter 8: "Investment"

Abel, Andrew, "Consumption and Investment," Chapter 14 in Benjamin Friedman and Frank Hahn (eds.) *Handbook of Monetary Economics*, North-Holland, 1990 (ScienceDirect)

Caballero, Ricardo J. "Aggregate Investment" *Handbook of Macroeconomics*, vol. 1B, Chapter 12, pp. 813-862 (ScienceDirect)

Ramey, Valerie A., and Kenneth D. West, "Inventories" *Handbook of Macroeconomics*, vol. 1B, Chapter 13, pp. 863-926 (ScienceDirect)


Dixit, Avinash, and Robert Pindyck, *Investment under Uncertainty*, 1994

**The Labor Market**

Romer, Chapter 9, "Unemployment"


Mortensen, Dale T., and Christopher A. Pissarides, "Job Reallocation, Employment Fluctuations, and Unemployment" *Handbook of Macroeconomics*, vol. 1B, Chapter 18, pp. 1171-1228 (ScienceDirect)

http://home.uchicago.edu/~shimer/wp/published/uv.pdf


1 Quiz

Answer all questions. Some contain ambiguities. State reasonable assumptions to resolve any ambiguity you find and then answer accordingly. Also note that the quiz tests your ability to recognize concepts by name and turn them into their mathematical expressions.

1. A consumer has a strictly concave utility function $u(c)$ and consumes for just two periods, 1 and 2, with no discount for the second period. The consumer has preferences that result in precautionary behavior. The consumer receives random wealth in the second period of $2 - \epsilon$ with probability 0.5 and $2 + \epsilon$ with probability 0.5. The consumer can borrow at an interest rate of zero to finance consumption in the first period.

(a) Write down the consumer’s first-order condition without using an expectation operator.

(b) Find the derivative $dc_1/d\epsilon$ at $\epsilon = 0$ and explain its value intuitively.

(c) Show that the same derivative is negative for $\epsilon > 0$ and relate this finding to precautionary behavior.

(d) Does the specific utility function $u(c) = -(1 + c)^{-1}$ result in precautionary behavior?

2. A competitive industry contains a large number of firms with identical production functions $y = Ak$, where $A$ is a fixed parameter. Demand for the product is $p\epsilon$, where $p$ is the output price. The rental price of capital is $z$.

(a) What is one firm’s demand for capital, as a function of $p$ and $z$?

(b) What are the equilibrium values of $p$ and total industry output and capital, $Y$ and $K$?

(c) Now suppose that the rental price of capital depends on the amount of capital used in the industry—the supply function for capital is $z = K^\phi$. Find the equilibrium values of $z$, $p$, $Y$, and $K$.

(d) How do $z$, $p$, $Y$, and $K$ change if productivity $A$ rises?

3. An employer derives revenue $Z$ from employing a worker. The worker’s opportunity cost is $C$ (called $U - V$ in class). They pick a wage $W$ before knowing the realizations of $Z$ and $C$. These are random variables distributed uniformly and independently on the square where both variables lie between zero and one.

(a) Given $W$, what is the probability of an inefficient layoff?

(b) What is the probability of an inefficient quit?

(c) What wage minimizes the probability of an inefficient separation?

(d) As a general matter, does minimizing the probability of an inefficient separation result in the optimal wage? What about in this special case? (intuitive answer only, no math)
2 Quiz Solutions

2.1 Problem 1

1. The consumers problem is:

\[ \max \mathbb{E} [u(c_1) + u(y - c_1)] \]

Substituting in the budget constraint directly:

\[ u(c_1) + .5 [u(2 - \epsilon - c_1) + u(2 + \epsilon - c_1)] \]

With resulting FOC:

\[ u'(c_1) = .5 [u'(2 - \epsilon - c_1) + u'(2 + \epsilon - c_1)] \]

2. Differentiate the above expression wrt \( \epsilon \):

\[ u''(c_1) \frac{\partial c_1}{\partial \epsilon} = .5 \left[ u''(2 - \epsilon - c_1) \left( -1 - \frac{\partial c_1}{\partial \epsilon} \right) + u''(2 + \epsilon - c_1) \left( 1 - \frac{\partial c_1}{\partial \epsilon} \right) \right] \]

Rearrange:

\[ \frac{\partial c_1}{\partial \epsilon} = 0 \]

At \( \epsilon = 0 \), the expression on the RHS is 0. Generically, the expression on the LHS in brackets is not zero so we conclude:

\[ \frac{\partial c_1}{\partial \epsilon} = 0 \]

The intuition for this result is that consumers are locally risk neutral. They are not risk averse to very small amounts of uncertainty.

3. Since \( u'' > 0 \) (the agent displays precautionary behavior), and \( \epsilon > 0 \)

\[ u''(2 + \epsilon - c_1) - u''(2 - \epsilon - c_1) > 0 \]

And since, generically, \( u'' < 0 \), it must be that \( \frac{\partial c_1}{\partial \epsilon} < 0 \). Increased uncertainty results in deferred consumption (increased savings in period 1) because of the precautionary motive.

4. Yes.

\[ u'(c) = (1 + c)^{-2} \]
\[ u''(c) = -2(1 + c)^{-3} \]
\[ u'''(c) = 6(1 + c)^{-4} > 0 \]
2.2 Problem 2

1. The price taking firms problem is:

$$\max pA k - zk$$

Which is $\infty$ if $pA > z$, indeterminate if $pA = z$ and 0 if $pA < z$

2. Thus in equilibrium:

$$p = \frac{z}{A}$$

From the demand function we have:

$$Y = p^{-\epsilon} = \left(\frac{z}{A}\right)^{-\epsilon}$$

And from the production function we have:

$$K = \frac{Y}{A} = z^{-\epsilon} A^{\epsilon-1}$$

3. We equate capital supply with capital demand to recover the rental price:

$$K_d = z^{-\epsilon} A^{\epsilon-1} = z^{\frac{1}{\phi}} = K_s$$

$$z = A^{\frac{\epsilon-1}{\phi + \epsilon}}$$

$$p = \frac{z}{A} = A^{\frac{\epsilon-1}{\phi + \epsilon}} = A^{\frac{-1}{\phi + \epsilon}} = A^{\frac{-1}{1 + \phi \epsilon}}$$

$$Y = p^{-\epsilon} = A^{\frac{\phi + \epsilon}{1 + \phi \epsilon}}$$

$$K = z^{\frac{1}{\phi}} = A^{\frac{\epsilon-1}{1 + \phi \epsilon}}$$

4. If $A \uparrow$ then variables will increase if the following conditions hold and weakly decrease if not (remain constant iff expressions = 0):

$$z: \frac{\epsilon - 1}{\frac{1}{\phi} + \epsilon} > 0$$

$$p: \frac{-\phi - 1}{1 + \phi \epsilon} > 0$$

$$Y: \frac{\epsilon \phi + \epsilon}{1 + \phi \epsilon} > 0$$

$$K: \frac{\epsilon - 1}{1 + \phi \epsilon} > 0$$
2.3 Problem 3

1. Inefficient layoff implies $Z < W$ and $Z > C$. The probability of that event (for a fixed wage is):

$$\int_0^W \int_0^Z 1 \, dc \, dz = \int_0^W zdz = \frac{1}{2}w^2$$

2. Inefficient quite implies $C > W$ and $Z > C$. The probability of that event (for a fixed wage is):

$$\int_0^1 \int_0^1 1 \, dz \, dc = \int_0^1 1 - C dc = 1 - \frac{1}{2} - w + \frac{1}{2}w^2 = \frac{1}{2}(1 - w)^2$$

3. Minimize the sum of the two probabilities:

$$\min_w \frac{1}{2}w^2 + \frac{1}{2}(1 - w)^2$$

With FOC:

$$w - (1 - w) = 0$$

$$w = \frac{1}{2}$$

(Second order condition holds)

4. In general parties will want to minimize expected loss of value from inefficient separation. However in this special case (by symmetry) $w = \frac{1}{2}$ also minimizes expected loss.
Consumption
Investment
Labor
Finance
Consumers:

\[
\max \{c(t)\} \int_0^\infty e^{-\rho \tau} \frac{c(t + \tau)^{1 - 1/\sigma}}{1 - 1/\sigma} d\tau
\]

subject to

\[
\int e^{-r \tau} [w(t + \tau) - c(t + \tau)] d\tau + A(t) = 0
\]
**Consumers:**

\[
\max\{c(t)\} \int_0^\infty e^{-\rho \tau} \frac{c(t+\tau)^{1-1/\sigma}}{1-1/\sigma} d\tau
\]

subject to
\[
\int e^{-r \tau} [w(t + \tau) - c(t + \tau)] d\tau + A(t) = 0
\]

**Variables:**

- \(c(t)\): consumption
- \(w(t)\): earnings
- \(A(t)\): assets (non-human wealth)
PARAMETERS:

ρ: Rate of time preference
σ: Intertemporal elasticity of substitution
r: Real interest rate
**First-order condition:**

\[
\frac{\partial U}{\partial c(t)} = \frac{\partial U}{\partial c(t+\tau)} = \frac{\text{Price at time } t \text{ of } c(t)}{\text{Price at time } t \text{ of } c(t + \tau)}
\]
**First-order condition:**

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\frac{\partial U}{\partial c(t)} \frac{\partial U}{\partial c(t+\tau)} = \frac{\text{Price at time } t \text{ of } c(t)}{\text{Price at time } t \text{ of } c(t + \tau)}
\]

or

\[
\frac{c(t)^{-1/\sigma}}{e^{-\rho \tau} c(t + \tau)^{-1/\sigma}} = \frac{1}{e^{-r \tau}}.
\]
**First-order condition:**

\[
\frac{\partial U}{\partial c(t)} \frac{\partial U}{\partial c(t+\tau)} = \frac{\text{Price at time } t \text{ of } c(t)}{\text{Price at time } t \text{ of } c(t+\tau)}
\]

or

\[
\frac{c(t)^{-1/\sigma}}{e^{-\rho \tau} c(t + \tau)^{-1/\sigma}} = \frac{1}{e^{-r \tau}}.
\]

Solve for \( c(t + \tau) \):

\[
c(t + \tau) = c(t)e^{\sigma(r-\rho)\tau}.
\]
Consumption Euler equation

\[ \dot{c}(t) = \sigma(r - \rho)c(t) \]
The planned consumption profile rises over time at exponential rate $\sigma r$ because of the interest reward to saving and declines at rate $\sigma \rho$ because of time preference.
Use the budget constraint to find $c(t)$

Let

$$H(t) = \int_{0}^{\infty} e^{-r\tau} w(t + \tau) d\tau,$$

human wealth.
Use the budget constraint to find \(c(t)\)

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\[
H(t) = \int_0^\infty e^{-r\tau} w(t + \tau) d\tau,
\]

human wealth.

The budget constraint is

\[
\int_0^\infty e^{-[r - \sigma(r - \rho)]\tau} c(t) d\tau = A(t) + H(t)
\]
Use the budget constraint to find $c(t)$

Let

$$H(t) = \int_0^\infty e^{-r\tau} w(t + \tau) d\tau,$$

human wealth.

The budget constraint is

$$\int_0^\infty e^{-[r - \sigma(r - \rho)]\tau} c(t) d\tau = A(t) + H(t)$$

so

$$c(t) = [r - \sigma(r - \rho)] [H(t) + A(t)]$$

$$= \alpha [H(t) + A(t)]$$
Life-cycle consumption model of Ando and Modigliani

\( \alpha \) is the propensity to consume out of wealth.
Life-cycle consumption model of Ando and Modigliani

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First special case: *Zero intertemporal substitution* \((\sigma=0)\):

\[
c(t) = r [H(t) + A(t)]
\]
Life-cycle consumption model of Ando and Modigliani

$\alpha$ is the propensity to consume out of wealth.

