ESSAYS ON THE THEORY OF WEALTH

by

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B.A., University of California
(1964)

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF

PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF

TECHNOLOGY

June 1967

Signature of Author. . . . .
Department of Economics, May 12, 1967

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Chairman, Departmental Committee on Graduate Students

ABSTRACT

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Submitted to the Department of Economics on May 12, 1967 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

This thesis consists of five essays on diverse aspects of the intertemporal competitive equilibrium in an idealized economy in which all markets are competitive and all participants anticipate future economic events correctly. In the first essay, a model is proposed in which the family is the basic consuming unit; various aspects of the allocation of wealth among the generations of a family are discussed. The resulting competitive equilibrium in a model with a simple production technology is described.

In the second essay some of the implications of the inheritance hypothesis of the first essay are discussed in the context of a family utility function. The problem of variable rates of impatience is considered in some detail.

In the third essay the macroeconomic model of the first essay is used to examine dynamic substitution effects which arise when fiscal policy changes. Some important anticipation effects are described, and it is shown that the price system acts to cushion the shocks in real variables caused by fiscal action.

In the fourth essay the problem of speculative booms in certain kinds of assets is considered, and a condition is derived which rules out these booms.

Finally, in the fifth essay, the effects of technical change and deterioration on the relations between the prices for durable capital goods is examined in detail, and a unified theory of deterioration, technical change, depreciation, and obsolescence is developed.

Thesis Supervisor: Robert M. Solow

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ACKNOWLEDGMENTS

I am grateful to the members of my committee, Professors Robert Solow, Peter Diamond, and Paul Samuelson, for their valuable comments, criticisms, and suggestions at every stage. I also benefitted from many useful conversations with Professors Karl Shell, Miguel Sidrauski, and Joseph Stiglitz, and with my colleagues Anthony Atkinson, William Nordhaus, Michael Rothschild, and Martin Weitzman.

The Ford Foundation provided generous financial support for this thesis; they have asked that they not be held responsible for its contents.

Mrs. Inez Crandall is to be thanked for her excellent last-minute typing job.

Robert E. Hall

INTRODUCTION AND SUMMARY

Wealth is the most important connection between today's economic equilibrium and the economic future. The familiar apparatus of capital theory -- present discounted value formulas and the equation of yield -- depends fundamentally on the assumption that it is wealth that forms this connection for the consumer. This argument was first presented by Irving Fisher almost half a century ago; on the consumer's side, it has undergone relatively little development since then. This series of essays investigates some of the properties of the dynamic competitive equilibrium in an economy in which both consumers and investors are concerned about the future in such a way that wealth has the important role which Fisher assigned it. To the extent that the essays have a further unifying theme, it is that while the moderately distant future may affect today's equilibrium in an important way, the far distant future is always irrelevant. This is revealed in two related properties of the equilibrium of a well-behaved economy: first, any equilibrium in an economy which lasts forever can be found as the limit as the horizon goes to infinity of the equilibria in a sequence of similar economies with finite horizons -- in other words, the fact that the economy lasts forever is not intrinsically important. Second, the sensitivity of today's behavior to future disturbances decreases to zero as the time interval before the disturbance becomes large. The second property is a consequence of the instability of the difference equations which determine the behavior. Because of the fundamental terminal or limiting conditions -- the budget constraint for the consumer and the non-speculation condition for the investor -- the instability of the difference equation implies stability of behavior. There has been a certain amount of confusion on this point recently, especially in connection with what has come to be called the Hahn problem.

These essays take only the first step in discussing the competitive equilibrium in this kind of model, that of deriving the price vectors which, if expected with certainty, would in fact bring about equality of supply and demand. In intertemporal problems there is an especially acute problem of specifying a mechanism by which a decentralized economy might find these prices. Although I have done some work on this problem (mentioned briefly in Chapter 1), it is still in an early stage and is not presented here, except in the form of the illustrations, which were drawn by a computer program using a modified tatonnement algorithm.

<u>Summary</u>

Chapter 1 investigates the competitive equilibrium in an economy composed of individuals who are connected with the future by their concern for the well-being of their children. Under a precise specification referred to as the inheritance hypothesis, various properties of the equilibrium are demonstrated. These are expressed in a competitive turnpike theorem, a theorem on the irrelevance of the distant future for the individual family, and remarks on the efficiency of the competitive equilibrium and on the general non-speculation condition.

Chapter 2 investigates the implications of a set of postulates on intertemporal preferences which turn out to be equivalent to the inheritance hypothesis. The important results are, first, that there exists

no intemporal utility function with a variable rate of impatience which meets all of the other postulates, and, second, that there exists a non-integrable set of marginal rates of substitution with both variable rates of impatience and generalized diminishing marginal rate of substitution. Further, the rate of impatience may decline with increasing consumption, in contrast to the alternative specifications appearing in the literature, all of which require increasing rates of impatience.

Chapter 3 examines the effects of discontinuous disturbances on the dynamic equilibrium of the model of Chapter 1, when these disturbances take the form of sudden jumps in tax rates. The equilibrium is shown to be highly stable; the economy always returns smoothly to its steady state after large finite disturbances. Further, important adjustments take place in anticipation of a tax change -- while the distant future is irrelevant, the near future enters the determination of the equilibrium in a fundamental way.

Chapter 4 is concerned with the equilibrium prices of assets in an economy with diverse assets. The notion of speculation in assets is defined, and in a Non-Speculation Theorem, it is shown that speculation in reproducible assets is ruled out if there is a market-clearing price at time zero for each productive factor considered as an asset. This shows that the equilibrium prices may be obtained as the limit of the solutions to a sequence of problems with finite horizons.

Finally, Chapter 5 examines in greater detail the problem of the determination of the price of a reproducible physical asset under general assumptions about deterioration and technical change. The non-speculation condition supplies a missing boundary condition which makes the problem determinate. In the last section, theorems are presented which show that it is possible to measure deterioration and technical change from data on prices.

Chapter 1

THE ALLOCATION OF WEALTH AMONG THE GENERATIONS OF A FAMILY WHICH LASTS FOREVER -- A THEORY OF INHERITANCE

Recent models of economic growth have been based on a variety of assumptions about consumption behavior. First, a large literature has grown out of the assumption that consumers make decisions by arbitrary rules, particularly the rule of consuming a fixed fraction of total income or that of consuming all wages and saving all profits. Second, in the past two or three years there has been a resurgence of interest in models of optimal accumulation in which consumption behavior is regulated by an authority which can see far beyond the lifetime of any individual and which maximizes a social welfare function defined over the consumption of present and future generations. Finally, an important series of papers by Samuelson (6), Diamond (3), and Cass and Yaari (1) have investigated competitive models in which individuals determine their consumption for two or more periods by maximizing a utility function subject to a wealth constraint. The results of these investigations are somewhat disturbing -- in particular, the competitive equilibrium interest rate may be permanently less than the rate of growth because of over-saving. This implies that the equilibrium is inefficient by a well-known theorem of Phelps and Koopmans (5). Further, as Diamond has shown, some seemingly neutral fiscal activities of the government may have an important effect on the equilibrium -- there may indeed be a "burden" of the public debt.

These models neglect an important aspect of the intertemporal decisions of the consumer, namely that a person usually cares not only about his own consumption but also about the well-being of his children. A father has a considerable amount of control over his sons' wealth because he can vary the size of the bequest which he makes to them. In this essay

I derive some of the implications of an hypothesis about the way in which a family makes decisions about inheritance. The hypothesis is that a father and his sons decide jointly how to allocate their wealth between the father's and the sons' consumption by maximizing a utility function in which each one's consumption appears as an argument. Interestingly, while the spirit of this hypothesis is similar to that of the competitive utility maximizing models of Samuelson-Diamond-Cass-Yaari, the properties of the resulting model are very much like those of the centrally-directed social welfare-maximizing models of optimal growth. In fact, I will demonstrate that this competitive model has the catenary turnpike property common to almost all models of optimal growth.

This essay treats a highly stylized economy in which there is one kind of output which may either be consumed or used as capital in production. Each person lives two periods, but only consumes and earns wages in the second period. At the beginning of the second period each person marries and has 1+n sons and 1+n daughters. A family consists of a man and his wife, their children, and all of their future descendents; however, only men enter the utility function. There is a perfect market for loans between any two periods. Production is carried out by profit-maximizing entrepreneurs who borrow from the public to finance all of their investments. Production is assumed to take place with constant returns to scale, and output is sold in a competitive market, so entrepreneurs earn no profit. Finally, all families are assumed to be identical.

1. Family demand functions with a finite horizon.

The assumptions of this essay about the family allocation process can be stated in two main hypotheses. First, we have the

Hypothesis on the allocation of wealth between father and son: However much wealth a father and his sons devote to their consumption, they divide it among themselves so as to maximize a joint utility function,

$$U(c_f, c_s) ,$$

where c_f is the father's consumption and c_s is the consumption of each of his sons. We further assume that U is quasi-concave.

Then, as a consequence of this maximization process, for any given amount of wealth which they spend in total, there is a unique demand function giving the father'a share, where the father is now identified as generation t:

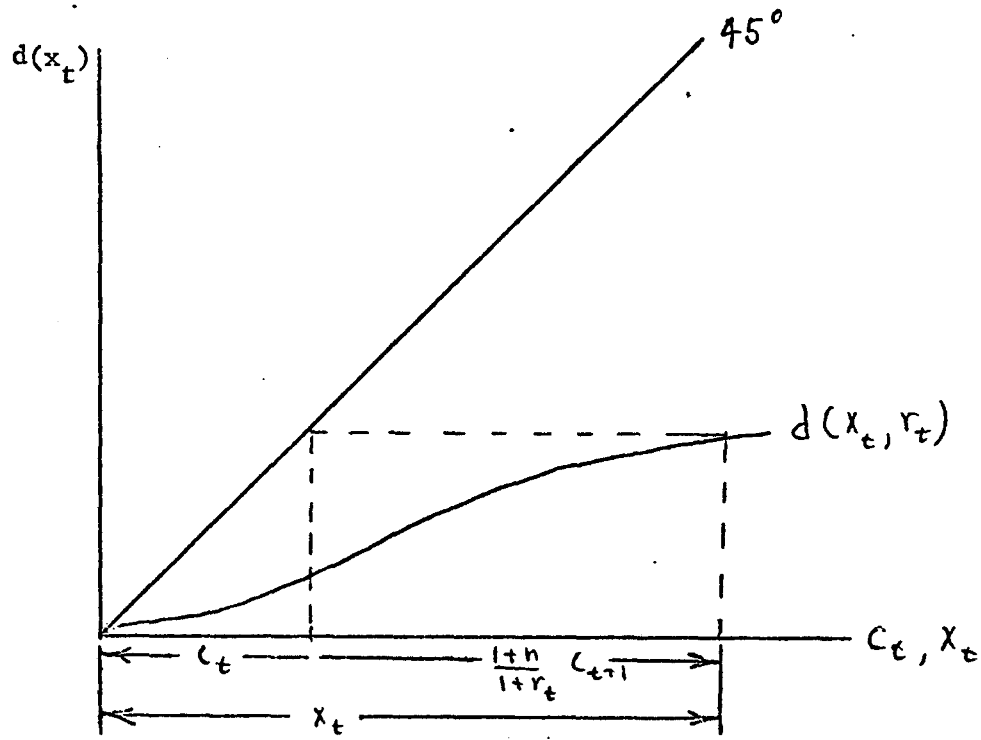
$$(1) \qquad c_t = d(x_t, r_t) ;$$

x_t is the present value of the consumption of father and sons and r_t is the interest rate on loans between period t and period t+1.

It is sensible to assume that the father's consumption is not an inferior good:

$$(2) \qquad \frac{\partial d}{\partial x_t} \geqq 0 .$$

Now if each pair of generations makes its decisions by this process, and if the decisions are consistent in that each son's planned consumption is the same as his actual consumption as a father, then the consumption of all future generations can be predicted exactly, given today's father's consumption. This follows by induction after establishing the uniqueness of the sons' consumption given the father's consumption, since each son subsequently becomes a father. For this purpose, consider the following diagram:



To determine the c_{t+1} corresponding to a particular c_t, draw a vertical line up from c_t to the 45^o line and extend it horizontally to its intersection with the $d(x_t, r_t)$ curve (the intersection is unique by the assumption that the curve does not turn down). The horizontal distance at the intersection is the total expenditure x_t corresponding to c_t; the consumption per son is given by

$$c_{t+1} = \frac{1 + r_t}{1 + n} (x_t - c_t) \quad . \tag{3}$$

Thus the father can predict the consumption of every generation corresponding to a particular value of his own consumption. Analytically, this can be seen by substituting

(4) $$x_t = c_t + \frac{1+n}{1+r_t} c_{t+1} \quad ,$$

in equation (1) to get an implicit difference equation

(5) $$c_t = d(c_t + \frac{1+n}{1+r_t} c_{t+1} \, , \, r_t) \quad ,$$

which can always be stated explicitly in the form

(6) $$c_{t+1} = g(c_t , r_t) \; .$$

Today's father can use this relation to predict the consumption of all future generations.

It remains to introduce an additional hypothesis to specify in what way today's father and sons care about future generations of the family, or, in other words, how they choose the part of the total family wealth which they will appropriate for themselves, x_t. This is the

Hypothesis on future generations: Today's father and sons spend the largest amount of wealth, x_t, that is consistent with the long-run family budget constraint requiring that terminal wealth not be negative.

Then the decision-making process of the family may be visualized in the following way: today's father and sons examine the various family consumption trajectories which correspond to alternative values of x_t, and pick the value of x_t whose consumption trajectory will exhaust family wealth at the terminal time, T. The easiest way to state the exhaustion of wealth is in terms of family assets (non-human wealth); if A_t denotes family assets per person measured at the beginning of period t, then

$$(7) \qquad A_{t+1} = \frac{1 + r_t}{1 + n} A_t + w_t - c_t \quad ,$$

where w_t is non-interest income (wages) per person in period t. Then the budget constraint is

$$(8) \qquad A_{T+1} = 0 \quad .$$

Because of the continuity of the functions involved, there is always a value of the father's consumption c_t corresponding to a trajectory which exactly meets this budget constraint.

The discussion so far has considered only the behavior and motivation of one pair of generations of a family, and has carefully avoided the notion that any individual made plans which are binding on future generations. It is interesting at this point to investigate the consequences of this kind of behavior for the family as a whole. In particular, we inquire whether or not this behavior is reasonable in the sense that it resembles the behavior which might be prescribed if the family in fact had a planner.