First special case: *Zero intertemporal substitution* ($\sigma=0$):

$$c(t) = r [H(t) + A(t)]$$

Second special case: *Unit intertemporal substitution or log utility* ($\sigma = 1$)

$$c(t) = \rho [H(t) + A(t)]$$

.
Assets

The consumer’s accumulated savings $A(t)$ are the result of the consumption decision, not an exogenous variable.
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Savings follow the accumulation equation, 

$$\dot{A}(t) = rA(t) + w(t) - c(t),$$

that is, savings grow by the amount that interest earnings and wage earnings exceed consumption.
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Savings follow the accumulation equation,

$$\dot{A}(t) = rA(t) + w(t) - c(t),$$

that is, savings grow by the amount that interest earnings and wage earnings exceed consumption.

Because human wealth can be written

$$H(t) = \int_t^\infty e^{-r(s-t)}w(s)ds,$$

$$\dot{H}(t) = rH(t) - w(t)$$

.
Human wealth, $H$

Human wealth is like a bank account that never receives any new deposits, but earns interest at rate $r$. When earnings are received, they deplete $H$ and augment $A$. 

$$w(t) = w_0 e^{\gamma t}$$

Assume that the growth rate, $\gamma$, is less than the interest rate, $r$. Human wealth is

$$H(t) = w_0 e^{\gamma t} \int_0^\infty e^{-(r-\gamma)\tau} d\tau = w_0 r - \gamma e^{\gamma t} \cdot 9$$
**Human wealth, $H$**

Human wealth is like a bank account that never receives any new deposits, but earns interest at rate $r$. When earnings are received, they deplete $H$ and augment $A$.

Suppose earnings grow exponentially:

$$w(t) = w_0 e^{\gamma t}.$$
Human wealth, $H$

Human wealth is like a bank account that never receives any new deposits, but earns interest at rate $r$. When earnings are received, they deplete $H$ and augment $A$.

Suppose earnings grow exponentially:

$$w(t) = w_0 e^{\gamma t}.$$

Assume that the growth rate, $\gamma$, is less than the interest rate, $r$. Human wealth is

$$H(t) = w_0 e^{\gamma t} \int_0^{\infty} e^{-(r-\gamma)\tau} d\tau$$

$$= \frac{w_0}{r - \gamma} e^{\gamma t}.$$
Consumption function

\[ c(t) = \alpha \left( A(t) + \frac{w_0}{r - \gamma} e^{\gamma t} \right). \]
**Consumption function**

\[ c(t) = \alpha \left( A(t) + \frac{w_0}{r - \gamma} e^{\gamma t} \right). \]

The *marginal propensity to consume* is

\[ \frac{\partial c(0)}{\partial w_0} = \frac{\alpha}{r - \gamma} = \frac{r - \sigma (r - \rho)}{r - \gamma} \]
**Consumption function**

\[ c(t) = \alpha \left( A(t) + \frac{w_0}{r - \gamma} e^{\gamma t} \right). \]

The *marginal propensity to consume* is

\[ \frac{\partial c(0)}{\partial w_0} = \frac{\alpha}{r - \gamma} = \frac{r - \sigma(r - \rho)}{r - \gamma} \]

The MPC will be less than one if

\[ \gamma < \sigma(r - \rho). \]

The consumer will spend more than one dollar out of each dollar of current earnings if earnings growth is present and either intertemporal substitution is low or the interest rate is not much above the rate of time preference.
Role of the interest rate

The MPC can be written

\[
\frac{(1 - \sigma)r + \sigma \rho}{r - \gamma},
\]

which is always a decreasing function of the interest rate except in the boundary case \( \rho = \gamma = 0 \).
STOCHASTIC ENVIRONMENT

Assume constant interest rate, \( r \). A consumer can consume a bit more today, \( x \), and satisfy her budget constraint by consuming \( (1 + r) \times \) less next period. The consumer optimizes over this (and other) margins:

\[
\max_x \left\{ u \left( c_t + x \right) + \mathbb{E}_t \left[ \frac{1}{1 + \rho} u \left( c_{t+1} - (1 + r) x \right) + \cdots \right] \right\}
\]


**Stochastic Environment**

Assume constant interest rate, \( r \). A consumer can consume a bit more today, \( x \), and satisfy her budget constraint by consuming \((1 + r) x\) less next period. The consumer optimizes over this (and other) margins:

\[
\max_x \left\{ u(c_t + x) + \mathbb{E}_t \left[ \frac{1}{1 + \rho} u(c_{t+1} - (1 + r) x) + \cdots \right] \right\}
\]

The first-order condition is

\[
u'(c_t + x) - \mathbb{E}_t \left[ \frac{1 + r}{1 + \rho} u'(c_{t+1} - (1 + r) x) \right] = 0
\]
**Stochastic environment**

Assume constant interest rate, \( r \). A consumer can consume a bit more today, \( x \), and satisfy her budget constraint by consuming \((1 + r) x\) less next period. The consumer optimizes over this (and other) margins:

\[
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\]

The first-order condition is

\[
u'(c_t + x) - \mathbb{E}_t \left[ \frac{1 + r}{1 + \rho} u'(c_{t+1} - (1 + r) x) \right] = 0
\]

When the consumer has chosen optimal consumption, \( x = 0 \), so

\[
\mathbb{E}_t \left[ u'(c_{t+1}) \right] = \frac{1 + \rho}{1 + r} u'(c_t)
\]
**Constant Absolute Risk Aversion—Exponential Utility**

For simplicity, take $\rho = r = 0$

$$u(c) = -\frac{1}{\theta}e^{-\theta c}$$
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$$u(c) = -\frac{1}{\theta} e^{-\theta c}$$

Euler equation:

$$\mathbb{E}_t e^{-\theta c_{t+1}} = e^{-\theta c_t}$$
Constant Absolute Risk Aversion—Exponential Utility

For simplicity, take $\rho = r = 0$

$$u(c) = -\frac{1}{\theta} e^{-\theta c}$$

Euler equation:

$$\mathbb{E}_t e^{-\theta c_{t+1}} = e^{-\theta c_t}$$

Assume income is a random walk:

$$y_t = y_{t-1} + \epsilon_t$$

$\epsilon_t$ iid and has mean zero.
Claim: Euler equation is

\[ c_{t+1} = \gamma + c_t + \epsilon_{t+1} \]
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\[ c_{t+1} = \gamma + c_t + \epsilon_{t+1} \]

\[ \gamma = \frac{1}{\theta} \log \left( \mathbb{E} e^{-\theta \epsilon} \right) \]
Claim: Euler equation is

\[ c_{t+1} = \gamma + c_t + \epsilon_{t+1} \]

\[ \gamma = \frac{1}{\theta} \log \left( \mathbb{E} e^{-\theta \epsilon} \right) \]

Because the exponential function is convex, Jensen’s inequality implies \( \mathbb{E} e^{-\theta \epsilon} \geq e^{-\theta \mathbb{E} \epsilon} = 1 \), so \( \gamma \geq 0 \). If \( \epsilon \) has any dispersion, \( \gamma > 0 \) and the consumer is strictly precautionary—she plans positive consumption growth when the interest rate equals the rate of time preference.
Proof of the claim

Substitute the conjectured functional form into the general Euler equation:

\[ E_t \left[ e^{-\theta (\gamma + c_t + \epsilon_{t+1})} \right] = e^{-\theta c_t}. \]
Proof of the claim

Substitute the conjectured functional form into the general Euler equation:

\[ \mathbb{E}_t \left[ e^{-\theta (\gamma + c_t + \epsilon_{t+1})} \right] = e^{-\theta c_t}. \]

This implies \( e^{-\theta \gamma} \mathbb{E} e^{-\theta \epsilon} = 1 \) and thus \( \gamma = \frac{1}{\theta} \log \left( \mathbb{E} e^{-\theta \epsilon} \right) \), as claimed.
Asset accumulation

The consumer’s assets or savings accumulate according to

\[ A_\tau = y_\tau - c_\tau + A_{\tau-1} \]
The consumer’s assets or savings accumulate according to

$$A_\tau = y_\tau - c_\tau + A_{\tau-1}$$

Subtract the Euler equation from the process for income to get

$$y_\tau - c_\tau = y_{\tau-1} - c_{\tau-1} - \gamma$$
**Asset Accumulation**

The consumer’s assets or savings accumulate according to

\[ A_\tau = y_\tau - c_\tau + A_{\tau-1} \]

Subtract the Euler equation from the process for income to get

\[ y_\tau - c_\tau = y_{\tau-1} - c_{\tau-1} - \gamma \]

Notice that this implies that

\[ y_\tau - c_\tau = - (\tau - t) \gamma + y_t - c_t \]

.
Asset accumulation, continued

Substitute this into the asset accumulation equation and cumulate to the terminal period, $T$:

$$A_T = -\gamma \sum (\tau - t) + (T - t + 1) (y_t - c_t) + A_{t-1}.$$
Asset accumulation, continued

Substitute this into the asset accumulation equation and cumulate to the terminal period, $T$:

$$A_T = -\gamma \sum (\tau - t) + (T - t + 1) (y_t - c_t) + A_{t-1}.$$ 

Asset accumulation is non-stochastic, as consumption adjusts fully to each change in income, because the change is permanent. Note that the summation is

$$\frac{(T - t) (T - t + 1)}{2}.$$
Asset accumulation, continued

Substitute this into the asset accumulation equation and cumulate to the terminal period, $T$:

$$A_T = -\gamma \sum (\tau - t) + (T - t + 1)(y_t - c_t) + A_{t-1}.$$  

Asset accumulation is non-stochastic, as consumption adjusts fully to each change in income, because the change is permanent. Note that the summation is

$$\frac{(T - t)(T - t + 1)}{2}.$$  

Now set terminal assets to zero and solve for current consumption:

$$c_t = y_t + \frac{A_{t-1}}{T - t + 1} - \gamma \frac{T - t}{2}.$$
Stochastic returns

Asset \( i \) has stochastic interest rate, \( r_{i,t} \). A consumer can consume a bit more today, \( x \), and satisfy her budget constraint by consuming \( (1 + r_{i,t+1}) x \) less next period.
**Stochastic returns**

Asset \( i \) has stochastic interest rate, \( r_{i,t} \). A consumer can consume a bit more today, \( x \), and satisfy her budget constraint by consuming \((1 + r_{i,t+1}) x\) less next period.

When the consumer has chosen optimal consumption, \( x = 0 \), and

\[
\mathbb{E}_t (1 + r_{i,t+1}) u'(c_{t+1}) = (1 + \rho) u'(c_t)
\]
Stochastic returns

Asset $i$ has stochastic interest rate, $r_{i,t}$. A consumer can consume a bit more today, $x$, and satisfy her budget constraint by consuming $(1 + r_{i,t+1})x$ less next period.

When the consumer has chosen optimal consumption, $x = 0$, and

$$
\mathbb{E}_t (1 + r_{i,t+1})u'(c_{t+1}) = (1 + \rho)u'(c_t)
$$

Write this as

$$
\mathbb{E}_t \frac{u'(c_{t+1})}{(1 + \rho)u'(c_t)} (1 + r_{i,t+1}) = 1
$$
Information aggregation

\[ E_t \frac{1 + r_{i,t+1}}{1 + \rho} u'(c_{t+1}) = u'(c_t) \]

implies that \( c_t \) encapsulates all information available at \( t \) that helps predict future consumption, in the sense of the expectation of future marginal utility multiplied by the return ratio divided by the impatience ratio.
Information aggregation

\[ E_t \frac{1 + r_{i,t+1}}{1 + \rho} u'(c_{t+1}) = u'(c_t) \]

implies that \( c_t \) encapsulates all information available at \( t \) that helps predict future consumption, in the sense of the expectation of future marginal utility multiplied by the return ratio divided by the impatience ratio.

Let \( R(t, t + \tau) \) be the realized asset return ratio from \( t \) to \( t + \tau \).

\[ E_t \frac{R(t, t + \tau)}{(1 + \rho)^\tau} u'(c_{t+\tau}) = u'(c_t) \]

so the consumer projects unchanging consumption over the indefinite future, again in the sense of expected marginal utility multiplied by the return ratio divided by the impatience ratio.
CONSUMPTION SURPRISES—RANDOM WALK

\[
\frac{1 + r_{i,t+1}}{1 + \rho} u'(c_{t+1}) - u'(c_t) = \epsilon_{t+1}
\]
Constant Relative Risk Aversion—Power Utility

\[ u(c) = \frac{c^{1-1/\sigma}}{1 - 1/\sigma} \]
CONSTANT RELATIVE RISK AVersions—POWER UTILITY

\[ u(c) = \frac{c^{1-1/\sigma}}{1 - 1/\sigma} \]

\( \rho = \text{rate of time preference continuously compounded} \)
Constant Relative Risk Aversion—Power Utility

\[ u(c) = \frac{c^{1-1/\sigma}}{1 - 1/\sigma} \]

\( \rho \) = rate of time preference continuously compounded

\( r_t \) = interest rate continuously compounded
Constant Relative Risk Aversion—Power Utility

\[ u(c) = \frac{c^{1-1/\sigma}}{1 - 1/\sigma} \]

\( \rho \) = rate of time preference continuously compounded

\( r_t \) = interest rate rate continuously compounded

Euler equation:

\[ \mathbb{E}_t \left[ e^{-\rho + r_t} c_{t+1}^{-1/\sigma} \right] = c_t^{-1/\sigma} \]
Assumptions:

\[ r_t \text{ is } N(\bar{r}_t, \gamma) \]
Assumptions:

\[ r_t \text{ is } N(\bar{r}_t, \gamma) \]

\[ z_{t+1} = \log c_{t+1} \text{ is } N(\bar{z}_{t+1}, v_t) \]
ASSUMPTIONS:

\[ r_t \text{ is } N (\bar{r}_t, \gamma) \]

\[ z_{t+1} = \log c_{t+1} \text{ is } N (\bar{z}_{t+1}, v_t) \]

\[ \text{Cov} (r_t, z_{t+1}) = \frac{\beta_t}{2} \]
If $y$ is $N(\mu, \theta)$, then

$$\mathbb{E} \left( e^y \right) = e^{\mu + \frac{1}{2} \theta}$$
If $y$ is $N(\mu, \theta)$,

$$E\left(e^y\right) = e^{\mu + \frac{1}{2} \theta}$$

We need

$$E_t\left(e^{-\rho + r_t - \frac{z_{t+1}}{\sigma}}\right)$$
FACT

If \( y \) is \( N(\mu, \theta) \),

\[
\mathbb{E} \left( e^{y} \right) = e^{\mu + \frac{\theta}{2}}
\]

We need

\[
\mathbb{E}_{t} \left( e^{-\rho + r_{t} - \frac{z_{t+1}}{\sigma}} \right)
\]

\[
\mathbb{E} \left( -\rho + r_{t} - \frac{z_{t+1}}{\sigma} \right) = -\rho + \bar{r}_{t} - \frac{\bar{z}_{t+1}}{\sigma}
\]

and

\[
V() = \gamma + \frac{v_{t}}{\sigma^{2}} - \frac{\beta_{t}}{\sigma}
\]

. 
\[
\exp \left( -\rho + \bar{r}_t - \frac{\bar{z}_{t+1}}{\sigma} + \frac{1}{2} \left( \gamma + \frac{\nu_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right) \right) = \exp \left( -\frac{1}{\sigma} \bar{z}_t \right)
\]
**Euler equation**

\[
\exp\left(-\rho + \bar{r}_t - \frac{\bar{z}_{t+1}}{\sigma} + \frac{1}{2} \left(\gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma}\right)\right) = \exp\left(-\frac{1}{\sigma} z_t\right)
\]

Take logs

\[
-\rho + \bar{r}_t - \frac{\bar{z}_{t+1}}{\sigma} + \frac{1}{2} \left(\gamma + \frac{V_t}{\sigma^2} - \frac{\beta_t}{\sigma}\right) = -\frac{z_t}{\sigma}
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Euler equation

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Take logs

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\]

or

\[
\bar{z}_{t+1} = z_t + \sigma (\bar{r}_t - \rho) + \frac{\sigma}{2} \left( \gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right)
\]
The modern econometric Euler equation

\[ z_{t+1} = z_t + \sigma (\bar{r}_t - \rho) + \frac{\sigma}{2} \left( \gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right) + \epsilon_{t+1} \]
Driving forces of log consumption growth

\[(1) \quad \sigma (\bar{r}_t - \rho) = \text{deferral from real interest rate}\]
Driving forces of log consumption growth

(1) \( \sigma (\bar{r}_t - \rho) = \) deferral from real interest rate

(2) \( \frac{\nu_t}{2\sigma} : \) deferral from precautionary behavior.
Driving forces of log consumption growth

(1) \( \sigma (\bar{r}_t - \rho) = \) deferral from real interest rate

(2) \( \frac{v_t}{2\sigma} \) : deferral from precautionary behavior.

A family with its back to the wall could plan consumption growth for both reasons.
Estimation

\[ z_{t+1} = z_t + \sigma (\bar{r}_t - \rho) + \frac{\sigma}{2} \left( \gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right) + \epsilon_{t+1} \]
Estimation

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\[ r_t = \bar{r}_t + \eta_{t+1} \]
Estimation

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z_{t+1} = z_t + \sigma(\bar{r}_t - \rho) + \frac{\sigma}{2} \left( \gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right) + \epsilon_{t+1}
\]

\[
r_t = \bar{r}_t + \eta_{t+1}
\]

OLS: \[
z_{t+1} - z_t - \sigma r_t = -\sigma \rho + \frac{\sigma}{2} \left( \gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right) + \epsilon_{t+1} + \sigma \eta_{t+1}
\]
Estimation

\[ z_{t+1} = z_t + \sigma (\bar{r}_t - \rho) + \frac{\sigma}{2} \left( \gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right) + \epsilon_{t+1} \]

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**OLS:**  
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**TSLS:**  
\[ z_{t+1} - z_t = \sigma r_t - \sigma \rho + \frac{\sigma}{2} \left( \gamma + \frac{v_t}{\sigma^2} - \frac{\beta_t}{\sigma} \right) + \epsilon_{t+1} + \sigma \eta_{t+1} \]
Finance in one slide

Let $m$ be the marginal rate of substitution,

$$m = \frac{u'(c_{t+1})}{(1 + \rho)u'(c_t)}$$

Read John Cochrane, *Asset Pricing*, to see how this maps into all the other ways of thinking about finance.
Finance in one slide

Let $m$ be the marginal rate of substitution,

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Then for all assets $i$,

$$\mathbb{E} \ m \cdot (1 + r_i) = 1$$
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Then for all assets $i$,

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Read John Cochrane, *Asset Pricing*, to see how this maps into all the other ways of thinking about finance.
Interest rate earned or paid is \( r(A) \).
Return depends on asset position

Interest rate earned or paid is $r(A)$.

Consume $x$ more units this period and

$$(1 + r(A_t - x))(A_t - x) - (1 + r(A_t))(A_t)$$

less next period.
Interest rate earned or paid is $r(A)$.

Consume $x$ more units this period and

$$(1 + r(A_t - x))(A_t - x) - (1 + r(A_t))(A_t)$$

less next period.

$$\frac{d c_{t+1}}{d x} |_{x=0} = -r'(A_t)A_t - (1 + r(A_t))$$

$$E_t \frac{1 + r(A_t) + r'(A_t)A_t}{1 + \rho} u'(c_{t+1}) = u'(c_t).$$
Asset constraint—borrowing limit

\[ \mathbb{E}_t \frac{1 + r_t + \lambda_t}{1 + \rho} u'(c_{t+1}) = u'(c_t) \]

\(\lambda_t\) is the Lagrangian multiplier on the constraint.
Quadratic preferences and certainty equivalence

Two-period case:

\[ U(c_1, c_2) = \mathbb{E} \left[ -\frac{1}{2} (\bar{c} - c_1)^2 - \frac{1}{2} (\bar{c} - c_2)^2 \right]. \]
Quadratic preferences and certainty equivalence

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U(c_1, c_2) = \mathbb{E} \left[ -\frac{1}{2}(\bar{c} - c_1)^2 - \frac{1}{2}(\bar{c} - c_2)^2 \right].
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No time preference or asset return: \( \rho = r = 0 \).
Quadratic preferences and certainty equivalence

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No time preference or asset return: \( \rho = r = 0 \).

Budget constraint with random wealth \( W \):

\[
c_2 = W - c_1
\]
Quadratic preferences and certainty equivalence

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No time preference or asset return: \( \rho = r = 0 \).

Budget constraint with random wealth \( W \):

\[ c_2 = W - c_1 \]

Euler equation (linear marginal utility):

\[ \bar{c} - c_1 - \mathbb{E} [\bar{c} - (W - c_1)] = 0. \]
Optimal consumption rule

Let

\[ \bar{W} = \mathbb{E}(W) \]

\[ c_1 = \frac{1}{2} \bar{W} \]
Optimal consumption rule

Let

\[
\bar{W} = \mathbb{E}(W)
\]

\[
c_1 = \frac{1}{2}\bar{W}
\]

\[
c_2 = W - \frac{1}{2}\bar{W}
\]
**Optimal Consumption Rule**

Let

\[ W = \mathbb{E} (W) \]

\[ c_1 = \frac{1}{2} W \]

\[ c_2 = W - \frac{1}{2} W \]

Random walk of consumption itself:

\[ \mathbb{E} (c_2) = c_1 \]
Certainty equivalence but risk aversion

\[ 2U = -(\bar{c} - \frac{1}{2}\bar{W})^2 - \mathbb{E} [\bar{c} - (W - \frac{1}{2}\bar{W})]^2 \]
Certainty equivalence but risk aversion

\[ 2U = - (\bar{c} - \frac{1}{2}\bar{W})^2 - \mathbb{E} \left[ \bar{c} - (W - \frac{1}{2}\bar{W}) \right]^2 \]

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CERTAINTY EQUIVALENCE BUT RISK AVERTION

\[ 2U = -\left( \bar{c} - \frac{1}{2} \bar{W} \right)^2 - \mathbb{E} \left[ \bar{c} - (W - \frac{1}{2} \bar{W}) \right]^2 \]

\[ = -\left( \bar{c} - \frac{1}{2} \bar{W} \right)^2 - \mathbb{E} \left[ \bar{c} - \frac{1}{2} \bar{W} - (W - \bar{W}) \right]^2 \]

\[ = -2\left( \bar{c} - \frac{1}{2} \bar{W} \right)^2 + 2\left( \bar{c} - \frac{1}{2} \bar{W} \right) \left[ \mathbb{E} (W - \bar{W}) \right] - \mathbb{E} (W - \bar{W})^2. \]
Expected utility is decreasing in the variance of wealth $V(W)$

$$2U = -2(\bar{c} - \frac{1}{2}\bar{W})^2 - V(W).$$
INVESTMENT
Capital stock

\[ K(t) = \int_0^t e^{-\delta(t-s)} I(s) ds \]
**Capital stock**

\[ K(t) = \int_0^t e^{-\delta(t-s)} I(s) \, ds \]

\[ \dot{K}(t) = I(t) - \delta \int_0^t e^{-\delta(t-s)} I(s) \, ds \]

\[ = I(t) - \delta K(t) \]
**CAPITAL STOCK**

\[ K(t) = \int_0^t e^{-\delta(t-s)} I(s) ds \]

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\[ = I(t) - \delta K(t) \]

Net investment = gross investment – replacement investment
In continuous time with a variable interest rate, the present value at time $t$ of a dollar to be received at time $s$ is

$$
\exp \left( - \int_t^s r(\tau) \, d\tau \right)
$$
Rental Price of Capital

In continuous time with a variable interest rate, the present value at time \( t \) of a dollar to be received at time \( s \) is

\[
\exp \left( - \int_t^s r(\tau) d\tau \right)
\]

The quantity \( r(t) \) is called the “force of interest.” It is an instantaneous interest rate.
The zero-profit condition for capital renters is

\[ p_K(t) = \int_t^\infty e^{-\int_t^s r(\tau)d\tau} e^{-\delta(s-t)} \rho(s) ds \]

Taking the derivative with respect to \( t \), we get

\[ \dot{p}_K(t) = (r + \delta)p_K(t) - \rho \]
The zero-profit condition for capital renters is

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Rental Price continued

The zero-profit condition for capital renters is

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Taking the derivative with respect to \( t \), we get

\[ \dot{p}_K = (r + \delta)p_K - \rho \]

or

\[ \rho = (r + \delta - \frac{\dot{p}_K}{p_K})p_K \]
**Theorem**

*Only the instantaneous force of interest at time \( t \), and not the longer-term interest rate, enters the rental price of capital.*
Theorem

Only the instantaneous force of interest at time $t$, and not the longer-term interest rate, enters the rental price of capital.