The discussion will draw upon the results of (4), which proposes a hypothesis which is equivalent to the present one in the special case where the intergenerational utility function has the special additive form

$$(9) \qquad U(c_t, c_{t+1}) = u(c_t) + (1 + n)v(c_{t+1}) \quad .$$

This form will be assumed in the following discussion.

The first important property of family consumption under the inheritance hypothesis is efficiency. A consumption trajectory is efficient if there is no generation whose consumption could be increased without violating the family budget constraint. Clearly with a finite horizon any trajectory which meets the budget constraint exactly is efficient; the real significance of this property is apparent only when the family is assumed to last forever. However, family consumption behavior based on arbitrary rules (such as a constant savings ratio) may fail to meet even this simple criterion.

The second important property is what Samuelson (7) calls reversibility: for any consumption trajectory there is a total family wealth and an interest rate trajectory which yields the consumption trajectory as the family demand. In other words, there are no parts of the consumption space which are permanently in the dark in the sense that they would never be the demand of a family in a competitive economy. If a surrogate family utility function exists, this property is equivalent to quasi-concavity of the function. In the present model this property always holds if the horizon is finite. The conditions for family equilibrium are

(10) $$s(c_t, c_{t+1}) \equiv \frac{v'(c_{t+1})}{u'(c_t)} = \frac{1 + r_t}{1 + n} \quad \text{and}$$

(11) $$W = \sum_{t=1}^{T} \prod_{\tau=1}^{t-1} \left(\frac{1+n}{1+r_\tau}\right) c_t \quad .$$

Thus the reversibility conditions are satisfied with

(12) $$r_t = (1 + n)s(c_t, c_{t+1}) - 1 \quad ,$$

and the value of W given in equation (11).

A third property, equivalent to the second if a surrogate family utility function exists, is diminishing marginal rate of substitution. This is important because it indicates a tendency for the family consumption plan to involve approximately equal consumption for all generations rather than concentrating on only a few generations. In (4) I have shown that diminishing marginal rate of substitution will hold in this model if the rate of change of the rate of impatience with respect to the consumption level is small in absolute value. The rate of impatience, $\rho(c)$, is defined by

$$(13) \qquad \rho(c) = (1 + n)s(c,c) - 1 ;$$

it is the interest rate at which the level of consumption c will remain constant. Diminishing marginal rate of substitution is guaranteed over any horizon T if

$$(14) \qquad \frac{|\rho'(c)|}{1 + \rho(c)} < \varepsilon$$

for a positive constant ε , independent of T.

A fourth property of possible interest is the existence of a surrogate family utility function $U^*(c_1,\ldots,c_T)$ with the property that all family consumption decisions could be portrayed as if they were made by maximizing this function subject to the family budget constraint. We find from (4) that in general there is no such surrogate family utility and hence no meaning can be given to the notion of family preferences among alternative consumption trajectories. This is neither a destructive, nor, in retrospect, a surprising conclusion. After all, the only connection that today's generations have with the future in this model is a concern for the financial integrity of the family; it would be surprising indeed if this were equivalent to having preferences between any pair of consumption trajectories even when the only difference between the trajectories was in the consumption of a generation far in the future.

There is one significant exception to this conclusion. If the rate of impatience is constant over all consumption levels, then there is, in fact, a surrogate utility with the familiar form

$$(15) \qquad U^*(c_1,\ldots,c_T) = \sum_{t=1}^{T} \left(\frac{1+n}{1+\rho}\right)^t u(c_t) \quad .$$

Then the process of allocating wealth between succeeding generations is exactly the same as would be implied by the Euler equation for maximizing (15); the budget constraint is exactly the transversality condition for this maximization. In this case the inheritance hypothesis amounts to assuming that the family has adopted as its behavioral rule, not the notion of maximizing a utility function, but an operationally identical rule which turns out to be the Euler equation and its transversality condition. This, I think, makes the notion of a family utility function of the special form (15) more acceptable to those who reject the idea of a family consciously maximizing a utility function on the grounds that there is no central authority within a family who makes and enforces consumption plans.

2. Consumption demand for a family which lasts forever.

Economic intuition suggests that the behavior of a family which expects to last a thousand years should be only infinitesimally different from one which expects to last forever. The hypotheses on family behavior proposed in Section 2 are not sufficiently strong to guarantee this irrelevance of the distant future, nor, in fact, are they strong enough to insure that the criteria of reasonable family demand behavior are met when the family lasts forever. Paradoxically, demand functions which are efficient and reversible for any horizon T, no matter how far distant, may be inefficient or irreversible when the horizon is infinitely distant. Not surprisingly, this is closely related to the problem of impatience. A similar paradox has been observed in models of optimal economic growth (for example, by Samuelson (8)), where it has been resolved by showing that impatience is a logical necessity if a true utility function is to exist (see Diamond (2)). Thus we may immediately conclude that the inheritance hypothesis implies a surrogate family utility function if and only if $\rho(c) > n$ and $\rho'(c) = 0$ for all c.

The difficulty with respect to the properties of efficiency and reversibility is the following: If there is a c such that $\rho(c) < n$, then either:

(i) if reversibility holds, the consumption trajectory $c_t = c$ for all t is inefficient, because the implied interest rate is less than the rate of growth, and a debt incurred by any generation vanishes in the limit*, or

(ii) reversibility fails. This will happen if $\rho'(c) < 0$; see Section 3 in this regard.

Thus the inheritance hypothesis must be strengthened in the following way:

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*The effect on the limit of family assets per person of one additional unit of consumption by generation t is $\lim_{T \to \infty} \left(\frac{1 + \rho(c)}{1 + n}\right)^{T-t} = 0$.

Hypothesis on impatience: Either both efficiency and reversibility hold for the consumption demand of a family which lasts forever, or (equivalently) the family is always impatient: $\rho(c) > n$. Efficiency is the more important of the first two properties, since it alone implies that the competitive equilibrium involves an interest rate whose limit is at least as large as the rate of growth.

Now we are prepared to discuss the full family wealth allocation problem over infinite time. Stated formally, the problem is to find a first generation consumption c_1 so that for given initial family assets A_1, $\lim_{t \to \infty} A_t \geqq 0$, where future consumption and assets are predicted by the pair of difference equations

(16) $$c_{t+1} = g(c_t, r_t)$$

(17) $$A_{t+1} = \frac{1 + r_t}{1 + n} A_t + w_t - c_t \quad .$$

As several authors have remarked, this problem may not have a sensible solution. For example, if

(18) $$g(c_t, r_t) = \frac{1 + r_t}{1 + n} c_t \quad ,$$

(this comes from a log-linear intergenerational utility), if r_t has the constant value $\bar{r}$; $\bar{r} > n$ and if $w_t = 0$ for all t, future consumption is

(19) $$c_t = \left(\frac{1 + \bar{r}}{1 + n}\right)^{t-1} c_1 \quad .$$

Family assets at time t are

(20) $$A_t = \left(\frac{1+\bar{r}}{1+n}\right)^{t-1}(A_1 - tc_1) \quad .$$

For any positive c_1, $\lim_{t\to\infty} A_t = -\infty$. The fact that some interest rate trajectories make it impossible for the family to allocate its wealth in a reasonable fashion should not cause us to reject this model of family behavior. Rather, it shows how the inheritance hypothesis restricts the form of the competitive equilibrium. For example, if families have a fixed rate of impatience, the competitive equilibrium capital stock will approach a steady state over time such that the net marginal product of capital will exactly equal the rate of impatience. This is discussed at greater length in Section 3.

3. General equilibrium in the inheritance model.

Suppose that a neoclassical technology prevails, in which output per person, y, is given by a smooth convex function of capital per person:

(21) $$y = f(k) \quad .$$

For convenience we assume that $f'(0)$ exists. Further, capital deteriorates geometrically at a rate δ , so investment for replacement is δk. After taking account of population growth and consumption, the capital stock per person held by the next generation is

(22) $$k_{t+1} = \frac{f(k_t) + (1-\delta)k_t}{1+n} - c_{t+1}$$

Consumption behavior is given by

(24) $$c_{t+1} = g(c_t, r_t) \quad \text{and}$$

the fundamental budget constraint $A_{t+1} = k_T = 0$. The analysis of this system will be carried out in terms of consumption and the interest rate, although it could also be done in terms of any of several pairs of variables.

The r,c) phase plane can be divided into two areas according to whether r is increasing or decreasing. The interest rate is unchanged from one period to the next only if k is unchanged, or

(25) $$f(k) = (\delta + n)k + (1 + n)c$$

In order to characterize this line in terms of r, we differentiate with respect to r:

(26) $$f'(k)\frac{dk}{dr} = (\delta + n)\frac{dk}{dr} + (1 + n)\frac{dc}{dr}\bigg|_{\Delta r=0} .$$

Now since $r = f'(k) - \delta$, $\frac{dk}{dr} = \frac{1}{f''(k)}$, and

(27) $$\frac{dc}{dr}\bigg|_{\Delta r=0} = \frac{f'(k) - \delta - n}{(1 + n)f''(k)}$$

$$= \frac{r - n}{(1+n)f''(k)} .$$

Thus the line slopes upward if $r < n$, reaches its maximum at $r = n$ (the golden rule), and declines if $r > n$; above the line capital is decreasing and r is increasing and <u>vice versa</u> below the line, as shown in Figure 1. A similar division of the phase plane is possible for the consumption equation: consumption is increasing whenever $r > \rho(c)$ and is decreasing whenever $r < \rho(c)$. The directions of movement everywhere in the phase plane are shown in Figure 2.

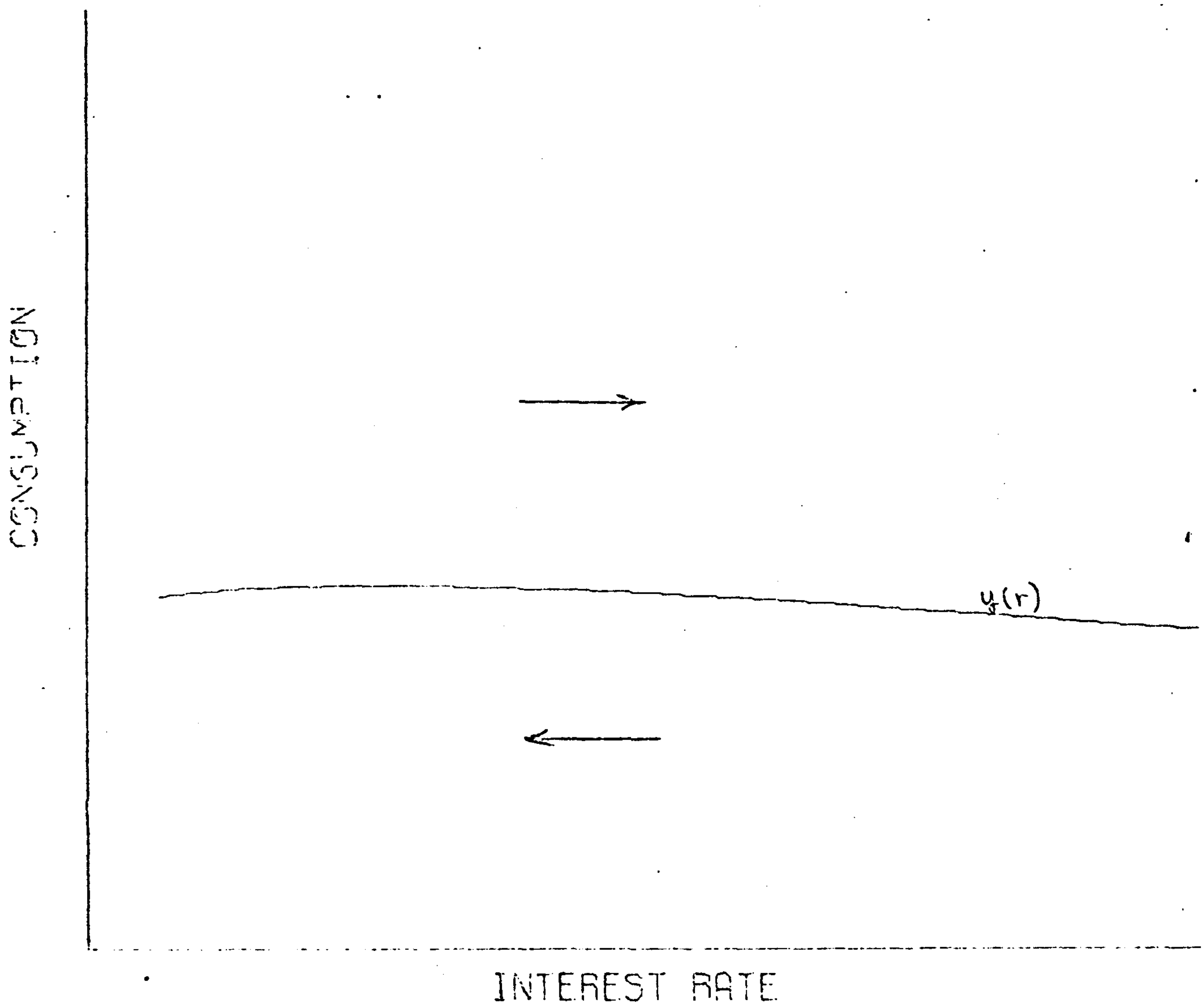

FIGURE 1

Direction of change in r_t over time

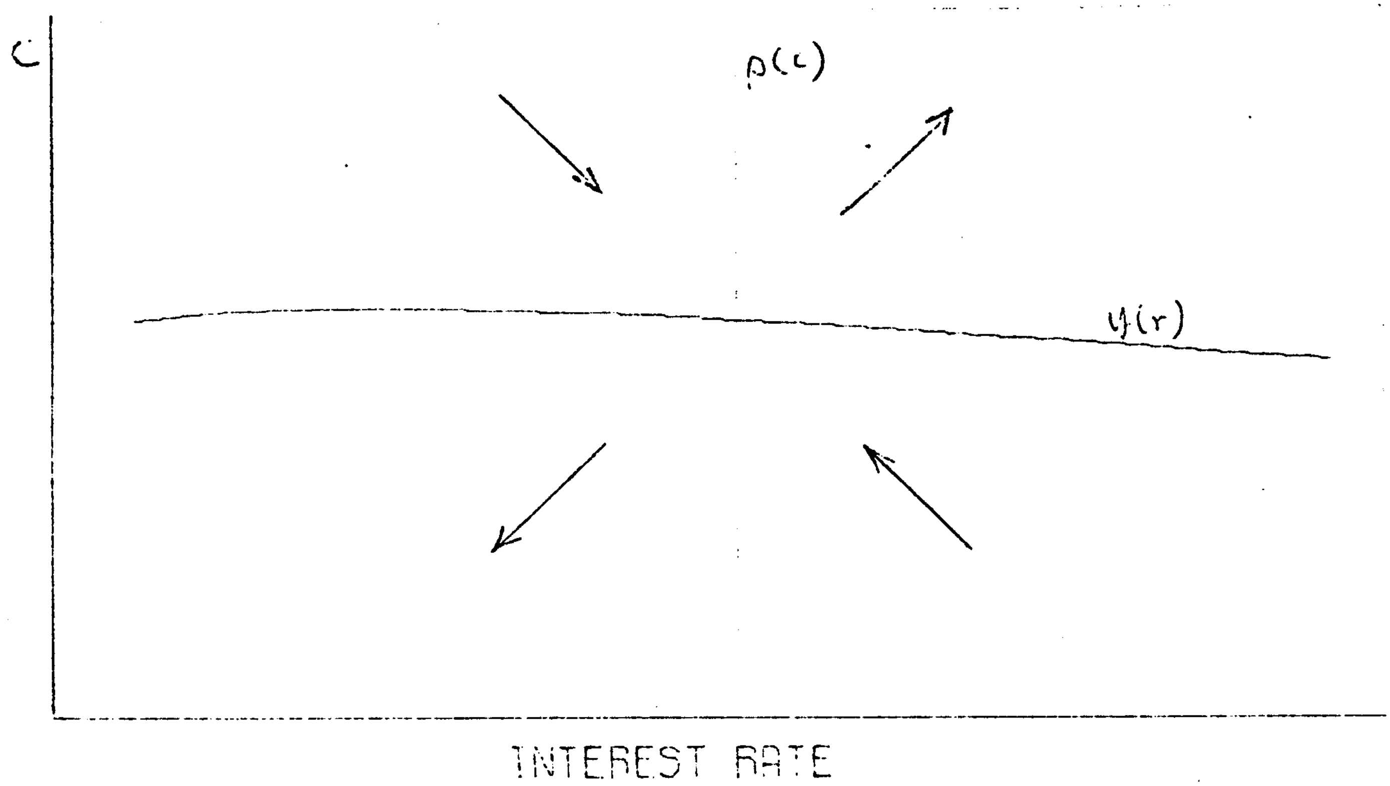

FIGURE 2

Direction of movement in the general phase plane

Since the absolute value of $\rho'(c)$ is restricted to small values, it is reasonable to assume that the two stationary loci meet at a unique stationary point (r^*,c^*). If so, $\rho(c)$ cuts $\Delta r=0$ from below, the stationary point is a saddle point, and from the catenary properties of a saddle point, the following result is established:*

<u>Competitive Turnpike Theorem</u>

As the end of the world becomes more distant (as the horizon T becomes large) the competitive equilibrium interest rate-consumption trajectory spends almost all of its time arbitrarily close to the point (r^*,c^*) where the rate of time preference is equal to the stationary interest rate.

Figures 3 and 4 illustrate trajectories for various T's, starting with the same initial capital stock.

The case of an economy which lasts forever is a simple extension of the previous case, except that the limiting value of family assets cannot be zero but is rather the steady state capital stock k^*. The only infinitely long (r,c) trajectories are those running along the top of the saddle, as shown in Figure 5. From any initial capital stock, the economy eventually approaches indefinitely close to the steady state point (r^*,c^*).

It remains to show that these trajectories are truly competitive equilibria. That is, we must show that given the interest rate and wage

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*Strictly speaking, the turnpike property is known to hold only for a system of differential equations approximating the difference equations (22) and (24).

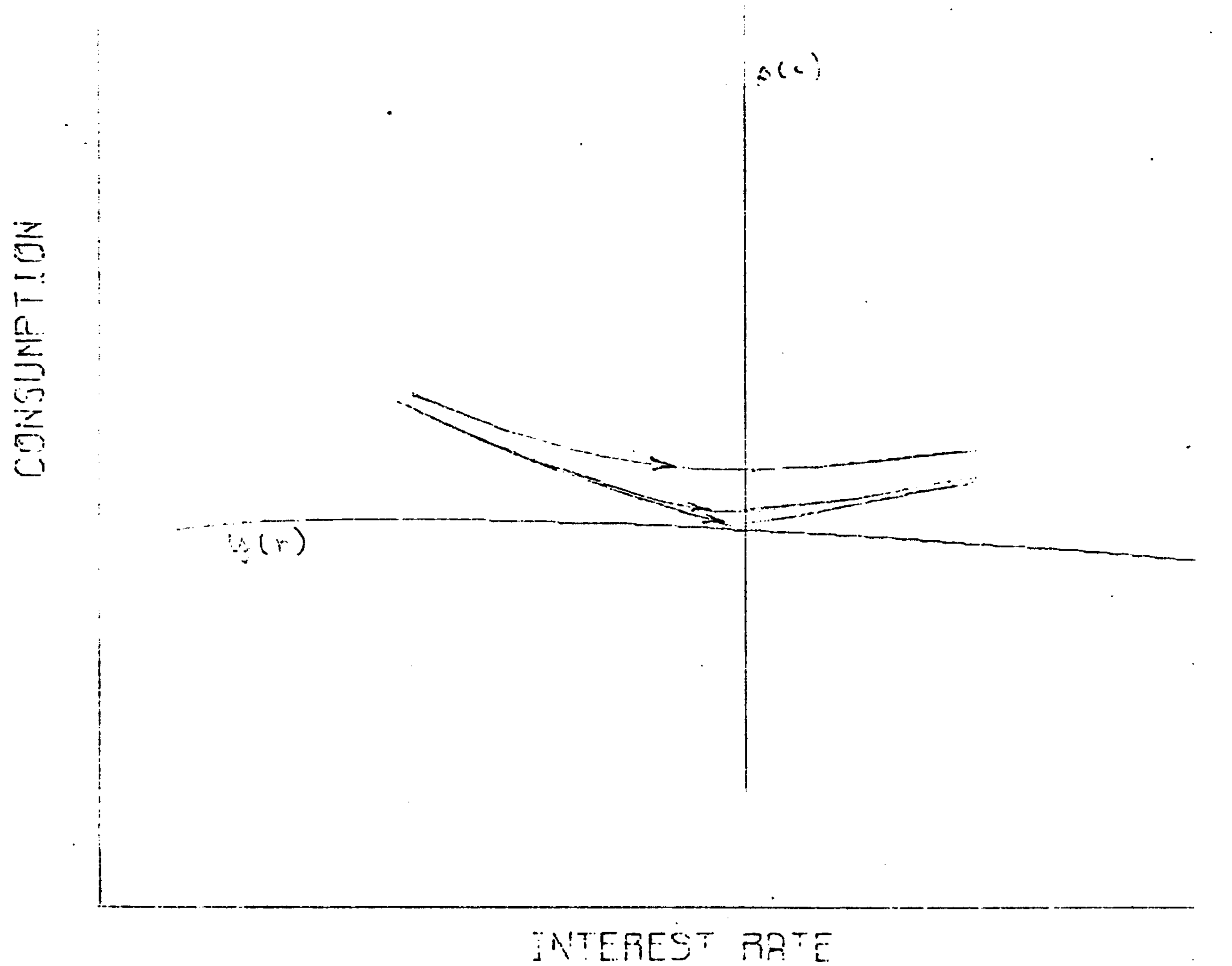

FIGURE 3

Some competitive equilibrium trajectories

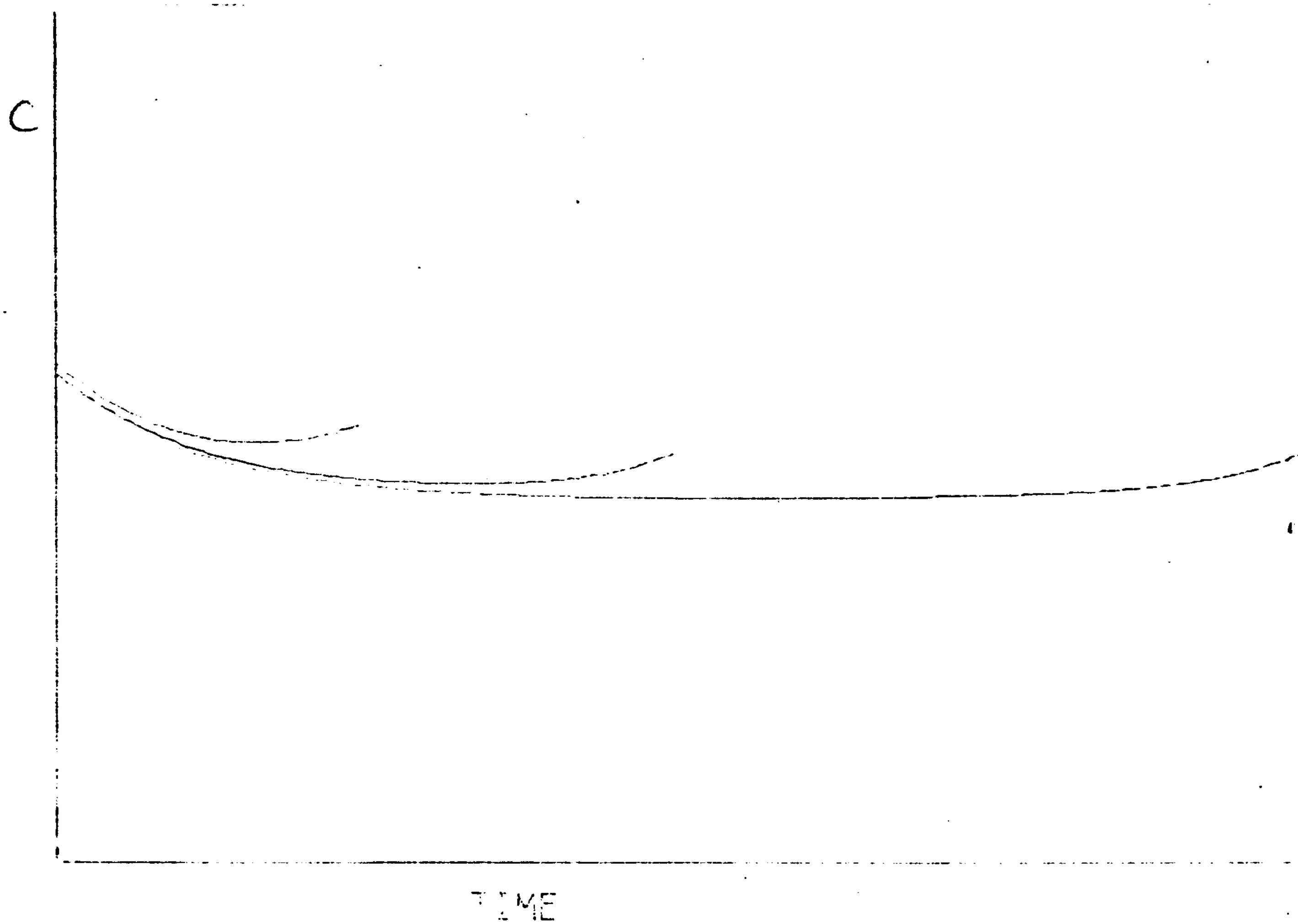

FIGURE 4

Consumption trajectories of Figure 3 plotted against time

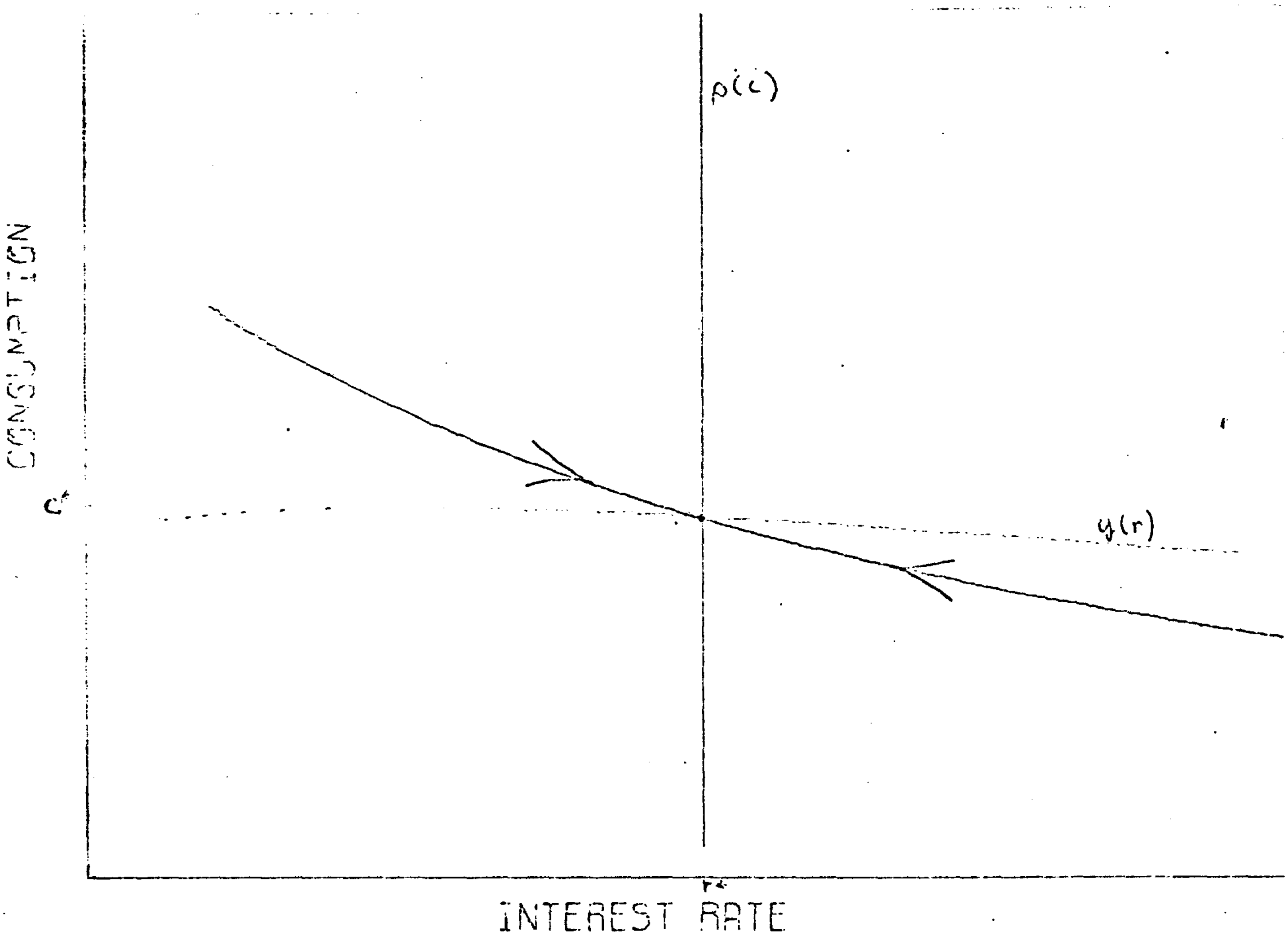

FIGURE 5

Unique infinitely long trajectories

trajectories derived from the phase plane analysis, family demand would in fact be the consumption trajectory shown. Since these trajectories satisfy the difference equation (24), the conditions on the marginal rate of substitution between consecutive generations are clearly met. Furthermore, if the horizon is finite, terminal family assets are zero and no generation could increase its consumption without decreasing the consumption of another generation. Thus the budget constraint is also met; we conclude that this trajectory is in fact the unique family demand. On the supply side, the assumption that the production function is convex guarantees that the supply of consumption goods is uniquely this trajectory.