The theorem rests on the zero-profit condition. In order for competition to enforce the zero-profit condition at each point in time, there must be immediate entry and exit from the capital rental business.
Theorem

Only the instantaneous force of interest at time $t$, and not the longer-term interest rate, enters the rental price of capital.

The theorem rests on the zero-profit condition. In order for competition to enforce the zero-profit condition at each point in time, there must be immediate entry and exit from the capital rental business.

To the extent that arbitrage takes time, the zero-profit condition may fail briefly and the interest rate over the period needed for arbitrage will matter.
**Theorem**

*Only the instantaneous force of interest at time t, and not the longer-term interest rate, enters the rental price of capital.*

The theorem rests on the zero-profit condition. In order for competition to enforce the zero-profit condition at each point in time, there must be immediate entry and exit from the capital rental business.

To the extent that arbitrage takes time, the zero-profit condition may fail briefly and the interest rate over the period needed for arbitrage will matter.

The term of the relevant interest rate is not set by the lifetime of the capital but by the time required for arbitrage.
Comments

If the firm that uses the capital has an intertemporally separable technology, so that its demand for current factors depends only on current factor prices, then investment depends only on the interest rates that determine the current rental price, as discussed above.
If the firm that uses the capital has an intertemporally separable technology, so that its demand for current factors depends only on current factor prices, then investment depends only on the interest rates that determine the current rental price, as discussed above.

If the technology is not time-separable—for example, if there are adjustment costs—then future rental prices are a determinant of today’s demand for capital. The future interest rate matters over the span of time when adjustment costs matter.
The interest rate relevant for investment is the rate for securities whose term matches the time needed to adjust the amount of capital in use in response to changing conditions. The term is not the lifetime of the capital good.
Industry equilibrium

$y$ is output, $p$ is the price of output, and the industry demand function is

$$y = dp^{-1}$$

where $d$ is a variable that determines the location of the demand function and which may change over time.
Industry equilibrium

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where $d$ is a variable that determines the location of the demand function and which may change over time.

Notice that $d = py$, so this specification says that industry revenue is held at the exogenous level $d$. 

.
Industry equilibrium, continued

Suppose further that the technology is Cobb-Douglas:

\[ y = An^\alpha k^{1-\alpha} \]
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Each firm in the industry is a price taker satisfying the first-order condition for capital,

\[ (1 - \alpha) An^\alpha k^{-\alpha} = \frac{\rho}{p}. \]
Industry equilibrium, continued

Suppose further that the technology is Cobb-Douglas:

\[ y = An^\alpha k^{1-\alpha}. \]

Each firm in the industry is a price taker satisfying the first-order condition for capital,

\[ (1 - \alpha) An^\alpha k^{-\alpha} = \frac{\rho}{p}. \]

This can be written

\[ (1 - \alpha) \frac{y}{k} = \frac{\rho}{p}. \]
Commentary

or

\[ k = (1 - \alpha) \frac{py}{\rho} = (1 - \alpha) \frac{d}{\rho}. \]
Commentary

or

\[ k = (1 - \alpha) \frac{py}{\rho} = (1 - \alpha) \frac{d}{\rho}. \]

The capital stock responds in positive proportion to shifts of industry demand, \( d \), and inversely to shifts in the rental price of capital, \( \rho \).
or

\[ k = (1 - \alpha) \frac{py}{\rho} = (1 - \alpha) \frac{d}{\rho}. \]

The capital stock responds in positive proportion to shifts of industry demand, \( d \), and inversely to shifts in the rental price of capital, \( \rho \).

The response is immediate.
If the technology is not time-separable—for example, if there are adjustment costs—then future rental prices are a determinant of today's demand for capital. The future interest rate matters over the span of time when adjustment costs matter.

To summarize, Proposition 1. The interest rate relevant for investment is the rate for securities whose term matches the time needed to adjust the amount of capital in use in response to changing conditions. The term is not the lifetime of the capital good.

Industry equilibrium

Suppose $y$ is output, $p$ is the price of output, and the industry demand function is $y_d p = \alpha d$ where $d$ is a variable that determines the location of the demand function and which may change over time. Notice that $d_p = 0$, so this specification says that industry revenue is held at the exogenous level $d$.

Suppose further that the technology is Cobb-Douglas: $y = A^n k^{\alpha}$. Each firm in the industry is a price taker satisfying the first-order condition for capital, $(1 - \alpha) k p^{-\alpha} = \rho$. This can be written $(1 - \alpha) y p^{-\alpha} = \rho$ or $(1 - \alpha)^{-1} y = \rho p^{\alpha} = \rho$. Thus the capital stock responds in positive proportion to shifts of industry demand, $d$, and inversely to shifts in the rental price of capital, $\rho$. The response is immediate. The impulse response functions for the capital stock and investment, for a step increase in industry demand, are:

Now consider a similar setup with adjustment costs. As before, let $K_p$ be the price of new capital and now let $q$ be the ratio of the price of installed capital to the price of new capital. Thus the effective price of capital used in production is $q p$ and the rental price of
Adjustment cost

Let $p_K$ be the price of new capital and $q$ be the ratio of the price of installed capital to the price of new capital.
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**Adjustment cost**
Let $p_K$ be the price of new capital and $q$ be the ratio of the price of installed capital to the price of new capital.

The effective price of capital used in production is $qp$.

The rental price of installed capital is

$$\rho = \left( r - \frac{\dot{q}}{q} - \frac{\dot{p}_K}{p_K} \right) qp_K.$$
ADJUSTMENT COST

Let $p_K$ be the price of new capital and $q$ be the ratio of the price of installed capital to the price of new capital.

The effective price of capital used in production is $qp$

The rental price of installed capital is

$$\rho = \left( r - \frac{\dot{q}}{q} - \frac{\dot{p}_K}{p_K} \right) qp_K.$$

Here we have assumed no depreciation, $\delta = 0$. 
The first-order condition remains the same with this new version of the rental price.

\[ k = (1 - \alpha) \frac{d}{\left( r - \frac{\dot{q}}{q} - \frac{\dot{p}_K}{p_K} \right) q p_K}. \]
The first-order condition remains the same with this new version of the rental price.

\[ k = (1 - \alpha) \frac{d}{\left( r - \frac{\dot{q}}{q} - \frac{\dot{p}_K}{p_K} \right) q p_K}. \]

The endogenous \( q \) appears in both levels and time derivatives in the demand for capital.

.
ADJUSTMENT COST FUNCTION

Let $c(I)$ be the adjustment cost of investment flow $I = \dot{k}$. 

Assume convex adjustment cost:

$$c'(I) \geq 0, \quad c'(0) = 0, \quad c''(I) > 0.$$ 

The flow of newly installed capital solves

$$\max I q K I - c(I) - p K I$$ 

so that,

$$c'(I) = p K (q - 1).$$
ADJUSTMENT COST FUNCTION

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Adjustment cost function

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The flow of newly installed capital solves

\[ \max_I qp_K I - c(I) - p_K I \]
ADJUSTMENT COST FUNCTION

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Optimal path

The variables $q$ and $K$ form a saddle-point system.
**Optimal path**

The variables $q$ and $K$ form a saddle-point system.

The locus of stationary values of $q$ is defined by

$$k = (1 - \alpha) \frac{d}{\left( r - \frac{\dot{p}_K}{p_K} \right) q p_K}$$
**Optimal path**

The variables $q$ and $K$ form a saddle-point system. The locus of stationary values of $q$ is defined by

$$k = (1 - \alpha) \frac{d}{(r - \frac{\dot{p}_K}{p_K}) qpK}$$

and the locus of stationary values of $k$ by

$$q = 1$$
installed capital is \( K \). Here we have assumed no depreciation, \( \delta = 0 \). The first-order condition remains the same with this new version of the rental price. Thus
\[
\frac{dK}{dp} = -\alpha.
\]
Now the endogenous \( q \) appears in both levels and time derivatives in the demand for capital.

Let \( c(I) \) be the adjustment cost of investment flow \( I \). Assume convex adjustment cost:
\[
0, (0), 0, (0)
\]

The flow of newly installed capital solves
\[
\max(K_I) = -c(I) - p \Rightarrow \frac{c(I) + p}{0} = 0.
\]
The variables \( q \) and \( K \) form a saddle-point system. The locus of stationary values of \( q \) is
\[
1
\]
and the locus of stationary values of \( k \) by \( q = 1 \). The phase diagram is:

\[
\begin{array}{c}
\text{PHASE DIAGRAM} \\
\end{array}
\]
Impulse response functions:

Consider a project that produces 1 unit of output per year. This year, output sells at price $P_0$, which is known before any decision has to be made. There are four possible decisions and outcomes about the project:

- Decide this year never to build the project.
- Decide this year to build the project.
- Deferring the decision this year and decide next year to build the project.
- Deferring the decision this year and decide next year not to build the project.

**Response of $k$ and $\dot{k}$**
Tobin’s $q$—Ratio of Market Value to Reproduction Cost of Plant and Equipment

Consider a project that produces 1 unit of output per year. This year, output sells at price $P_0$, which is known before any decision has to be made. There are four possible decisions and outcomes about the project:

- Decide this year never to build the project.
- Decide this year to build the project this year.
- Defer the decision this year and decide next year to build the project.
- Defer the decision this year and decide next year not to build the project.
InVESTMENT UNDER UNCERTAINTY, WITH IRREVERSIBILITY

Investment under Uncertainty, with Irreversibility

Reference: Avinash Dixit and Robert S. Pindyck, Investment under Uncertainty, 1994, pp. 40-41

Often called real option theory
Consider a project that produces 1 unit of output per year. This year, output sells at price $P_0$, which is known before any decision has to be made.
Setup

Consider a project that produces 1 unit of output per year. This year, output sells at price $P_0$, which is known before any decision has to be made.