With an infinite horizon, the limiting value of family assets is not zero, as it would be if the family could exhaust its wealth, but rather is the value of the steady state capital stock k^*. However, if all debts must eventually be paid back (that is, if the present value of one unit of income per person, $\prod_{\tau=1}^{t}(\frac{1+n}{1+r_\tau})$, goes to zero in the limit) no generation could increase its consumption by even the smallest amount without causing eventual family bankruptcy. Thus, even though family assets eventually always have a large positive value close to k^*, the budget constraint is met, and the trajectory is the true family demand. The requirement that debts must be paid back is crucial. For example, if there is a surrogate family utility function with a rate of impatience equal to the rate of growth n, the present value function does not go to zero and there can be no competitive equilibrium. Given the interest rate trajectory from the phase plane analysis, the family will not choose the consumption trajectory shown there, but rather will choose one with higher consumption for one or more generations and for which the limiting value of family assets is not k^* but zero. This is one of a variety of difficulties which arise in an economy in which the interest rate goes to n sufficiently quickly for the present value function not to have limit zero. Most of the hard problems of optimal growth theory are related to this problem; similarly, the interesting aspects of the study of the efficiency of an economy which lasts forever arise only in this case.

One property of the family's allocation problem deserves further attention; it is stated in the

Theorem on the irrelevance of the distant future

Along competitive equilibrium interest rate and wage trajectories the consumption of the present generation of a family becomes increasingly insensitive to their desired level of assets for generation t, as t increases, provided $\rho'(c) > n$. Their sensitivity decreases with increasing impatience and increases with an increasing rate of change of impatience with respect to consumption.

proof:

The sensitivity of the present generation to future asset levels is measured as the reciprocal of the derivative of A_t with respect to c_1. The system of difference equations governing the allocation of family wealth is

$$(28) \qquad c_{t+1} = g(c_t, r_t)$$

$$A_{t+1} = \frac{1 + r_t}{1 + n} A_t + w_t - c_t .$$

Differentiating with respect to c_1, we get

(29) $$\frac{dc_{t+1}}{dc_1} = \frac{\partial g(c_t, r_t)}{\partial c_t} \frac{dc_t}{dc_1}$$

(30) $$\frac{dA_{t+1}}{dc_1} = \frac{1 + r_t}{1 + n} \frac{dA_t}{dc_1} - \frac{dc_t}{dc_1} \quad ,$$

with initial conditions $\frac{dc_1}{dc_1} = 1$ and $\frac{dA_1}{dc_1} = 0$.

Now the function $g(c_t, r_t)$ is defined implicitly by

(31) $$\frac{v'(c_{t+1})}{u'(c_t)} = \frac{1 + r_t}{1 + n} \quad ,$$

so

(32) $$\frac{v''(c_{t+1})}{u'(c_t)} \frac{\partial g}{\partial c_t} = \frac{v'(c_{t+1})u''(c_t)}{[u'(c_t)]^2} \quad ,$$

or

(33) $$\frac{\partial g}{\partial c_t} = \frac{u''(c_t)\ v'(c_{t+1})}{v''(c_{t+1})u'(c_t)} \quad .$$

Now c_t approaches the limit $c*$, and from (4),

(34) $$\rho'(c*) = [1 + \rho(c*)][\frac{v''(c*)}{v'(c*)} - \frac{u''(c*)}{u'(c*)}] \quad .$$

Thus if $\rho'(c*) < 0$, there is a T such that $t > T$ implies

(35) $$\frac{u''(c_t)}{u'(c_t)} < \frac{v''(c_{t+1})}{v'(c_{t+1})} \quad , \text{ or}$$

$$\frac{\partial g}{\partial c_t} > 1 \quad .$$

Similarly, if $\rho'(c^*) > 0$, eventually $\frac{\partial g}{\partial c_t} < 1$.

In the asset equation, eventually $\frac{1 + r_t}{1 + n} > 1$, since r_t approaches the limit $\rho(c^*)$, which is greater than n. Because of its simple recursive form, the properties of the system (29) and (30) may be seen by inspection. Since $\frac{\partial g}{\partial c_t}$ is always positive and $\frac{1 + r_t}{1 + n}$ is eventually strictly greater than one, $\frac{dA_t}{dc_1}$ becomes indefinitely negative with increasing t. If $\rho'(c^*) < 0$, the contribution of the term $-\frac{dc_t}{dc_1}$ also becomes indefinitely large, while in the opposite case, its contribution is eventually zero. But in either case, $\frac{dc_1}{dA_t}$ has the limiting value zero, and the theorem is established.

The property stated in this theorem is also observed in all optimal growth problems with catenary motions; it is sometimes referred to as instability, but this is extremely misleading, since its behavioral implication is one of stability, not instability.

The assumption that the family faces interest rate and wage trajectories which will turn out to be the equilibrium trajectories is crucial in this theorem. Along other trajectories, there may not be a solution to the allocation problem. This will almost always be true if $\rho'(c) < 0$,. since any stationary point of equations (28) is an unstable node, with roots

$$\lambda_1 = \frac{1 + r}{1 + n} \quad , \quad \lambda_2 = \frac{\partial g}{\partial c} \quad ,$$

both of which exceed 1. Usually no trajectory can reach such a stationary point of an asymptotically autonomous system. Along an equilibrium trajectory, however, the interest rate changes over time exactly fast enough to allow the (A_t, c_t) trajectory to reach the unstable node. This means that a tâtonnement procedure would probably not be able to find the equilibrium, since it would require families to solve the allocation problem with interest rate trajectories which are not equilibria. In fact, computational experiments have indicated that the family must take account of the effect of its allocation of wealth (and the identical allocation of all other families) on the interest rate, in order to obtain a tâtonnement procedure that is likely to work. It is possible to show that there is an interest rate adjustment of a simple form which can be applied by each family and which guarantees convergence to the competitive equilibrium -- this adjustment converts the family's allocation problem into one with strictly catenary properties.

4. Some implications of the inheritance hypothesis

The most important difference between this and other models of competitive equilibrium with individually-directed saving is that each person is required to see some distance into the future because he is sensitive to future economic developments. This has a number of important implications. First, since the family has a rate of preference for the father's consumption at least equal to the rate of growth, the possibility of an inefficient competitive equilibrium is ruled out.

Second, in this model the equilibrium is independent of the size of the government's debt, so there is no burden of the debt. In Diamond's model, the equilibrium is sensitive to the size of the debt because the market capitalizes all of the interest payments which a bond yields but the individual takes account of only the tax payments to finance the interest which are levied during his lifetime. This asymmetric effect makes him spend more and save less, driving up the interest rate. Under the inheritance hypothesis there is no asymmetry because the individual does not distinguish between his own wealth and the wealth of the future generations of his family. The equilibrium is independent of any transfer of wealth between generations and in particular is independent of the transfer implied by taxing and paying interest.

Third, this model resolves a perplexing question about the competitive equilibrium price for an asset which cannot be produced, such as land. Nothing in the equilibrium condition for the market for such an asset prevents an upward speculative movement in its price which lasts forever. This is, if p_t is the equilibrium price for the asset, then

$$p_t + s_o \prod_{\tau=0}^{t}(1 + r_\tau) \qquad (36)$$

is also an equilibrium price, where s_0 is any positive constant. Under my hypothesis, however, no price which goes to infinity is a general equilibrium price, because families would then have infinite wealth in the limit, allowing additional consumption for at least one generation. In this way, speculative booms in non-reproducible assets can be ruled out.

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Chapter 2

INTERTEMPORAL PREFERENCES WITH VARIABLE RATES OF IMPATIENCE

Stated most generally, the problem of specifying intertemporal preferences among alternative vectors of consumption of a single good in various periods of time differs from the atemporal problem with many kinds of consumption goods only in the dimension of the consumption space. However, the study of economies in which the participants individually or collectively maximize an intertemporal utility function has achieved much more dramatic and powerful results than the study of atemporal economies. This has come about because the investigators of intertemporal problems have felt that the strong assumptions they make about the form of intertemporal preferences still capture the essential properties of the problem; few economists would accept the same assumptions for an atemporal analysis. In fact, almost all investigators have assumed that intertemporal preferences can be stated in a utility function of the form proposed by Ramsey,

$$\sum_{t=1}^{T} u(c_t, t) \ . \tag{1}$$

This hypothesis has been extremely fruitful, first in problems of optimal growth where it specifies the social welfare function, and, more recently, in models of intertemporal competitive equilibrium, where it specifies the utility function of the individual or family.

In this essay, I will examine the problem of generalizing the Ramsey utility function along lines which would retain the properties of the Ramsey function which yield such useful results in the theory of intertemporal equilibrium. The approach of this investigation is close to the opposite of that taken by Koopmans in his famous paper on intertemporal utility functions (5); Koopmans specializes the most general utility function in accordance with postulates which state restrictions which are natural for intertemporal preferences. Unfortunately, the recursive functional form which he obtains does not yield an easy characterization of the implied demand functions, so the study of general equilibrium with these preferences is quite difficult. The unpublished papers of Beals and Koopmans (1) and Uzawa (8) are apparently the only studies of this problem. Both of them report the somewhat unnatural result that the rate of impatience must be a rising function of consumption -- in fact, this property underlies all of Uzawa's conclusions about family demand functions. One of the purposes of this essay is to show that under an alternative specification the rate of impatience may decline with increasing consumption.

1. Postulates on intertemporal preferences

The following postulates express the restrictions which I seek to impose on intertemporal preferences.

> I. Independence. The marginal rate of substitution between c_t and c_{t+1} is a function $s(c_t, c_{t+1}, t)$, independent of the consumption level in any other period.

This centrally important postulate enables us to characterize the demand for consumption with a difference equation of the form

(2) $c_{t+1} = g(c_t, r_t, t)$,

(where r_t is the interest rate and $c_{t+1} = g(c_t, s(c_t, c_{t+1}, t)-1, t)$), subject to a budget constraint on terminal or limiting assets.

II. Stationarity. The marginal rate of substitution function s is independent of time.

While this is not crucial, it simplifies the discussion and the notation.

III. Variable impatience. The rate of impatience is not a constant function of consumption. The rate of impatience $\rho(c)$ is defined as the marginal rate of substitution less one when c_t and c_{t+1} are equal:

$$\rho(c) = s(c,c) - 1.$$

We require $\rho'(c)$ not identically zero.

IV. Declining marginal rate of substitution. The rate of substitution between c_1 and any linear combination of the other c_τ's declines with increasing amounts of the linear combination.

While these postulates appear at first to define a reasonable class of utility functions, the appearance is deceptive. In fact, no utility function can meet all of these conditions, as shown in the following

Theorem (Debreu-Koopmans)

A utility function meeting postulates I, II, and IV has a constant rate of impatience; further, the only function meeting them is the Ramsey utility function.

proof:

This is a direct consequence of a result of Debreu (2) cited in this connection by Koopmans (5, Sec. 14). An alternative proof for the differentiable case is useful and is given here.

Consider the first three periods, with consumptions c_1, c_2, and c_3. The integrability conditions on the marginal rate of substitution between c_1 and c_2 and between c_1 and c_3 (See, e.g., Samuelson, (6)) require that

$$- \frac{s_2(c_1,c_2)}{s(c_1,s_2)} = \frac{s_1(c_2,c_3)}{s(c_2,c_3)} \quad , \tag{3}$$

for all c_1, c_2, and c_3. Since the left side is independent of c_1, a simple argument shows that

$$c(c_1,c_2) = a(c_1)b(c_2) \quad .$$

Substituting in (3), we find that

$$\frac{a'(c)}{a(c)} = - \frac{b'(c)}{b(c)} \quad . \tag{4}$$

Now

$$\rho(c) = s(c,c)$$
$$= a(c)b(c), \quad \text{so}$$

$$\rho'(c) = a'(c)b(c) + a(c)b'(c)$$
$$= 0 \quad \text{from equation (4).}$$

Now let $d(c) = \frac{1}{a(c)}$ and observe that

$$\frac{d'(c)}{d(c)} = \frac{b'(c)}{b(c)}$$

Then by taking logs and integrating, we get

$$d(c) = (1 + \rho)b(c) \quad ,$$

where ρ is an arbitrary constant greater than -1. Define a function $u(c)$ by

$u'(c) = d(c)$. Then

$$s(c_1,c_2) = (1 + \rho)^{-1} \frac{u'(c_2)}{u'(c_1)} , \tag{5}$$

exactly and uniquely the marginal rate of substitution for the Ramsey utility function

$$U = \sum_t (1 + \rho)^{-t} u(c_t) . \tag{6}$$

The main point of this theorem is that the assumption of independence, which makes possible the characterization of general equilibrium in terms of trajectories in a phase plane, implies that the utility function has the restrictive Ramsey form. Three alternatives present themselves at this point. First, following Koopmans, we might abandon the hypothesis of independence at the cost of additional effort in characterizing the intertemporal demand function. Conversely, we might interpret this theorem as showing that a fixed rate of impatience is in a sense more general than we had thought previously, and therefore continue to postulate the Ramsey utility function.

A third alternative, pursued in the remainder of this essay, is to take the radical step of abandoning the notion that individuals, families, or societies have consistent preference orderings among all consumption vectors. This assumption appeared above in the requirement of integrability. But demand functions can be derived directly from the difference equation (2), without any reference to a utility function. This procedure is of course subject to all the criticism which has been directed at the Cassel-Hicks-Allen approach (in particular by Samuelson in (7)). However, it has somewhat greater plausibility in intertemporal cases than in atemporal ones, since it can be portrayed as a kind of myopia with regard to the future. If individuals choose between this period's consumption and next period's by maximizing a utility function defined over just the two, and if their concern for future consumption is limited to taking care to leave enough wealth to insure that this process can be continued forever, then the present hypothesis is appropriate. This ides is discussed in (3).

2. Diminishing marginal rate of substitution

This section is devoted to showing that a variable rate of impatience is consistent with diminishing marginal rate of substitution in the absence of integrability. One of the important conclusions of this investigation is that there is no restriction on the sign of $\rho'(c)$ -- the rate of impatience may either rise or fall with an increasing consumption level. The curious and counterintuitive result of Beals and Koopmans (1) and Uzawa (8) -- that the rate of impatience must rise with increasing consumption -- does not carry over to this model. The only restriction found here is that the absolute value of $\rho'(c)$ must be less than some positive critical value.

If R_t is the marginal rate of substitution between consumption in period one and consumption in period t, it shows general declining marginal rate of substitution if the rate of substitution between consumption in period one and any linear combination of future consumptions declines with additional amounts of the linear combination, provided the linear combination lies in the tangent hyperplane defined by $\left\{R_t, t=1,\ldots,T\right\}$ (this will be the tangent to the indifference curve with integrability). This definition was first stated by Hicks and Allen (4). It requires that

$$(7) \qquad \sum_t \sum_\tau a_t a_\tau \frac{\partial R_t}{\partial c_\tau} < 0 \quad \text{whenever } \sum_t a_t R_t = 0.$$

The discussion of the possibility of a variable rate of impatience will be carried out under the special assumption that the rate of substitution between adjacent consumptions can be factored into two functions of one variable each:

$$(8) \qquad s(c_t, c_{t+1}) = \frac{v'(c_{t+1})}{u'(c_t)} \quad .$$

This corresponds to the assumption that the two-period utility function mentioned above has the special additive form

$$(9) \qquad U(c_t, c_{t+1}) = u(c_t) + v(c_{t+1}).$$

We assume $u'' < 0$, $v'' < 0$, and $\frac{v''}{v'} < -\beta$ for some positive constant β. There is no apparent obstacle to generalizing the results for any two-period utility function.

Further, in cases where we consider the limit as the horizon T goes to infinitely long consumption trajectories we require diminishing rate of substitution only on trajectories which approach limits. This is a reasonable step since the competitive equilibrium in an economy in which consumption decisions are made in this way always involves consumption trajectories with limits.

Now the first period rates of substitution are given by

$$R_t = \frac{v'(c_2)\ v'(c_3)\ \dots\ v'(c_t)}{u'(c_1)\ u'(c_2)\ \dots\ u'(c_{t-1})}\ , \qquad t > 1; \tag{10}$$

$$R_1 = 1\ .$$

The values of $\frac{\partial R_t}{\partial c_1}$ are as follows:

$$\frac{\partial R_t}{\partial c_1} = -\frac{u''(c_1)}{u'(c_1)}\ R_t\ . \tag{11}$$

$$\frac{\partial R_t}{\partial c_\tau} = 0 \qquad \text{if } \tau > t. \tag{12}$$

$$\frac{\partial R_t}{\partial c_t} = \frac{v''(c_t)}{v'(c_t)}\ R_t\ . \tag{13}$$

$$\frac{\partial R_t}{\partial c_\tau} = \frac{v''(c_\tau)}{v'(c_\tau)} - \frac{u''(c_\tau)}{u'(c_\tau)}\ R_t\ , \tag{14}$$

for $2 \leq \tau \leq t$.

The crucial step in relating this to the rate of impatience is to observe that

$$\rho(c) = \frac{v'(c)}{u'(c)} - 1\ , \text{ so}$$

(15) $$\rho'(c) = \frac{u'(c)v''(c) - v'(c)u''(c)}{[u'(c)]^2}$$

$$= \frac{v'(c)}{u'(c)} \left[\frac{v''(c)}{v'(c)} - \frac{u''(c)}{u'(c)}\right]$$

$$= [1 + \rho(c)] \left[\frac{v''(c)}{v'(c)} - \frac{u''(c)}{u'(c)}\right]$$

Thus equation (14) has the more revealing form

(16) $$\frac{\partial R_t}{\partial c_\tau} = \frac{\rho'(c_\tau)}{1 + \quad (c\ )} R_t \quad .$$

The diminishing marginal rate of substitution requirement is

(17) $$\sum_{t=1}^{T} \sum_{\tau=1}^{T} a_t a_\tau \frac{\partial R_t}{\partial c_\tau} < 0 \quad , \quad \text{or}$$

$$-a_1 \frac{u''(c_1)}{u'(c_1)} \sum_{t=2}^{T} a_t R_t + \sum_{t=2}^{T} a_t^2 \frac{v''(c_t)}{v'(c_t)} R_t$$

$$+ \sum_{t=3}^{T} a_t R_t \sum_{\tau=2}^{t-1} a_\tau \frac{\rho'(c_\tau)}{1 + \rho(c_\tau)} < 0 \ ,$$

whenever $\sum_{t=1}^{T} a_t R_t = 0$

Now from the tangent plane restriction,

$$a_1 = - \sum_{t=2}^{T} a_t R_t \quad ,$$

so the first term of the inequality (17) can be written

(18) $$\frac{u''(c_1)}{u'(c_1)} \left[\sum_{t=2}^{T} a_t R_t \right]^2$$

which is negative by hypothesis. Similarly, the second term is also negative. It remains to show that there is a restriction on

$$\frac{\rho'(c)}{1 + \rho(c)}$$

which will guarantee that the third term cannot be large enough to make the whole expression zero or positive. For a particular choice of the horizon T, it is easy to find such a restriction, since the ratio of the last term in (17) to the sum of its first two terms is bounded; if ϵ is the reciprocal of this bound, we can require

(19) $$\frac{|\rho'(c)|}{1 + \rho(c)} < \epsilon \ .$$

Such a bound can be derived by considering the problem of maximizing

(20) $$\sum_{t=3}^{T} a_t R_t \sum_{\tau=2}^{t-1} a_\tau$$

subject to any arbitrary normalization of the a_t's; it is easiest to take

(21) $$\sum_{t=2}^{T} a_t^2 R_t = 1 \ .$$

Now let β be less than or equal to the smallest value attainable by $\left|\frac{v''(c)}{v'(c)}\right|$. Then the second term,

$$\sum_{t=2}^{T} a_t^2 \frac{v''(c_t)}{v'(c_t)} R_t$$

is less than $-\beta$, by the normalization. Finally, since (21) bounds every a_t, and R_t is bounded, the double sum (20) is bounded -- there exists a B(T) such that

$$(22) \qquad \sum_{t=3}^{T} a_t R_t \sum_{\tau=2}^{t-1} a_\tau < B(T) \quad .$$

Thus, the condition

$$(23) \qquad \frac{|\rho'(c)|}{1+\rho'(c)} < \frac{\beta}{B(T)}$$

guarantees that the marginal rate of substitution is always declining.

The interesting and important question, however, is whether or not there is a positive bound on $\rho'(c)$ which is independent of the horizon T. To show that there is such a bound requires a stronger argument than the one presented above, since nothing so far rules out the undesirable possibility

$$(24) \qquad \lim_{T \to \infty} B(T) = \infty$$

This might prevent flexible impatience for the case of an infinitely distant horizon.

It is possible to show that the basic expression (17) is negative both for the case of persistent impatience, $\lim_{t \to \infty} R_t = 0$, and for the alternative cases $\lim_{t \to \infty} R_t > 0$ or R_t unbounded. In view of the conclusions of (3), however, only the case of persistent impatience will be considered. We will show that under conditions of persistent impatience there is a bound on

(25) $$\sum_{t=3}^{T} a_t R_t \sum_{\tau=2}^{t-1} a_\tau$$

which is independent of T; in other words, the expression (25) cannot be made indefinitely large by choosing a large enough T.

The problem of maximizing (25) subject to the constraint (21) can be stated as one of finding a stationary point of the Langrangean expression

(26) $$L = \sum_{t=3}^{T} a_t R_t \sum_{\tau=2}^{t-1} a_\tau + \frac{\lambda}{2} [1 - \sum_{t=2}^{T} a_t^2 R_t] \quad .$$

Setting the first derivative of L with respect to each a_s equal to zero, we have

(27) $$\lambda R_s a_s = \sum_{t=s+1}^{T} a_t R_t + R_s \sum_{\tau=2}^{s-1} a_\tau \quad .$$

Now we let

(28) $$x_s = \frac{1}{R_s} \sum_{t=s}^{T} a_t R_t \qquad \text{and}$$

(29) $$y_s = \sum_{\tau=2}^{s-1} a_\tau \quad .$$

Then the conditions for the maximum can be written as a system of three linear difference equations:

(30) $$a_s = \frac{1}{\lambda + 1} [(1 - \frac{R_{s-1}}{R_s}) a_{s-1} + \frac{R_{s-1}}{R_s} x_{s-1} + y_{s-1}]$$

(31) $$x_s = - \frac{R_{s-1}}{R_s} a_{s-1} + \frac{R_{s-1}}{R_s} x_{s-1}$$

(32) $$y_s = a_{s-1} + y_{s-1} \quad ,$$

for $s \geqq 3$; $a_2 = \frac{x_2}{1 + \lambda}$ and $y_2 = 0$.

The maximum trajectory is found by locating an x_2 which meets the boundary condition $x_T = 0$ and a λ such that the constraint $\Sigma a_t^2 R_t = 1$ is met. As T becomes large, the coefficients of the system (30), (31), and (32) approach constants, since $\frac{R_{t-1}}{R_t} = s(c_{t-1}, c_t)$ and $\lim_{t \to \infty} s(c_{t-1}, c_t) = 1 + \rho(\bar{c})$, where $\bar{c} = \lim c_t$. Then for large s, the solution approaches the solution to a linear system with constant coefficients; from the properties of such solutions, we know that

$$(33) \qquad \lim_{s \to \infty} \frac{a_{s+1}}{a_s}$$

exists if the characteristic roots of the system are real and positive. It is straightforward but tedious to show that if the roots of this system are not real and positive, then expression (25) is uniformly bounded. Thus we may assume

$$(34) \qquad \lim_{s \to \infty} \frac{a_{s+1}}{a_s} = \alpha .$$

Now from the constraint, (21), we can see that

$$(35) \qquad \frac{\alpha^2}{1 + \rho} \leqq 1,$$

since otherwise (21) would diverge; we conclude that

$\alpha \leqq \sqrt{1 + \rho}$; thus $\frac{\alpha}{1 + \rho} \leqq \frac{1}{\sqrt{1 + \rho}}$ and finally $\frac{\alpha}{1 + \rho} < 1$.

Next we rewrite the maximand as

$$(36) \qquad \sum_{\tau=2}^{T} a_\tau \sum_{t=\tau+1}^{T} a_t R_t \quad .$$

Now the inner summation is eventually close to a convergent geometric series, and $a_t R_t > 0$ for all τ, so that there exists a constant K such that

(37) $$\sum_{t=\tau+1}^{T} a_t R_t \leqq K\, a_\tau R_\tau \,.$$

But then by multiplying by a_τ and summing, we have

(38) $$\sum_{\tau=2}^{T} a_\tau \sum_{t=\tau+1}^{T} a_t R_t \leqq K \sum_{\tau=2}^{T} a_\tau^2 R_\tau$$
$$\leqq K$$

Thus $B(T) \leqq K$, and a uniform restriction on $\rho(c)$ can be written

(39) $$\frac{|\rho'(c)|}{1 + \rho(c)} < \frac{\beta}{K}$$

The results of this section may be stated in a

Theorem on variable rates of impatience

There exists a function $s(c_t, c_{t+1})$ giving the marginal rate of substitution between c_t and c_{t+1} for which the implied rate of substitution between c_1 and c_t has a general diminishing marginal rate of substitution and for which the rate of impatience is not a constant. Furthermore, there is no restriction on the sign of $\rho'(c)$.

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Chapter 3

THE DYNAMIC EFFECTS OF FISCAL POLICY IN A SIMPLE ECONOMY

Recent experience with the manipulation of fiscal instruments for the purpose of stabilizing the economy suggests that economists' understanding of the dynamic effects of fiscal changes is somewhat limited. In particular, economic models have failed to take account of the ways by which the economy anticipates policy changes which are announced some time before they take effect. The case in point is the recent temporary suspension of the investment tax credit: the expectation that the credit would be restored, either as scheduled or earlier (as apparently will be the case), may have strongly inhibited investment in equipment during the period of the suspension. None of the estimates prepared in September 1966 by applying econometric models took account of this problem; this includes (5), although it was the most bearish.

The purpose of this essay is not to attempt the difficult task of formulating a model which would give better answers to the questions of fiscal policy makers. Rather, it examines some dynamic fiscal problems in a simple, unrealistic economic model whose distinguishing feature is that all the participants in the economy are assumed to make plans by looking into the future. The value of this exercise, I think, is that it reveals the important differences between the behavior of a model in which consumers and investors correctly predict and take account of the future and that of models in which expectations are based only on past experience and behavior is determined by arbitrary rules which do not look into the future. Since these differences are so large, it appears that the old models used for fiscal policy based on naive or adaptive expectations may need to be revised in a fundamental way. Furthermore, fiscal policy changes seem to be the most natural source of large perturbations in a competitive equilibrium model.

Consequently, a careful analysis of fiscal problems is one of the best ways to demonstrate the properties of this kind of competitive economy. Among the most important of these properties are stability and fulfillment of all expectations even when there are large discontinuous changes in the external forces acting on the economy.

The model which I propose to consider is described in (2) and is similar in some ways to ones employed by Phelps in connection with problems of static fiscal policy and by Sidrauski in connection with monetary theory (6,7); its formal properties are very much the same as those of the popular one sector model of optimal economic growth. Essentially it assumes that consumers decide on today's consumption by formulating a future consumption plan both for themselves and their heirs, knowing future interest rates and wages. Investors always equate the net marginal product of capital to the interest rate and pay wage earners their marginal product. There is a unique competitive equilibrium in which these two rules are consistent -- this essay describes first the changes in the steady state and then the full dynamic equilibrium for several kinds of changes in fiscal policy.