There are four possible decisions and outcomes about the project:

1. Decide this year never to build the project.
2. Decide this year to build the project this year.
3. Defer the decision this year and decide next year to build the project.
4. Defer the decision this year and decide next year not to build the project.
If the project is built, it yields a present discounted value

$$(1 + u)MP_0 \text{ with probability } q$$

$$(1 - d)MP_0 \text{ with probability } 1 - q$$
If the project is built,

it yields a present discounted value

\[(1 + u)MP_0\text{ with probability } q\]

\[(1 - d)MP_0\text{ with probability } 1 - q\]

\(M\) is a capitalization factor with a value like 5 or 10; it embodies discounting as well as capturing the length of the payoff from the project.
Cost

The project costs $I$ whether it is built this year or next year.
The project costs $I$ whether it is built this year or next year. Assume that this year’s revenue exceeds the interest cost on the investment:

$$P_0 > \frac{r}{1 + r} I.$$
Payoffs

Let $V_N$ be the value associated with making the decision about whether or not to build the project this year:

$$V_N = \max \left[ 0, -I + P_0 + q(1 + u)MP_0 + (1 - q)(1 - d)MP_0 \right]$$
**Payoffs**

Let $V_N$ be the value associated with making the decision about whether or not to build the project this year:

$$V_N = \max \left[ 0, -I + P_0 + q(1 + u)MP_0 + (1 - q)(1 - d)MP_0 \right]$$

Let $V_W$ be the expected value associated with deferring the decision until next year:

$$V_W = q \max \left[ 0, \frac{-I}{1+r} + (1 + u)MP_0 \right] + (1 - q) \max \left[ 0, \frac{-I}{1+r} + (1 - d)MP_0 \right]$$
Four relevant prices are shown on the horizontal axis:

- A is the cutoff for investing next year if the good outcome occurs.
- B is the cutoff for investing this year.
- C is the point where the expected values of the two decision times are equal.
- D is the point where investment will occur with a deferred decision even if the bad outcome occurs.

Now the optimal decision strategy is apparent: If $P < A$, forget the project now. If $P \leq C$, defer the decision and invest only if the good outcome occurs. If $P > C$, invest this year.
Relevant prices

$A$ is the cutoff for investing next year if the good outcome occurs:

$$P_{0A} = \frac{I}{(1 + r)(1 + u)M}$$
Relevant prices

$A$ is the cutoff for investing next year if the good outcome occurs:

$$P_{0A} = \frac{I}{(1 + r)(1 + u)M}$$

$B$ is the cutoff for investing this year:

$$P_{0B} = \frac{I}{1 + q(1 + u)M + (1 - q)(1 - d)M}.$$
More relevant prices

Assume that $P_{0A} < P_{0B}$. $C$ is the point where the expected values of the two decision times are equal:

$$P_{0C} = \frac{(1 - \frac{q}{1+r}) I}{1 + (1 - q)(1 - d) M}$$
More relevant prices

Assume that \( P_{0A} < P_{0B} \). \( C \) is the point where the expected values of the two decision times are equal:

\[
P_{0C} = \frac{\left(1 - \frac{q}{1+r}\right) I}{1 + (1 - q)(1 - d)M}
\]

\( D \) is the point where investment will occur with a deferred decision even if the bad outcome occurs:

\[
P_{0D} = \frac{I}{(1 + r)(1 - d)M}
\]
Optimal decision strategy

If $P_0 < P_{0A}$, forget the project now.
Optimal decision strategy

If $P_0 < P_{0A}$, forget the project now.

If $P_{0A} \leq P_0 < P_{0C}$, defer the decision and invest only if the good outcome occurs.
Optimal decision strategy

If $P_0 < P_{0A}$, forget the project now.

If $P_{0A} \leq P_0 < P_{0C}$, defer the decision and invest only if the good outcome occurs.

If $P_0 > P_{0C}$, invest this year.
The Bad News Principle

Note that the formula for $P_{0C}$ does not depend on how good the good news might be next year; $u$ does not appear in the formula. This establishes
The Bad News Principle

Note that the formula for $P_{0C}$ does not depend on how good the good news might be next year; $u$ does not appear in the formula. This establishes

Bernanke’s Theorem: The decision whether or not to invest this year does not depend on how good the good news might be, but only on how bad the bad news might be. The value of waiting comes entirely from avoiding the effect of bad news. All of the value of good news is captured either by investing now or investing next year.
Effect of increased uncertainty

Consider the family of distributions of subsequent value whose means are $P_0 M$. 
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Further restrict the family to those satisfying $u = \alpha d$—that is, the good news is proportional to the bad news, with a prescribed constant of proportionality, $\alpha$. 
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In this family, the parameter $d$ is a measure of dispersion. Higher values of $d$ mean there is more uncertainty about the future payoff of the project.
Increased uncertainty, continued

The restriction on the mean of the subsequent payoff turns out to restrict the probability of the good outcome:

\[ q = \frac{1}{1 + \alpha} \]
Increased uncertainty, continued

The restriction on the mean of the subsequent payoff turns out to restrict the probability of the good outcome:

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Putting this into the formula for the switchover price gives:

\[ P_{0C} = \frac{(1 + \alpha - \frac{1}{1+r}) I}{1 + \alpha + \alpha(1 - d)M} \]

This is an increasing function of dispersion, \( d \).
Increased uncertainty, continued

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This is an increasing function of dispersion, \( d \).

Theorem. Deferral of investment is more likely if there is more uncertainty.
Relation between project selection and investment

Firm:

$$\min \sum_{s=t}^{\infty} R_{s,t}(C_s(Q_s, K_0, \ldots, K_s) + p_{k,s}K_s)$$
Relation between project selection and investment

Firm:

$$\min \sum_{s=t}^{\infty} R_{s,t}(C_s(Q_s, K_0, \ldots, K_s) + p_{k,s}K_s)$$

FONC (project selection):

$$\sum_{s=t}^{\infty} R_{s,t}M_{s,t} = p_{k,t}$$

$$M_{s,t} = -\frac{\partial C_s}{\partial K_t}$$

The marginal investment project just meets the present-value criterion.
The investment equation

FONC for next period:

\[
\sum_{s=t+1}^{\infty} R_{s,t} M_{s,t+1} = \frac{p_{k,t}}{1 + r_t}
\]

Key assumption: different ages of capital are perfect substitutes with deterioration:

\[
M_{s,t} = M_{s,t+1} (1 + \delta)
\]
**The investment equation**

FONC for next period:

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\sum_{s=t+1}^{\infty} R_{s,t} M_{s,t+1} = \frac{p_{k,t}}{1 + r_t}
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Key assumption: different ages of capital are perfect substitutes with deterioration:

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THE INVESTMENT EQUATION

FONC for next period:

\[ \sum_{s=t+1}^{\infty} R_{s,t} M_{s,t+1} = \frac{p_{k,t}}{1 + r_t} \]

Key assumption: different ages of capital are perfect substitutes with deterioration:

\[ M_{s,t} = \frac{M_{s,t+1}}{1 + \delta} \]

So

\[ \sum_{s=t+1}^{\infty} R_{s,t} M_{s,t} = \frac{p_{k,t+1}}{(1 + \delta)(1 + r_t)} \]
Subtract to get

\[ M_{t,t} = p_{k,t} - \frac{p_{k,t+1}}{(1 + \delta)(1 + r_t)}, \]

the investment equation involving only the next future period.
FIXED COSTS OF ADJUSTMENT

Dynamic program:

\[ V(K) = \pi(K) + \max_I \left[ -C(I) - F(I) + \frac{1}{1 + r} V(K') \right] \]

\[ K' = (1 - \delta)K + I \]
**Fixed Costs of Adjustment**

Dynamic program:

\[ V(K) = \pi(K) + \max_I \left[ -C(I) - F(I) + \frac{1}{1 + r} V(K') \right] \]

\[ K' = (1 - \delta)K + I \]

- \( \pi(K) \) is the period profit from capital \( K \).
- \( I \) is investment, which cannot be negative (irreversible).
- \( F(I) \) is a fixed cost \( \bar{F} \) if \( I > 0 \) with \( F(0) = 0 \).
- \( C(I) \) is the variable cost of investment \( I \).
Behavior

Let

\[ V^*(K) = \max_I \left[ -C(I) + \frac{1}{1 + r} V(K') \right], \]

the future value from investing, putting aside the fixed cost.
Behavior

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Let

\[ V^0(K) = \frac{1}{1 + r} V((1 - \delta)K), \]

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Then the condition for positive investment is

\[ V^*(K) \geq V^0(K) + \bar{F}. \]
Given $N$ and $K$, let $I$ satisfy

$$K = (1 - \delta)^N K + I$$
Given $N$ and $K$, let $I$ satisfy

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Invest $I$ every $N$ periods and zero the rest of the time.
Dynamic program becomes

\[ V(K) = \sum_{i=0}^{N-1} \left( \frac{1}{1 + r} \right)^i \pi((1 - \delta)^i K) \]
\[ + \left( \frac{1}{1 + r} \right)^N (V(K) - \bar{F} - C([1 - (1 - \delta)^N]K)) \]
Dynamic program becomes

\[ V(K) = \sum_{i=0}^{N-1} \left( \frac{1}{1+r} \right)^i \pi((1-\delta)^i K) + \left( \frac{1}{1+r} \right)^N (V(K) - \bar{F} - C([1 - (1 - \delta)^N] K)) \]

Thus we can solve for \( V \) explicitly as

\[ V(K) = \sum_{i=0}^{N-1} \left( \frac{1}{1+r} \right)^i \pi((1-\delta)^i K) - \left( \frac{1}{1+r} \right)^N (\bar{F} + C()) \frac{1 - \left( \frac{1}{1+r} \right)^N}{1 - \left( \frac{1}{1+r} \right)^N} \]
Dynamic program becomes

\[ V(K) = \sum_{i=0}^{N-1} \left( \frac{1}{1+r} \right)^i \pi((1 - \delta)^i K) \]

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Thus we can solve for \( V \) explicitly as

\[ V(K) = \frac{\sum_{i=0}^{N-1} \left( \frac{1}{1+r} \right)^i \pi((1 - \delta)^i K) - \left( \frac{1}{1+r} \right)^N (\bar{F} + C())}{1 - \left( \frac{1}{1+r} \right)^N} \]

Maximize over \( N \) and \( K \), then set \( I = [1 - (1 - \delta)^N]K \).
Applications—$(s, S)$ rules

Inventory investment: $\bar{F}$ is cost of restocking. Wait until inventory falls to $(1 - \delta)^N K$ then buy $I = [1 - (1 - \delta)^N]K$ to bring stock up to $K$. 
Applications—\((s, S)\) rules

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Truck or car purchasing: \(\bar{F}\) is cost of selling current vehicle and going to dealer to get a new one. \(\delta\) is rate of decline of net benefit of ownership. Wait until benefit falls to \((1 - \delta)^N K\) then trade up by \(I = [1 - (1 - \delta)^N]K\).
Important non-investment applications

Tobin’s model of currency demand: $\bar{F}$ is cost of going to bank (ATM). $\delta$ is rate of spending currency. Wait until money in wallet falls to $(1 - \delta)^N K$ then withdraw $I = [1 - (1 - \delta)^N]K$ to bring wallet up to $K$. 

Menu costs: $\bar{F}$ is cost of changing price, $K$. $\delta$ is rate of inflation. Wait until real price falls to $(1 - \delta)^N K$ then raise price by $I = [1 - (1 - \delta)^N]K$ to bring to $K$. 
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Menu costs: $\bar{F}$ is cost of changing price, $K$. $\delta$ is rate of inflation. Wait until real price falls to $(1 - \delta)^N K$ then raise price by $I = [1 - (1 - \delta)^N]K$ to bring to $K$. 