1. Steady-state Tax Effects

The steady state in this model is found at the intersection of the impatience curve, $\rho(c)$ and the net output curve y(r) (c is consumption per person and r is the interest rate; the single output of the economy is taken as numéraire). This is illustrated in Figure 1. At the steady state point (r*,c*) consumption is constant because the interest rate equals the rate of impatience. The capital stock is constant because all net output is devoted to consumption; hence the interest rate is also constant.

I will examine the effects of five kinds of taxes in this economy: a fixed levy or recurring lump-sum tax, an interest tax, an income tax, a consumption tax, and an investment tax credit or negative tax on investment. By assuming that all government expenditures are purely wasteful I will avoid the important question of the influence of government expenditures on production or consumption.

First, a fixed levy, paid once a year, simply reduces the net output available for consumption at a given interest rate, as shown in Figure 2. If $\rho(c)$ is constant consumption after the tax is less than consumption before the tax by the amount of the tax -- gross output is unchanged.

Second, we consider the more complicated case of an interest tax. The tax changes the relation between the interest rate and the capital stock (interest rates will always be measured after taxes). As a concrete example, one might think of a corporate income tax in an economy where all production is carried out by corporations which borrow only by selling shares; the interest rate r would be measured by adding dividends to capital gains and dividing by the share price.

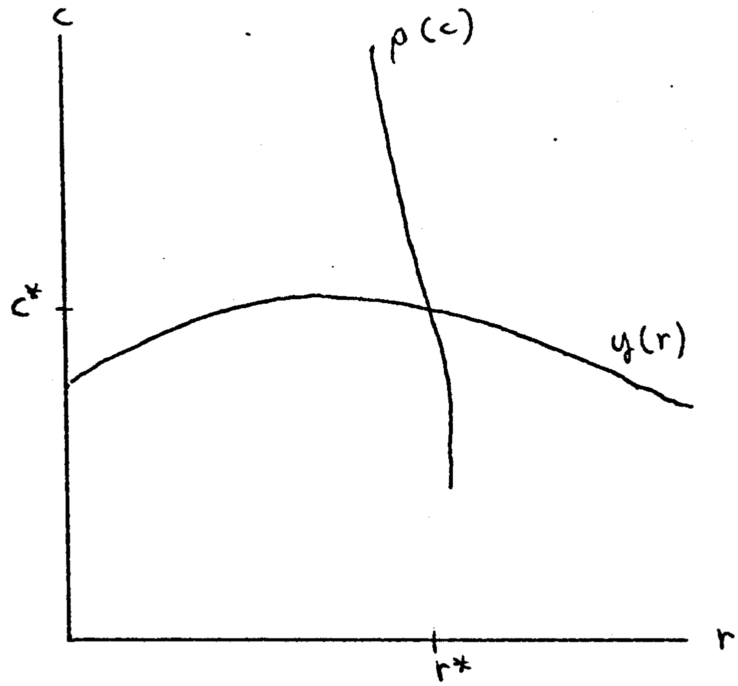

FIGURE 1

The steady state

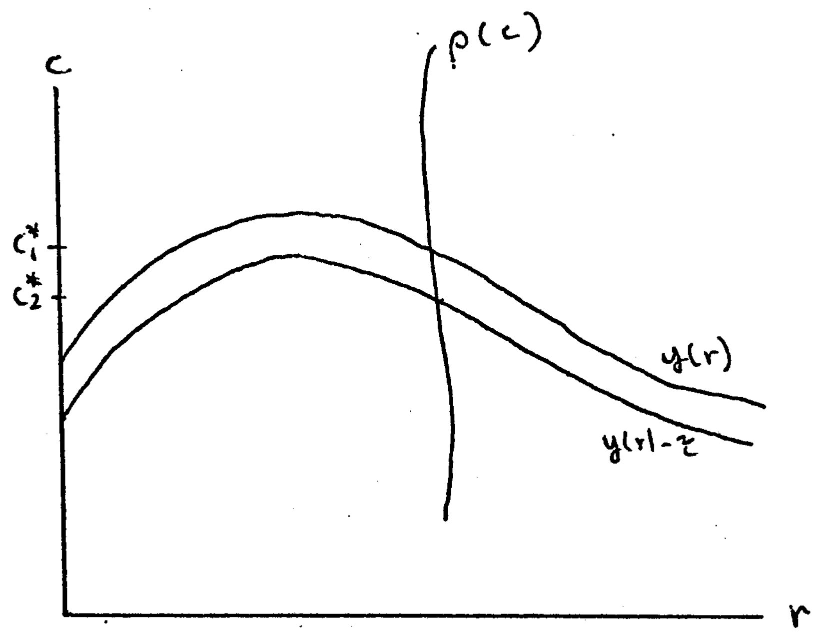

FIGURE 2

Fixed levy

Before the tax is imposed r is related to capital, k, by

(1) $$r = f'(k) - \delta .$$

If the tax laws provide for true economic depreciation deductions, the interest rate when a tax at rate τ is imposed is

(2) $$r = (1 - \tau)(f'(k) - \delta) .$$

The analysis could easily be extended along the lines of (4) if depreciation deductions are allowed at a rate which is faster or slower than the true rate.

Equations (1) and (2) show that the new net output curve $y_2(r)$ lies to the left of the old curve $y_1(r)$. The steady state consumption curve $c_2(r)$ is $y_2(r)$ less the tax yield:

(3) $$c_2(r) = y_2(r) - \tau rk ,$$

as shown in Figure 3. Note that the interest tax reduces net output before taxes: c_2^* is less than c_1^* by an amount larger than the tax yield. A sizeable element of deadweight loss results from an interest tax.

This diagram answers the important question of the long run shifting of the corporate income tax in this kind of economy: if $\rho(c)$ is constant, the tax is shifted 100 per cent (the rate of profit is unaffected by imposing a tax on profit); if $\frac{d\rho}{dc} > 0$, shifting is between 0 and 100 per cent, and if $\frac{d\rho}{dc} < 0$, shifting is greater than 100 per cent.

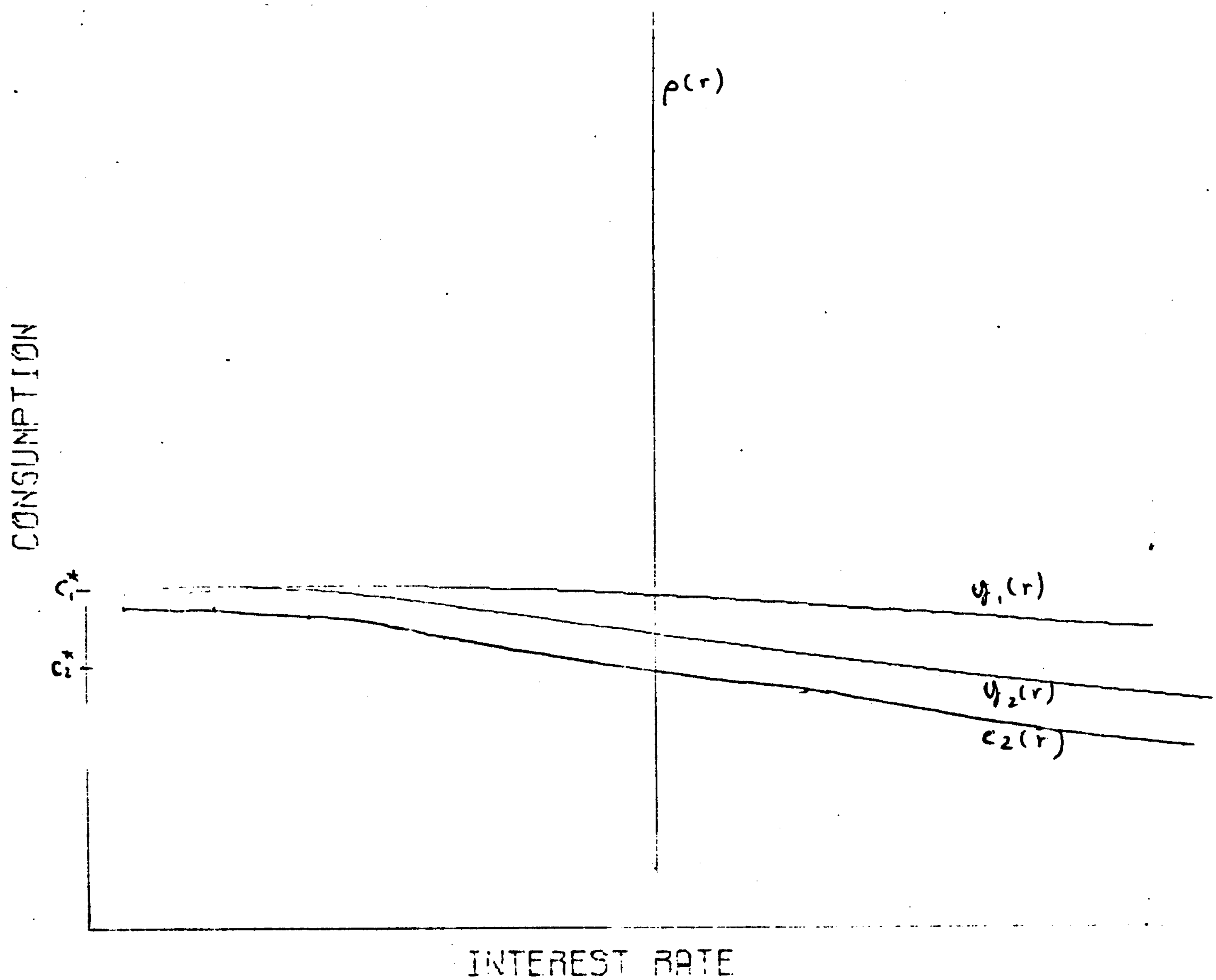

FIGURE 3

Interest tax

Third, an income tax is a combination of the interest tax just discussed and a wage tax. Since labor is supplied inelastically in this model, a wage tax is non-distorting in the same way as a fixed levy. The net output curve $y_2(r)$ is pushed to the left by the interest tax while the consumption curve $c_2(r)$ lies below $y_2(r)$ by the total amount of the tax:

$$c_2(r) = (1 - \tau)y_2(r) \tag{4}$$

This is shown in Figure 4.

Fourth, we consider a consumption tax. In a comparison of alternative steady states a consumption tax is very much like a fixed levy or a wage tax. The net output curve $y(r)$ is unchanged; this is illustrated in Figure 5.

Finally, an investment credit of the kind in force in the United States for 1962 and 1963 reduces the rent on capital in proportion to the amount of the credit:

$$f'(k) = (1 - v)(r + \delta) \quad , \tag{6}$$

where v is the investment credit rate, .07 for 1962 and 1963. The effect of the credit is to shift the net output curve in the following way: $y_2(r) = y_1((1 - v)r - v\delta)$. The consumption curve lies above $y_2(r)$ by the amount of the credit for replacement investment:

$$c_2(r) = y_2(r) + v\delta k \quad . \tag{7}$$

These curves are shown in Figure 6. The investment credit raises steady-state consumption by more than the amount of the credit. In its long-run effects it is much the same as a negative interest tax, but as I shall show, its short run effects around the time of announcement are quite different.