STOCHASTIC CASE

Dynamic program:

\[ V(K, x) = \pi(K, x) + \max_I \left[ -C(I) - F(I) + \mathbb{E}_{x'} mV(K', x') \right] \]

\[ K' = (1 - \delta)K + I \]
STOCHASTIC CASE

Dynamic program:

\[
V(K, x) = \pi(K, x) + \max_I \left[ -C(I) - F(I) + \mathbb{E}_{x'} mV(K', x') \right]
\]

\[
K' = (1 - \delta)K + I
\]

The shock must be Markov: the distribution of \( x' \) is a function only of \( x \).
As before

Let

\[ V^*(K, x) = \max_I \left[ -C(I) + \mathbb{E}_{x'} mV(K', x') \right], \]

the future value from investing, putting aside the fixed cost.
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Let

\[ V^0(K, x) = \mathbb{E}_{x'} mV((1 - \delta)K, x') , \]

the future value from zero investment.
As before

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\[ V^*(K, x) = \max_I [-C(I) + \mathbb{E}_{x'} mV(K', x')] , \]

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Let

\[ V^0(K, x) = \mathbb{E}_{x'} mV((1 - \delta)K, x'), \]

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Then the condition for positive investment is

\[ V^*(K, x) \geq V^0(K, x) + \bar{F}. \]
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\[ V^*(K, x) = \max_I \left[ -C(I) + \mathbb{E}_{x'} \ mV(K', x') \right] , \]

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the future value from zero investment.

Then the condition for positive investment is

\[ V^*(K, x) \geq V^0(K, x) + \bar{F} . \]

\[ V^*(K, x) = V^0(K, x) + \bar{F} \] defines \( K \), the value of \( K \) where investment becomes beneficial. \( K > \underline{K} \) is the zone of inaction.
Stochastic evolution of $K$

After an investment, $K$ is well inside the zone of inaction.
Stochastic evolution of $K$

After an investment, $K$ is well inside the zone of inaction. Each period, $K' = (1 - \delta)K$, so $K'$ will eventually drop below the zone of inaction and another investment will occur.
**Stochastic evolution of \( K \)**

After an investment, \( K \) is well inside the zone of inaction.

Each period, \( K' = (1 - \delta)K \), so \( K' \) will eventually drop below the zone of inaction and another investment will occur.

A shock \( x \) that raises the marginal payoff from capital will raise \( K \) and thus raise the zone of inaction and may cause \( K \) to be below the zone, in which case investment will occur sooner than without the shock.
Many firms with idiosyncratic shocks

\[ x = a + \epsilon_i. \]

\( a \) affects all firms and \( \epsilon_i \) is idiosyncratic to firm \( i \); both iid.
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\( a \) affects all firms and \( \epsilon_i \) is idiosyncratic to firm \( i \); both iid.

\( K_i \) is distributed across firms within their zones of inaction. This distribution is a state variable of dimension equal to the number of firms (infinity in many models).
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If a large mass of firms is close to \( K \), a small positive aggregate shock \( a \) will cause a large volume of investment and vice versa.
Many firms with idiosyncratic shocks

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If a large mass of firms is close to \( K \), a small positive aggregate shock \( a \) will cause a large volume of investment and vice versa.

Hard to generalize about the effects of fixed investment costs on the sensitivity of investment to aggregate shocks. Same for non-investment applications.
Two-sided case

There is another cost \( F \) for disinvesting.
Two-sided case

There is another cost $F$ for disinvesting.

Then there will be another critical value of $K, \bar{K}$, above which disinvestment will be advantageous.
Two-sided case

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Then there will be another critical value of $K$, $\bar{K}$, above which disinvestment will be advantageous.

The zone of inaction is $K < K < \bar{K}$. 
Labor
INTERTEMPORAL LABOR SUPPLY UNDER UNCERTAINTY

Let $u(c, h)$ be period utility when consuming goods $c$ and working hours $h$ and let $\beta$ be the worker’s subjective discount ratio.
Intertemporal labor supply under uncertainty

Let $u(c, h)$ be period utility when consuming goods $c$ and working hours $h$ and let $\beta$ be the worker’s subjective discount ratio.

Let $\tilde{p}_{s,t}$ be the Arrow-Debreu time-0 price of a unit of consumption at time $t$ and state of the world $s$ and let $\tilde{w}_{s,t}$ be the price of an hour of work—the wage.
**INTERTEMPORAL LABOR SUPPLY UNDER UNCERTAINTY**

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Period zero has only one state and output in that state is numéraire: $p_{1,0} = 1$. 
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Period zero has only one state and output in that state is numéraire: $p_{1,0} = 1$.

The probability of state $s$ at time $t$ is $\pi_{s,t}$ with $\sum_s \pi_{s,t} = 1$. 
CHOICE

At time zero, the worker makes a lifetime contingent choice to maximize expected utility,

$$\sum_t \sum_s \beta^t \pi_{s,t} u(c_{s,t}, h_{s,t})$$

subject to

$$\sum_t \sum_s (\tilde{p}_{s,t} c_{s,t} - \tilde{w}_{s,t} h_{s,t}) = 0.$$
At time zero, the worker makes a lifetime contingent choice to maximize expected utility,

$$\sum_{t} \sum_{s} \beta^t \pi_{s,t} u(c_{s,t}, h_{s,t})$$

subject to

$$\sum_{t} \sum_{s} (\tilde{p}_{s,t} c_{s,t} - \tilde{w}_{s,t} h_{s,t}) = 0.$$

Suppose that the uncertainty facing the worker is purely personal, so that a competitive insurance company will be risk-neutral and offer contingent claims on consumption and hours at

$$\tilde{p}_{s,t} = \beta^t \pi_{s,t}$$

$$\tilde{w}_{s,t} = \tilde{p}_{s,t} w_{s,t}.$$

Here $w_{s,t}$ is the real wage.
**First-order conditions**

Let $\lambda$ be the Lagrangian associated with the budget constraint. The first-order conditions for the maximization of expected utility are

$$\beta^t \pi_{s,t} u_c = \lambda \beta^t \pi_{s,t}$$

$$-\beta^t \pi_{s,t} u_h = \lambda \beta^t \pi_{s,t} w_{s,t}$$

or

$$u_c = \lambda$$

$$-u_h = w_{s,t} \lambda.$$
**First-order conditions**

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$$
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$$

or

$$
u_c = \lambda \\
u_h = w_{s,t} \lambda.
$$

The first is the Borch-Arrow condition, calling for the equality of marginal utility of consumption across all states (and time periods). The second is analogous—it is called the Frisch labor-supply function in labor economics.
No insurance

Now suppose that workers can’t buy insurance and can’t borrow against future income. They make sequential decisions and face the constraint $c_{s,t} \leq W_t + y_{s,t}$. Here $W$ is the worker’s non-human wealth (savings).
No insurance

Now suppose that workers can’t buy insurance and can’t borrow against future income. They make sequential decisions and face the constraint $c_{s,t} \leq W_t + y_{s,t}$. Here $W$ is the worker’s non-human wealth (savings).

Suppose that the uncertain state follows a Markoff process:

$$\text{Prob}[s_{t+1} = s' | s_t = s] = \phi_{s,s'}$$
No insurance

Now suppose that workers can’t buy insurance and can’t borrow against future income. They make sequential decisions and face the constraint $c_{s,t} \leq W_t + y_{s,t}$. Here $W$ is the worker’s non-human wealth (savings).

Suppose that the uncertain state follows a Markoff process:

$$\text{Prob}[s_{t+1} = s' | s_t = s] = \phi_{s,s'}$$

Use dynamic programming to find the worker’s optimal choice of consumption and hours. Let $V_{s,t}(W)$ be the worker’s expected value at the beginning of period $t$, emerging from state $s$ with non-human wealth $W$. 


Bellman equation

\[ V_{s,t}(W) = \max_{c,h} \left[ u(c, h) + \sum_{s'} \phi_{s,s'} \beta V_{s',t+1} \left( \frac{W - c + w_{s,t} h}{\beta} \right) \right]. \]

The division by \( \beta \) reflects the interest that the carried-forward wealth earns: \( 1 + r = 1/\beta \).
Bellman equation

\[ V_{s,t}(W) = \max_{c,h} \left[ u(c, h) + \sum_{s'} \phi_{s,s'} \beta V_{s',t+1} \left( \frac{W - c + w_{s,t}h}{\beta} \right) \right] . \]

The division by \( \beta \) reflects the interest that the carried-forward wealth earns: \( 1 + r = 1/\beta \).

The first-order conditions are:

\[ u_c = \sum_{s'} \phi_{s,s'} V_{s',t+1} (W_{t+1}) \]

and

\[ -u_h = w_{s,t} \sum_{s'} \phi_{s,s'} V_{s',t+1} (W_{t+1}) . \]
Relation to insured case

If the value function is linear in $W$ with a constant coefficient over $s$, the first-order conditions are the same as in the insured case, with $\lambda = V'$. 
**Relation to insured case**

If the value function is linear in $W$ with a constant coefficient over $s$, the first-order conditions are the same as in the insured case, with $\lambda = V'$. Insurance matters because of the curvature of $V$. The curvature arises for the reasons we discussed in connection with consumption behavior—it is especially pronounced for people with powerful precautionary tendencies.
A utility function

Suppose people order consumption-hours pairs according to the period utility,

\[ u(c, h) = \frac{1}{1 - 1/\delta} \left( \kappa + \log c - \frac{\gamma}{1 + 1/\psi} h^{1+1/\psi} \right)^{1-1/\delta}. \]
A utility function

Suppose people order consumption-hours pairs according to the period utility,

\[ u(c, h) = \frac{1}{1 - 1/\delta} \left( \kappa + \log c - \frac{\gamma}{1 + 1/\psi} h^{1+1/\psi} \right)^{1-1/\delta}. \]

The kernel inside the parentheses governs the marginal rate of substitution between consumption and work within a month and state of the world. Because consumption enters as the log, these preferences imply that hours of work are independent of the wage if wage income in the same month and same state is the only way the consumer finances consumption. There is a concave transformation of the kernel before adding across months or states. \( \delta \) controls the concavity—it is the intertemporal and inter-state elasticity of substitution with respect to the kernel.
Separability between $c$ and $h$

If the parameter $\delta$ is infinite—a widely used specification with separability between $c$ and $h$—then the Frisch consumption demand is

$$u_c(c, h) = \frac{1}{c} = \lambda \text{ or } c = \frac{1}{\lambda},$$

constant over states of the world and time periods. This property has created the impression that the observed behavior of consumption is inconsistent with insurance or life-cycle smoothing. It is known that consumption is lower in states or at times when people are not working. But that finding is consistent with the impression only in the special case of separability.
Frisch Complementarity

Consumption and non-work as Frisch complements if consumption rises when the wage rises (work rises and non-work falls).
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If $\delta$ is finite, consumption and non-work are unambiguously Frisch complements. People consume more when wages are high because they work more and consume less leisure.
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If $\delta$ is finite, consumption and non-work are unambiguously Frisch complements. People consume more when wages are high because they work more and consume less leisure.

Nothing can be deduced about limited insurance or consumption smoothing from the reaction of consumption to changes in work opportunities, without further structure.
Calibration

Calibrate in a stationary, non-stochastic setting where consumption $c$ and hours $h$ are 1. The budget constraint implies that the value of earnings, $y$, is also one. From the first-order condition for balancing $c$ against $h$ in the face of a wage, $w = 1$ (so the budget constraint is $c = wh$), I infer that $\gamma = 1$ under this normalization, so I will omit $\gamma$ in what follows.

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Curvature parameters

The three remaining parameters of preferences are the intercept in the kernel, $\kappa$, the overall substitution parameter, $\delta$, and the inverse elasticity of marginal disutility of work, $\psi$. Calibrate to three basic properties of consumer behavior.
Risk aversion and intertemporal substitution

The first property is risk aversion and intertemporal substitution in consumption. With additively separable preferences across states and time periods, the coefficient of relative risk aversion (CRRA) and the intertemporal elasticity of substitution are reciprocals of one another. A substantial body of research suggests that the CRRA is fairly close to 2. Another body of research on the intertemporal elasticity of substitution suggests that 0.5 is a reasonable value.

\[
\text{CRRA} = 1 + \frac{1}{\delta} \left( \kappa + \log \frac{c}{\gamma} - \frac{1}{1 + \frac{1}{\psi h}} \right) - 1
\]

The calibration sets this expression equal to 2.
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The CRRA in the preferences above is

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The calibration sets this expression equal to 2.
The second property is the Frisch elasticity of labor supply. Another large body of research finds values for this elasticity around 0.8. The second condition sets the Frisch elasticity to this value. In the case of $\delta = \infty$, that elasticity is the parameter $\psi$, but the expression is much more complicated otherwise.
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It is the elasticity of $h$ with respect to $w$ in the two-equation Frisch system:

$$\frac{1}{c} \left( \kappa + \log c - \frac{\gamma}{1 + 1/\psi} h^{1+1/\psi} \right)^{-1/\delta} = \lambda$$

$$h^{1/\psi} \left( \kappa + \log c - \frac{\gamma}{1 + 1/\psi} h^{1+1/\psi} \right)^{-1/\delta} = \lambda w.$$
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This elasticity equals 0.8 in the calibration.
The third property is the relation between hours of work and consumption. A substantial body of work has examined what happens to consumption when a person stops working, either because of unemployment following job loss or because of retirement, which may be the result of job loss. A consensus of this research is that people consume about 20 percent less when their work falls to zero, compared to their consumption when working a normal amount. Incorporate this property in the calibration by requiring

\[ \frac{1}{0.8} (\kappa + \log 0.8)^{-1/\delta} = \lambda \].
Voluntary work reduction?

Notice that this calibration does not require a stand on whether people who are not working have chosen that condition voluntarily, against other available choices. The equation in the previous slide holds when the choice is either voluntary or involuntary. Some of the research on the effects of unemployment and retirement on consumption have interpreted the decline as the result of frictions in capital and insurance markets. But the decline may arise from the Frisch complementarity of non-work and consumption, not from failures of insurance and capital markets.
In some models, workers pick among available jobs according to their values. The value of a job requiring $h$ hours of work and paying $y$ is, in consumption units, $v(h, y) = \max_c \left[ \frac{1}{\lambda} \left( \kappa + \log c - \frac{\gamma h^{1+1/\psi}}{1+1/\psi} \right)^{1-1/\delta} - c + y \right]$. 
JOB VALUE AS A FUNCTION OF HOURS AND EARNINGS
The previous slide shows the shape of $v(h, y)$. The value of a job rises linearly with its earnings, $y$. It falls with increasingly negative slope with rising hours. The value of a job with zero hours and zero earnings is -0.2. The contour line rising up and to the right from the corner is the boundary between the jobs that a worker would pick in preference to zero hours and zero earnings. The contour line above it is the boundary between working zero hours with benefits of $y = 0.4$ and working positive hours with higher earnings. This line would apply if the replacement rate for benefits were 40 percent and the benefits were not integrated with the insurance covering the worker—the benefits are joint revenue to the insurer and the worker, not a transfer from insurer to worker.
Implications

If not working involves no benefits, the minimum acceptable level of earnings for a normal amount of work \((h = 1)\) is 0.52. To induce a worker to accept a job with 20 percent more than normal hours \((h = 1.2)\), earnings need to be at least 0.75. On the other hand, if not working involves benefits of 0.4, the minimum acceptable level of earnings for a normal amount of work is that much higher, at 0.92. Workers are close to the point of indifference between work and non-work. Labor supply is perfectly elastic in this case. It occurs when benefits are at 0.48. This figure is not realistic for the United States, where typical replacement rates for unemployment benefits are generally less than 20 percent.
These findings shed some light on the decline in value the occurs when a worker loses a job. When working a normal schedule ($h = 1$) and receiving normal earnings ($y = 1$), absent any benefits, workers enjoy a value of 0.32 consumption units. After a job loss, without benefits ($h = y = 0$), the value drops considerably to -0.16.
The Employment Relationship

Topics:

- Bilateral efficiency of the relationship
- Mechanism design
- The fixed-wage governance scheme
- The Nash bargain
Basic analysis of efficient continuation or separation

Let

- $Z =$ Present value of worker’s contribution to revenue (called productivity for short).
- $U =$ Present value of a worker’s remaining career at a time when unemployed
- $V =$ Present value of the $U$ that will prevail at the time a new job ends, as of the time of taking the job.
Nature of the employment opportunity

Employer has unconstrained potential number of jobs.
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$Z$ is present value of worker’s marginal revenue product.
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Employer has unconstrained potential number of jobs. 

$Z$ is present value of worker’s marginal revenue product.

Employer does not consider other workers for a particular job, because other workers would also add $Z$ to the present value of revenue.
Worker’s alternative opportunities

Worker does have other opportunities, but they may require some search time.
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The asymmetry between employers and workers reflects the fact that labor is the sole primary factor in the economy and earns Ricardian rents.
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Thus a worker’s earnings are expected to be close to $Z$. 


Setup

Assume that there is free entry to job creation, so the firm attributes no value to a vacant job.
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Criterion for efficient continuation: $Z + V \geq U$
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Criterion for efficient continuation: \( Z + V \geq U \)

Notice that the wage plays no role in the criterion for efficiency. The wage may play an important role in governance regimes that try to achieve efficiency, however.
Joint value of continuation, $Z+V$

Worker stays, efficiently

Worker departs, efficiently

Joint value from separation, $U$
Another way to look at efficiency

Criterion: $Z \geq U - V$
Another way to look at efficiency

Criterion: $Z \geq U - V$

Value from working at least as large as worker’s opportunity cost
Another way to look at efficiency

Criterion: $Z \geq U - V$

Value from working at least as large as worker’s opportunity cost

In general, $Z$ private to employer and $U - V$ private to worker.
Efficiency again

Job value, $Z$

Efficient retention

Efficient layoff

45° line

Opportunity cost, $U-V$
Mechanism design

See Mas-Colell, Whinston, and Green, Chapter 23, especially pp. 894-897.
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The Myerson-Satterthwaite theorem discussed there implies that there is no perfect solution to the governance problem if both productivity $Z$ and opportunity cost $U - V$ are private information.
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Question becomes how to structure the governance of the employment relationship to achieve at least approximate efficiency.
Double Vickrey auction

Employer bids wage $W_E$ and worker bids $W_K$.

- If $W_E \geq W_K$, employment occurs or continues; worker receives $W_E$ and employer pays $W_K$.
- Dominant strategy of employer is to bid $W_E = Z$, productivity. Employer's bid has no effect on how much employer pays and any lower bid would result in a lost opportunity to gain profit by employing the worker.
- Dominant strategy of worker is to bid $W_K = U - V$, opportunity cost. Worker's bid has no effect on how much worker earns and any higher bid would result in a lost opportunity to gain a wage that exceeds the opportunity cost.
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Dominant strategy of worker is to bid $W_K = U - V$, opportunity cost. Worker’s bid has no effect on how much worker earns and any higher bid would result in a lost opportunity to gain a wage that exceeds the opportunity cost.
Why we never see the double Vickrey auction

The wage paid falls short of the wage received, so the parties need to line up an insurance company to make up the difference.
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The insurance company faces an information problem similar to the one facing the worker and employer—needs to know the distributions of $Z$ and $U - V$. 
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Double Vickrey does not satisfy the assumptions of Myerson-Satterthwaite, which exclude third parties that have an interest in the outcome. M-S mechanism has balanced budget *ex post*, not just *ex ante*.
Also called the Nash demand auction or the Chatterjee-Samuelson auction
Split-the-difference auction

Also called the Nash demand auction or the Chatterjee-Samuelson auction

Same bidding as in double Vickrey
Also called the Nash demand auction or the Chatterjee-Samuelson auction

Same bidding as in double Vickrey

If \( W_E \geq W_K \), employment occurs or continues; wage paid and received is \( .5W_E + .5W_K \).
Nash equilibrium of
split-the-difference auction

No dominant strategies
Nash equilibrium of split-the-difference auction

No dominant strategies

Continuum of equilibria
Nash equilibrium of split-the-difference auction

- No dominant strategies
- Continuum of equilibria
- Any employment that occurs is efficient, but many equilibria fail to generate employment that would be efficient
Worker’s opportunity cost, $U - V$, is predetermined (not necessarily constant over time, but predictable as of the time when the employment relationship is established).

Let the firm make a unilateral choice about whether to continue employment. The firm agrees in advance to a program of wage payments whose present value is $W$. The firm will choose unilaterally to terminate the worker unless the firm's benefit from continuing, $Z - W$, is positive.
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**Efficiency**
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The firm will make the efficient decision if the wage equals the opportunity cost, or \( W = U - V \).

The firm considers its profit, \( Z - W \), whereas efficiency considers the surplus, \( Z + V - U \), and the two are equal if the wage, \( W \), causes the firm to internalize the worker’s opportunity cost, \( U - V \).
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Note that the firm has a call option on the worker’s services.
Properties of Employer-call-option contract

Best if the worker’s marginal product at the firm is variable, unpredictable, and unverifiable, while the worker’s opportunity cost is known in advance or is contractible.
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Employer captures all of the joint surplus.
Quits

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These quits could be highly inefficient.

At a minimum, need to build in a wage premium to head off quits, but at the expense of incurring some inefficient layoffs.
Reverse case: Only opportunity cost private

Wage is the known productivity.
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Worker has put option—can choose to work at the given wage, if it exceeds opportunity cost
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Worker receives all the surplus.
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Taxi drivers. Does not seem to be a common governance mechanism.
Fixed wage with both productivity and opportunity cost private

Fixed wage with both productivity and opportunity cost private


Tradeoff between inefficient layoffs and inefficient quits results in the choice of the fixed wage somewhere in the middle between the average value of productivity and the average value of the opportunity cost.
GENERAL CASE

Diagram showing the relationship between contract wage, opportunity cost ($U-V$), and various labor market outcomes:

- Efficient retention
- Inefficient quit
- Efficient quit
- Efficient layoff
- Inefficient layoff

The diagram includes a 45° line indicating efficient layoffs and the point where efficient layoff and quit meet.
Other aspects of the fixed-wage contract

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On the other hand, when a wage turns out to be higher than both productivity and opportunity cost, the worker will suffer layoff inefficiently and the parties would be better off by agreeing on a lower wage, below productivity but above opportunity cost. This operates against commitment.
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On the other hand, when a wage turns out to be higher than both productivity and opportunity cost, the worker will suffer layoff inefficiently and the parties would be better off by agreeing on a lower wage, below productivity but above opportunity cost. This operates against commitment.

There’s no good overall solution when productivity and opportunity cost are both private.
Offer matching

In some markets (notably the market for economists), wages are committed to a fixed path with one exception: Employers match wage offers from other employers when the wage remains below the worker’s productivity.
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Offer matching generates another inefficiency—excess effort to receive offers just to gain higher pay.
Observable values and Nash bargain

Assume productivity and opportunity cost are observable to both sides.
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Nash bargaining solution to the bargaining problem. Bargained wage is the equally weighted average of the reservation wages of the two parties.
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Employer’s reservation wage is the value lost from its outside option of not filling the job—recall that filling the job with another applicant is not an option, because that applicant could fill another job just as well. Thus the employer reservation wage is productivity $Z \cdot 123$. 

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Nash wage bargain, continued

\[ W = \frac{1}{2}(Z + U - V) \]
Nash wage bargain, continued

\[ W = \frac{1}{2}(Z + U - V) \]

An advantageous bargain is possible whenever employment is efficient. All separations are efficient.
Division of the surplus

The surplus from the match is $S = Z - (U - V)$. 