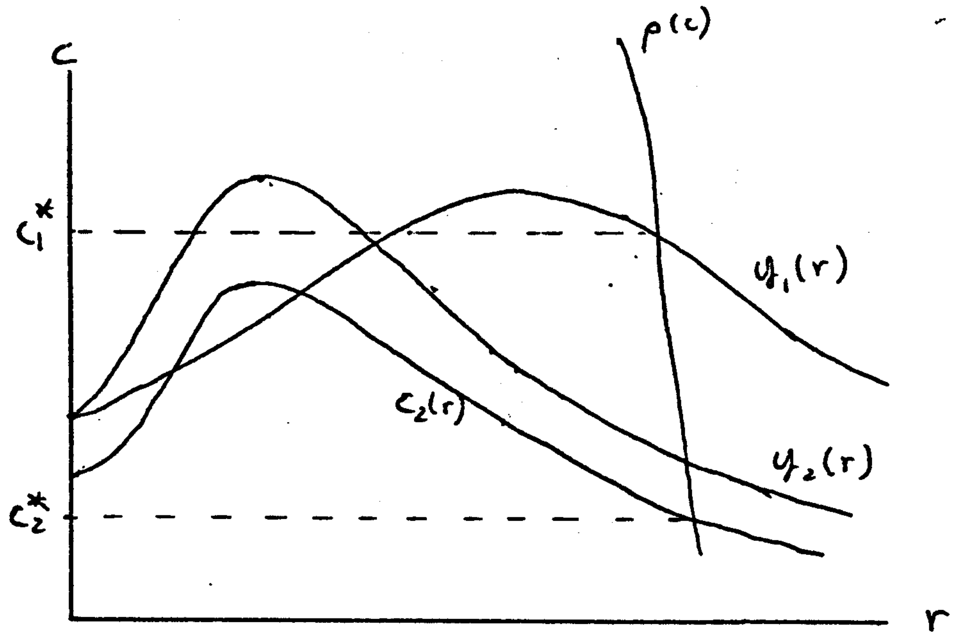

FIGURE 4

Income tax

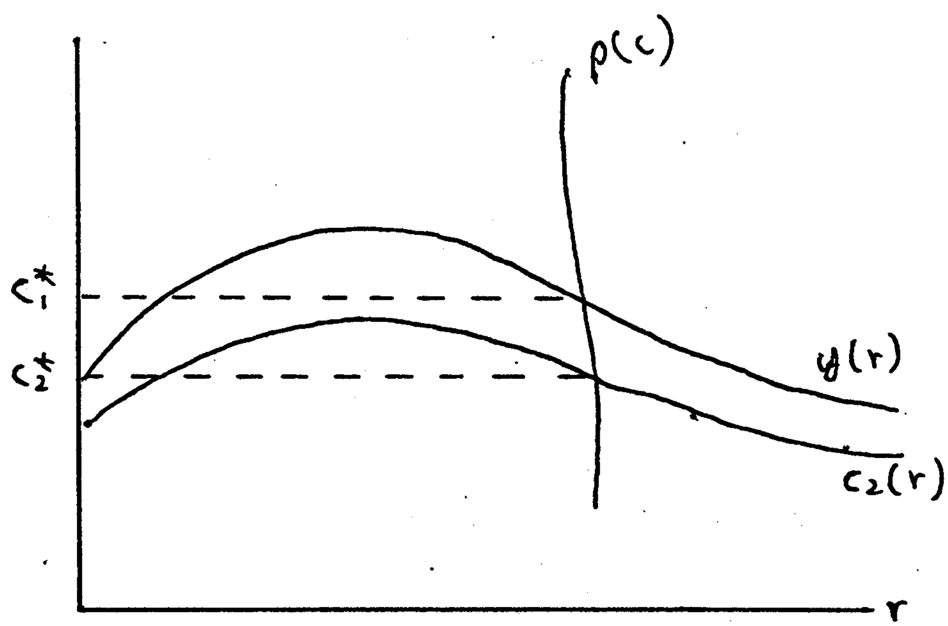

FIGURE 5

Consumption tax

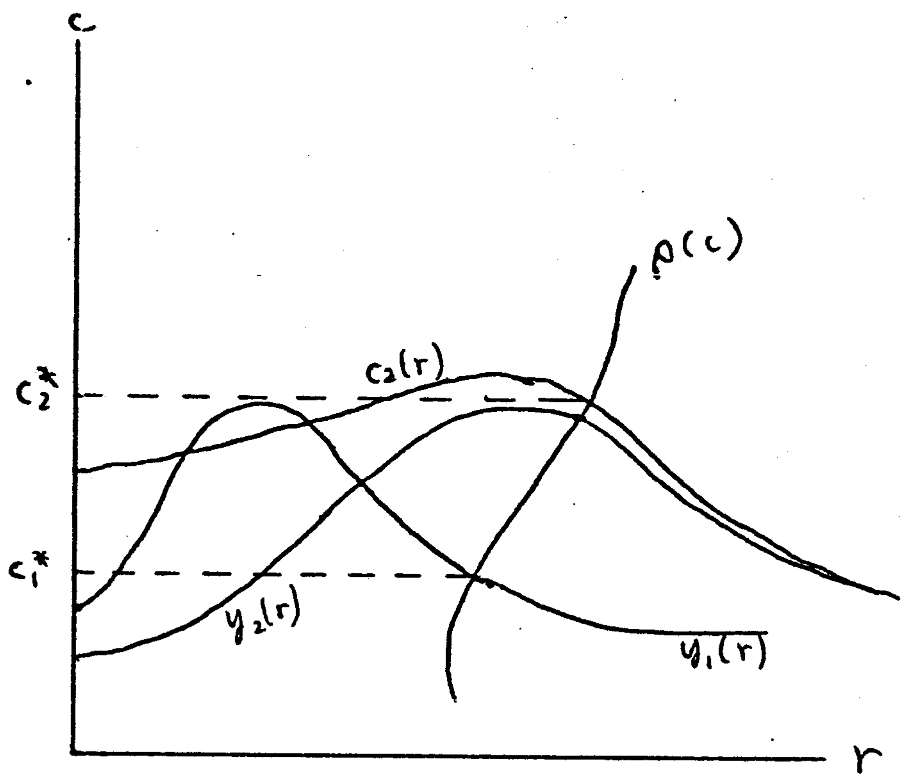

FIGURE 6

Investment tax credit

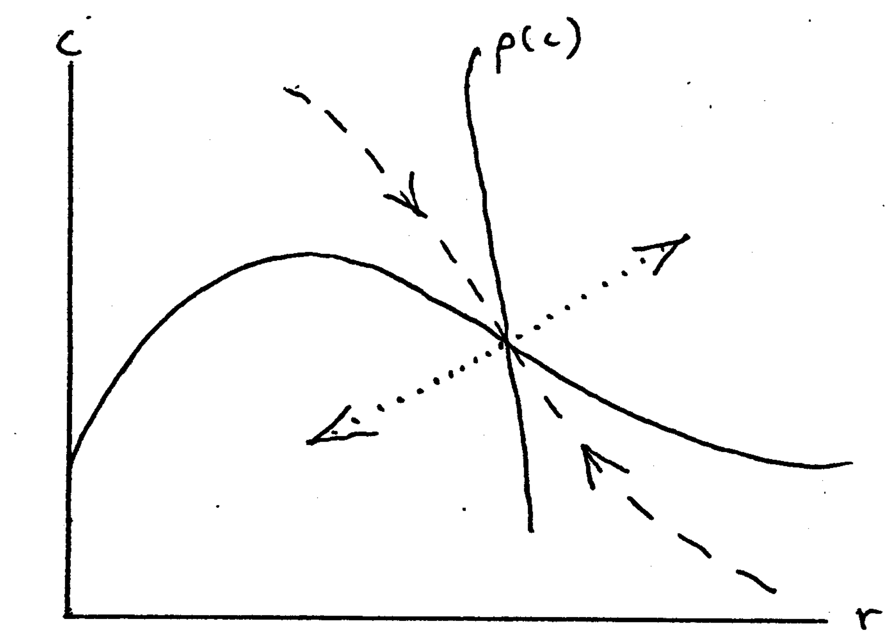

FIGURE 7

General phase diagram

2. Dynamic effects before and after tax changes

This section presents derivations of the competitive equilibrium trajectories which would prevail in an economy during the transition between the steady state without any taxes and that with various of the taxes discussed in the first section. In all cases the tax changes are assumed to be instantaneous jumps to a positive rate which remains constant for the rest of time. Further, except for the fixed levy, only the differential effects of each kind of tax will be considered -- the proceeds are assumed to be distributed as transfer payments. The method of analysis could handle more complicated fiscal policies, but it would be difficult to present the results as graphs.

In all cases my method will be to reinterpret the graphs of the first section as phase diagrams in the (r,c) space and to find the unique trajectory which goes from the before-tax steady state (r_1^*,c_1^*) to the after-tax steady state (r_2^*,c_2^*) . This trajectory will trace out the competitive equilibrium of the economy from the infinite past to the infinite future.

The general phase diagram for this model has consumption decreasing over time if the interest rate is less than the rate of impatience $\rho(c)$ and increasing otherwise. Similarly, if consumption is greater than net output y(r), the interest rate is increasing and vice versa. Thus the diagram has the following form as shown in Figure 7. The two dashed arrows are the unique infinitely long trajectories going from any initial interest rate to the steady state interest and consumption point (r*,c*). The dotted arrows show the locus of points which the economy can reach after starting at (r^*,c^*) in the infinite past; they correspond to the dashed arrows of the same system with time reversed. I will call the dashed arrows the converging arms of the system and the dotted arrows the emerging arms. The competitive equilibrium trajectory in an economy with a tax imposed at time T follows the emerging arms of the before-tax system before T and the converging arms of the after-tax system after T. In some cases the trajectory is continuous at T while in others the imposition of the tax causes a discontinuity in r or c.

a. Fixed levy.

The first case to consider once again is the fixed levy. In Figure 8 I have redrawn Figure 2 to include the appropriate emerging and converging arms. The economy follows the upper heavy arrow before the time T in anticipation of the tax; toward the end of the period, consumption begins to decline as the economy accumulates an extra buffer of capital. When the tax is imposed, it is first paid entirely by drawing down the buffer. However, consumption continues to fall until eventually the economy comes indefinitely close to the new steady state (r_2^*,c_2^*). The interest rate and consumption trajectories have the time profiles in the period around T shown in Figure 9. Notice that the time path of consumption is perfectly smooth -- investment rather than consumption absorbs the shock of the imposition of the tax. This is not the case with some of the other taxes, however.

b. Interest tax

The discussion of the dynamic effects of an interest tax must begin with a digression on the implications of the irreversibility of investment.

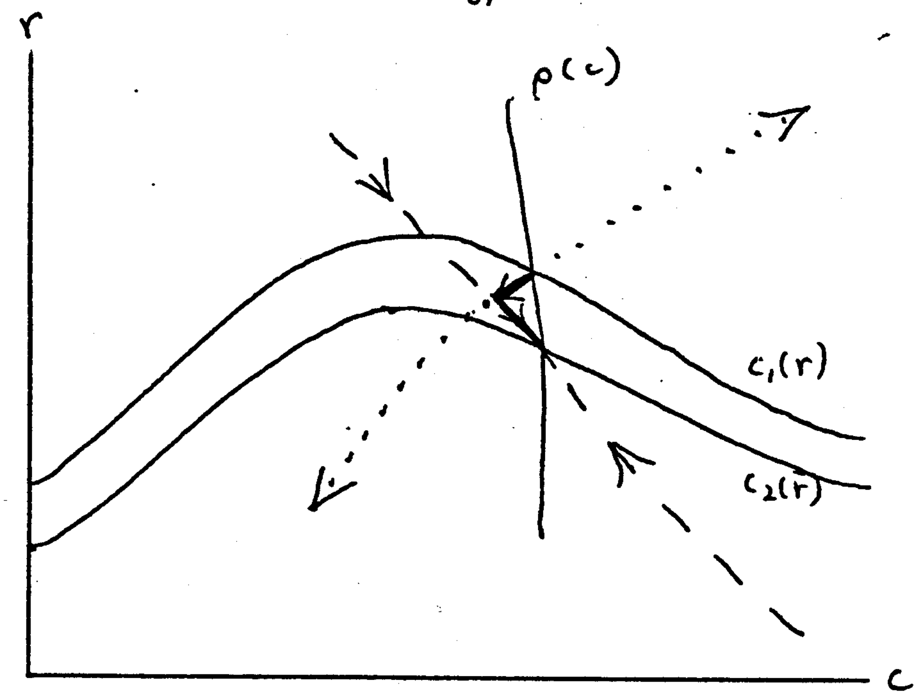

FIGURE 8

Trajectories for the fixed levy

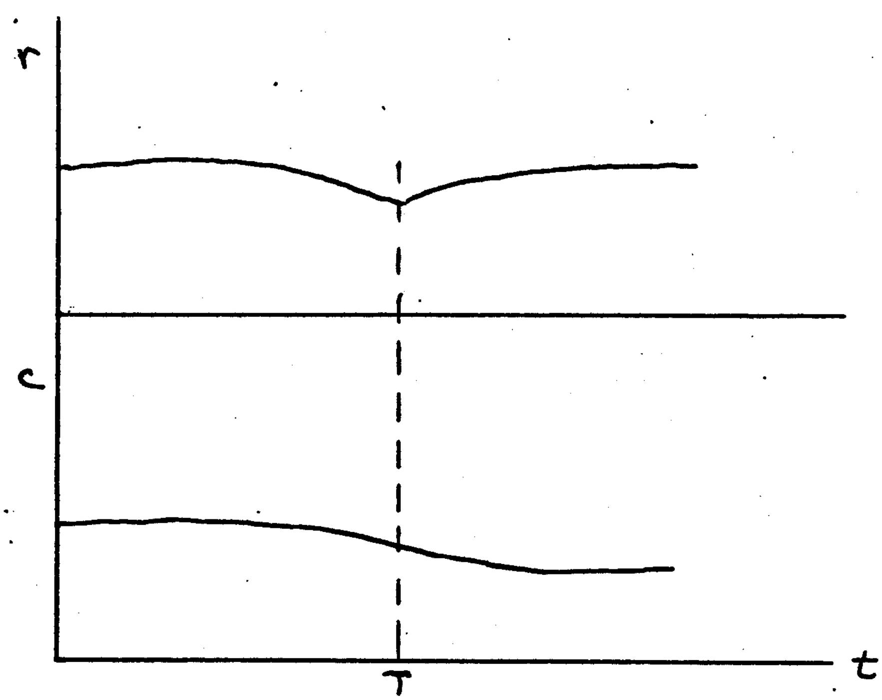

FIGURE 9

Interest rate and consumption as functions of time for the fixed levy

This is one realistic assumption which the model is capable of incorporating and whose implications are in themselves fairly interesting.

The first step in understanding the operation of the economy when investment is irreversible is to examine the model with reversible investment in the capital-consumption phase plane rather than the interest rate-consumption phase plane. The change is purely formal since the interest rate is a monotonic function of the capital stock:

(8) $$r = f'(k) - \delta \ .$$

The new system is shown in Figure 10. The trajectories drawn with dots and dashes are the same as those shown in Figure 7.

Points in this plane above the production function curve f(k) involve negative gross investment. If these points are ruled out by the hypothesis of irreversibility, all trajectories must lie on or below f(k). In that case the infinitely long arms will have the form shown in Figure 11. The parts of trajectories which are forced to lie on f(k) are called blocked segments in Arrow's terminology (1).

Although Figure 11 gives a correct portrayal of the behavior of consumption and investment with irreversibility, it obscures the way that the competitive system behaves in order to bring this about. In particular, it is not obvious that the kinked trajectories of Figure 11 can be supported by a competitive equilibrium interest rate. To see that they can be supported it is necessary to return to the (r,c) phase plane.