If the worker can find another job at the same wage fairly soon instead of taking this job, alternative earnings if this job is not taken will be close to $Z$, the surplus will be fairly small, and $W$ will be close to $Z$. The magnitude of the surplus depends on how much search friction there is in the labor market, not on the fundamental productivity of workers.
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The surplus from the match is \( S = Z - (U - V) \).

The wage is \( W = U - V + \frac{1}{2}S \), so the worker gets half the surplus. The employer gets the other half.
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Other aspects of the Nash bargain

There is a 50 percent diminution of match-specific investments by either side—this applies to both up-front investments and marginal investments.
OTHER ASPECTS OF THE NASH BARGAIN

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Each side has an incentive to reduce the other side’s outside option value, to capture more of the joint value.
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UNEMPLOYMENT

Topics:
- Turnover
- Efficiency wage model
- Diamond-Mortensen-Pissarides model
- Sticky-wage model
TURNOVER

\[ f = \text{Job-finding rate, per period probability of finding a new match} \]
**Turnover**

\[ f = \text{Job-finding rate, per period probability of finding a new match} \]

\[ s = \text{Separation rate, the probability that match becomes unproductive in a given period} \]
Values of $U$, $V$, and $Z$

$r =$ Discount rate, risk adjusted
VALUES OF $U$, $V$, AND $Z$

$r =$Discount rate, risk adjusted

$$V = \frac{1}{1 + r} \left[(1 - s)V + \delta U\right]$$
Values of $U$, $V$, and $Z$

$r =$ Discount rate, risk adjusted

$$V = \frac{1}{1 + r} [(1 - s)V + \delta U]$$

$$U = \frac{1}{1 + r} [(1 - f)U + f \cdot (W + V)]$$
VALUES OF $U$, $V$, AND $Z$

$\text{Discount rate, risk adjusted}$

\[ V = \frac{1}{1 + r} \left[ (1 - s)V + \delta U \right] \]

\[ U = \frac{1}{1 + r} \left[ (1 - f)U + f \cdot (W + V) \right] \]

Substitute the Nash value for $W$ and rewrite as

**Job value relation:** $Z + V = \frac{s}{r+s} U + Z$

**Unemployment value relation:** $Z + V = \frac{2r+f}{f} U$
JOB AND UNEMPLOYMENT VALUES

\[ Z+V \]

\[ f<1 \]

Unemployment value relation

45° line

Job value relation

Glue

\[ U \]
Glue: Match-specific capital or job surplus

The glue is the amount of the match-specific capital, the surplus or difference between $Z + V$ and $U$. 
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Under 50-50 Nash bargaining, the worker owns half the glue and the employer the other half.
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Under 50-50 Nash bargaining, the worker owns half the glue and the employer the other half.

The more friction in the market (lower \( f \)), the more glue.
Efficiency wage theory

Lower $f$ creates more glue.
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To make termination costly to workers and to prevent them from shirking, $f$ must be low and so unemployment must be high.
Lower $f$ creates more glue.

To make termination costly to workers and to prevent them from shirking, $f$ must be low and so unemployment must be high.

This requires a model where $f$ can vary. Suppose $f = \phi(s)$, a negatively-sloped function of $s$, the separation rate.


Suppose workers will shirk unless match capital is high enough so that their sacrifice by shirking is less than their gain.
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To sustain the employment relationship, the job-finding rate must be below some critical level, \( f^* \).
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To sustain the employment relationship, the job-finding rate must be below some critical level, $f^*$. The job destruction rate will be 1 for $f > f^*$, any value $s \leq d \leq 1$ for $f = f^*$, and $s$ for $f < f^*$.
Efficiency wage model

\[ f(s) \]

\[ f^* \]

\[ \phi(s) \]
Labor-Market Equilibrium

The value of the outside option of the job-seeker when bargaining over the wage with a prospective employer is $U - V$. 

Workers produce output with a present value $Z$ over the course of the job. We will be concerned with the response of unemployment and other endogenous variables to changes in $Z$, the driving force of fluctuations.

Add one more element: When a worker is searching rather than working, the worker receives unemployment benefits and enjoys not having to work, with a combined flow value of $\lambda$. 


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Matching

Matching of employers and workers results from non-contractible pre-match effort by employers—help-wanted advertising and other recruiting costs. We describe the mechanism in terms of vacancies, though this concept need be nothing more than a metaphor capturing recruiting effort of many kinds. The key variable is $\theta$, the ratio of vacancies to unemployment.
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The job-finding rate depends on $\theta$ according to

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The recruiting rate is

$$\rho(\theta) = \frac{\phi(\theta)}{\theta},$$

which is assumed to be decreasing.
There is free entry on the employer side, so that employer pre-match cost equals the employer’s expected share of the match surplus in equilibrium. Employers control the resources that govern the rate of job finding. The incentive to deploy the resources is the employer’s net value from a match, $Z - W$. Recruiting to fill a vacancy costs $c$ per period.
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Employers create vacancies, drive up the vacancy/unemployment ratio $\theta$, and drive down the recruiting rate $\rho(\theta)$ to the point that satisfies the zero-profit condition.
The model has five endogenous variables, the worker’s value of being unemployed, $U$, her value of employment after the prospective job, $V$, the job-finding rate, $f$, the vacancy/unemployment ratio, $\theta$, and the present value of wage payments, $W$. It has five equations.
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In a stationary equilibrium, the flow rate of workers into unemployment is $(1 - u)s$ and the flow rate out of unemployment is $uf$; equilibrium requires that these rates must be equal. So, the unemployment rate is:

$$u = \frac{s}{s + f}.$$
**Flexible wage**

In this model with Nash bargaining, the wage is the average of productivity $Z$ and the worker’s opportunity cost, $U - V$. The wage is highly responsive to changes in productivity because $Z$ and $U - V$ move together—the worker’s opportunity cost $U - V$ depends sensitively on the wages of other jobs. Further, if unemployment rises, the wage will fall because the worker’s opportunity cost falls. For both of these reasons, a reduction in $Z$ results in correspondingly large changes in $W$ but only tiny changes in unemployment.
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This flexible-wage property of the standard model is the point of a paper by Shimer in the AER, March 2005.
PARAMETERS

Measure time in months and calibrate to a separation rate of 3 percent per month and an unemployment rate of 5.5 percent. These imply a job-finding rate of 52 percent per month. From the Job Openings and Labor Turnover Survey, take the vacancy/unemployment ratio, $\theta$ to be 2. Normalize $Z$ to 1 at the calibration point. Take the discount rate to be $r = 0.05/12$. Take the flow value of unemployment compensation and leisure to be $\lambda = 0.4(r + s)$, 40 percent of flow productivity.
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Solve the model for the cost of the employer’s pre-match recruiting, $c$, to fit the job-finding rate. The value is $c = 0.036$, about a month of wages.
Matching

Take the job-finding function to be

$$\phi(\theta) = \phi_0 \theta^{0.5},$$

so the recruiting rate function is

$$\rho(\theta) = \phi_0 \theta^{-0.5}.$$
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Calibrate the efficiency parameter $\phi_0$ to the job-finding rate and vacancy/unemployment ratio.
Equilibrium values

At the calibrated equilibrium, the wage is $W = 0.965$ and the job-seeker’s value while unemployed is $U = 7.61$. 
**EQUILIBRIUM DIAGRAM**

The next slide shows the determination of the equilibrium in the standard model in a diagram with the vacancy/unemployment ratio, $\theta$, on the horizontal axis and the wage, $W$, on the vertical. The downward-sloping curve depicts values where firms earn zero profits from hiring. The upward-sloping curve describes the equilibrium of the rest of the model, including the Nash bargain for the wage. The equilibrium is stable in the following sense: When the vacancy/unemployment ratio is below the equilibrium, the wage determined in the model leaves hiring profits for employers. As they expand hiring, they raise the vacancy/unemployment ratio and move the labor market toward equilibrium.
Determination of the Wage

Vacancy/Unemployment Ratio, $\theta$

Wage, $W$

Bargaining Equilibrium

Zero Profit

144
Response to changes in $Z$

To see how the bargaining equilibrium curve shifts as $Z$ changes, take the derivative of $W$ with respect to $Z$ in the system of equations underlying the curve, keeping $\theta$ constant. The derivative is

$$\frac{dW}{dZ} = \frac{r + f + s}{2r + f + 2s}.$$
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At the calibrated values, the derivative is 0.94. In a recession, $W$ falls directly by the 0.5 coefficient of $Z$ and by another 0.44 because the worker’s opportunity cost, $U - V$, falls. The derivative of the zero-profit value of $W$, again holding $\theta$ constant, is 1. Hence the two curves in the diagram shift downward by about the same amount and conditions in the labor market, measured by $\theta$, hardly change. This is Shimer’s point, repeated.
Sticky Wages

Now consider the same model, except that the wage $W$ is fixed. Suppose that, initially, its fixed value lies in the bargaining set of the wage bargain, $[U - V, Z]$. Then $Z$ falls. The market has to slacken, with lower $\theta$, to keep employers at the point of zero profit. Unemployment rises sharply. The sticky-wage model delivers large changes in unemployment from small changes in $Z$.

Notice that bilateral efficiency is retained as long as $Z$ does not fall far enough to be below $W$, in which case employers would not hire workers with whom they are matched, even though efficiency calls for them to hire.
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Based on John Cochrane, *Asset Pricing*
Price of an asset with payoff $x$

$$p_t = \mathbb{E}_t \left[ \delta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right]$$
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Examples

Stock: \( p_t = \mathbb{E}_t \left[ m_{t+1}(p_{t+1} + d_{t+1}) \right] \)
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One-period bond: \( p_t = E_t \left[ m_{t+1} \right] \).
Risk-free return: $1 = R^f_t \mathbb{E}_t [m_{t+1}]$
More examples

Risk-free return: $1 = R^f \ E_t [m_{t+1}]$

Option: $p_t = E_t [m_{t+1} \max(S_{t+1} - K, 0)]$
Risk-free rate

\[ R_f = \frac{1}{\mathbb{E}(m)} \]
Risk-free rate

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With log-normal consumption,
**Risk-free rate**

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With log-normal consumption,

\[ \log(R^f) = \delta + \gamma \mathbb{E}_t (\Delta \log c_{t+1}) - \frac{\gamma^2}{2} \sigma_t^2 (\Delta \log c_{t+1}) \]
Risk and price

\[ p = \mathbb{E} (mx) = \mathbb{E} (m) \mathbb{E} (x) + \text{cov}(m, x) \]
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\[ p = \text{discounted expected return plus risk adjustment} \]
Risk premium over safe rate

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No benefit to acquisition of assets or spinning assets off unless their returns change as a result of change in control
$\beta$

$$E(R^i) = R^f + \frac{\text{cov}(m, R^i)}{V(m)}(-R^f V(m))$$

Expected return is the risk-free rate plus the coefficient of the regression of the return on the stochastic discounter multiplied by a factor that is the same for all assets. $\lambda$ is the price of risk and $\beta_i$ is asset $i$’s risk.
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One-year real returns on one-year Treasury bills and from holding the S&P 500 stock portfolio for one year (including dividends), 1959 through 2009.
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Form the discounter from consumption of nondurable goods per person, with $\gamma=2$. 

Find the value of the utility discount factor from the condition that the discounted average real return ratio on bills is one. This turns out to be 1.0064, a problem. The observed real return is too small to square with the observed growth of consumption per person.

The average value of the discounted return on the stock market is 1.014, higher than the CCAPM value of one. But the standard error is 0.022, so the discrepancy could easily arise from sampling error.
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The average value of the discounted return on the stock market is 1.014, higher than the CCAPM value of one. But the standard error is 0.022, so the discrepancy could easily arise from sampling error.
The value of the coefficient of relative risk aversion $\gamma$ that results in a discounted stock-market return ratio of one is 11.25, implausibly high.