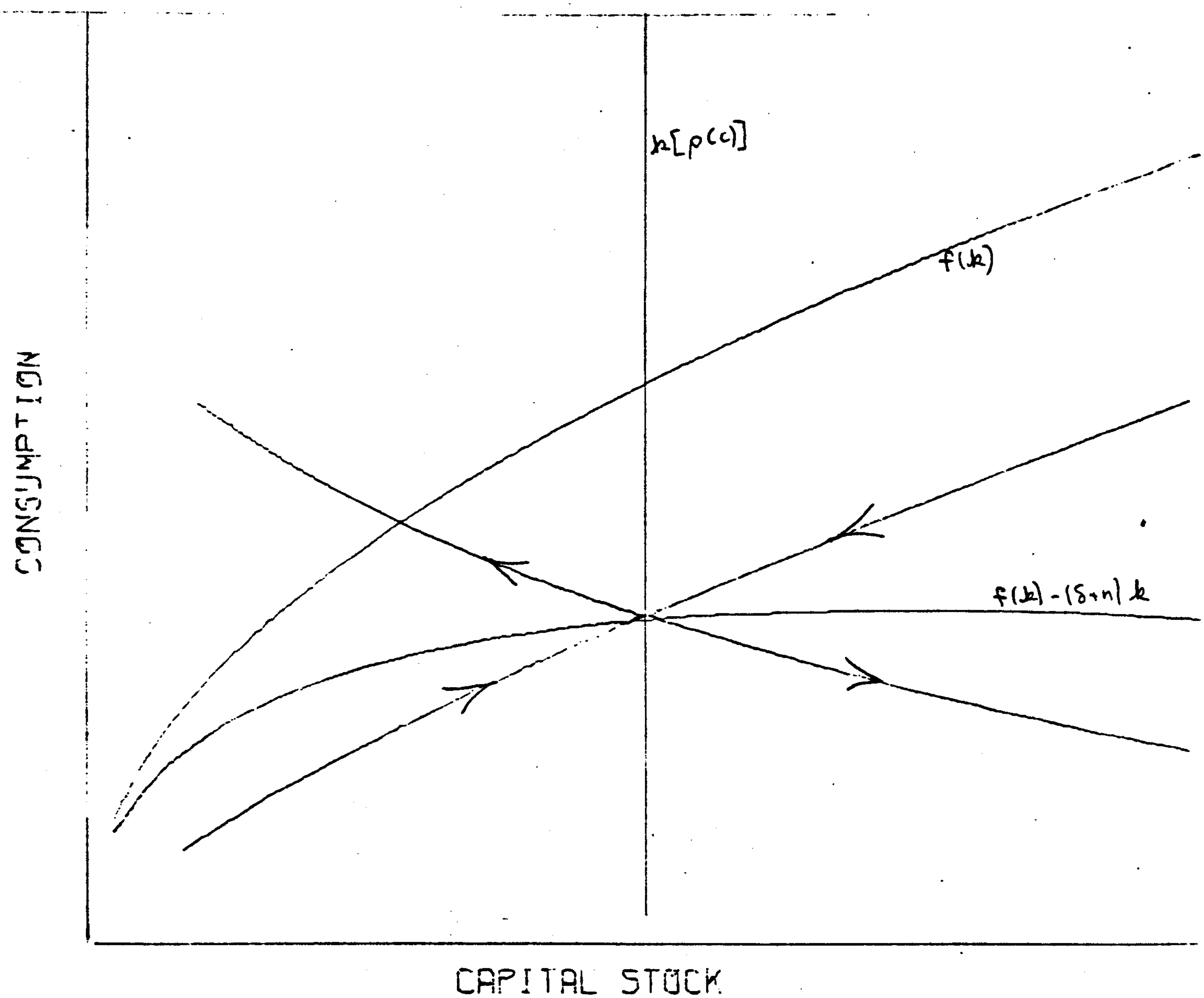

FIGURE 10

The basic model in the (k,c) plane

The old system

$$\frac{dc}{dt} = g(c,r) \tag{9}$$

$$r = f'(k) - \delta$$

$$\frac{dk}{dt} = f(k) - (\delta + n)k - c$$

cannot possibly govern the economy if investment is irreversible, since some of its trajectories enter the forbidden region where $c > f(k(r))$. Which equation fails on blocked segments? The answer is that if capital cannot be consumed after it is installed, it is not necessarily a perfect substitute for consumption goods; hence its price in terms of consumption goods may be less than one during periods when gross investment is zero. If the price of capital, p, is less than one, then the old relation between the interest rate and the capital stock,

$$f'(k) = r + \delta, \tag{10}$$

becomes the more general equation of rent which takes account of capital gains and losses:

$$f'(k) = p(r + \delta) - \frac{dp}{dt} \quad . \tag{11}$$

On blocked segments the price of capital is less than one because capital is redundant. Trajectories which move toward a segment with positive investment have $\frac{dp}{dt} > 0$ so that $p = 1$ exactly at the moment investment resumes.

On a blocked interval, the capital stock declines exponentially from its value when the interval was entered:

(12) $$k(t) = k_o e^{-(\delta+n)(t-t_o)} \quad ,$$

where $k_o = k(t_o)$, the initial capital stock. Consumption is the output from this stock:

(13) $$c(t) = f(k(t)) \quad .$$

The rate of change of consumption is

(14) $$\frac{dc}{dt} = f'(k)\frac{dk}{dt}$$

$$= -f'(k)(n + \delta)k \quad .$$

But the rule for consumption behavior requires

(15) $$\frac{dc}{dt} = g(c,r) \quad .$$

Thus in a blocked interval the interest rate is found by solving

(16) $$g(f(k),r) = -f'(k)(n + \delta)k.$$

Note that this restricts the blocked segments of all trajectories to a single line in the (r,c) phase plane -- this line is defined by equation (16) and corresponds to the f(k) curve in Figure 11.* Thus the (r,c) phase diagram for the system with irreversible investment has the form shown in

*It is possible to show that the blocked trajectory always passes to the right of the intersection of the upper converging arm of the system and the gross output curve.

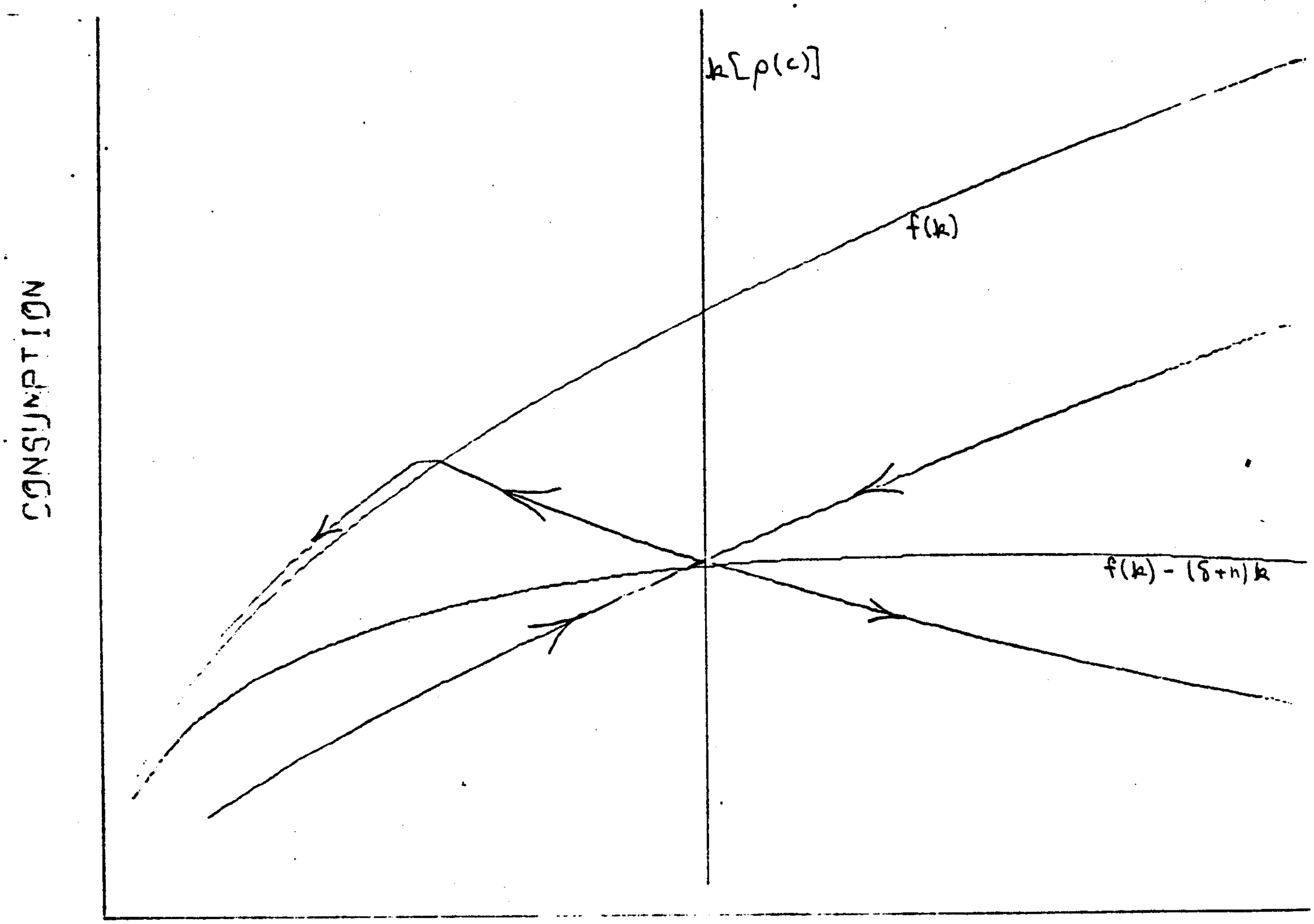

FIGURE 11

The (k,c) plane with irreversible investment

Figure 12. Along the blocked segment of a trajectory the price of capital p(t) can be found by integrating the rent equation (11) and applying the boundary condition p(t) = 1 at the time that the trajectory enters or leaves the segment of positive investment. While this price does not play an important part in the analysis of a one-sector model, it would be critical in a model with several kinds of capital. See (3) in this regard.

The upper converging arm in Figure 12 begins with redundant capital with price p less than one. The price of capital rises along the blocked segment, exactly compensating the owners of capital for the higher interest rate caused by the blocking. Just as the price reaches one, the interest rate jumps down to its free value, $r = f'(k) - \delta$. Similarly, along the upper emerging arm consumption increases and investment gradually decreases until it reaches zero just at the gross output curve f(k(r)). Then the interest rate jumps down to compensate holders of capital for the capital losses they sustain along the blocked trajectory. The end of the world must come just as the price of capital becomes zero -- otherwise this negative speculative boom could not be consistent with competitive equilibrium. The point on the blocked trajectory where p = 0 will be above the horizontal axis; one of the properties of economies with irreversible investment is that they will have a positive capital stock with zero value at the end of the world.

With this preparation it is possible to discuss taxes which cause serious jolts in the economy around the time of their introduction.

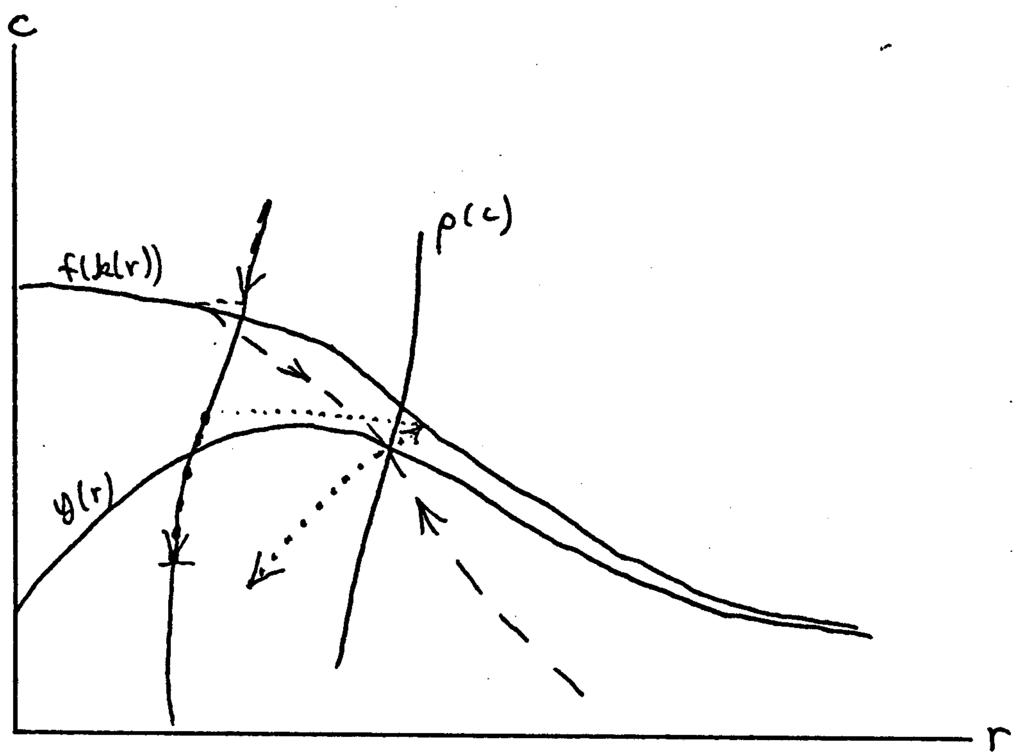

FIGURE 12

The (r,c) phase plane with irreversible investment

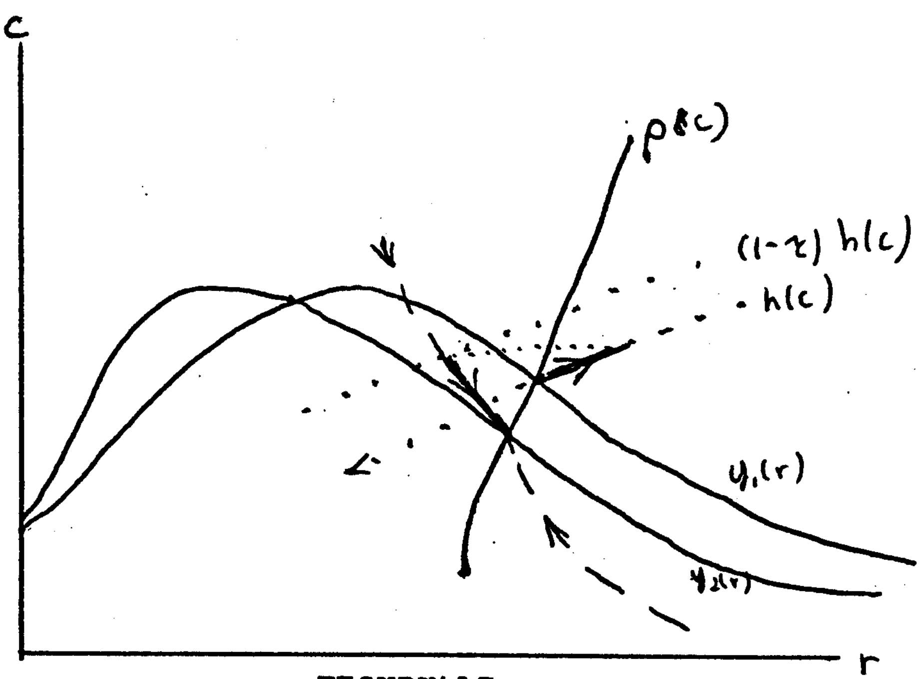

FIGURE 13

Interest tax, reversible investment

The first of these is the interest tax, whose immediate effect is to reduce the interest rate in proportion to the tax. In an economy with reversible investment, the locus of (r,c) points at which the economy could find itself just after the imposition of an interest tax is a line proportionally to the left of the emerging arms of the pre-tax system. If the function defined by the emerging arms is $r = h(c)$, then the locus is $r = (1 - \tau)h(c)$. The point where this locus crosses the upper converging arm of the after-tax system is the point where the competitive equilibrium trajectory joins the converging arm at time T. This trajectory is shown in Figure 13. Before the tax is imposed, consumption increases, drawing down the capital stock. After the tax the interest rate drops and the trend of consumption referses abruptly, gradually converging to the new lower steady state.

If investment is irreversible, the kind of trajectory described above may still hold, particularly if the new tax is small; if the trajectory for the reversible case never enters the region of negative gross investment, it is also the competitive equilibrium trajectory for the irreversible case. On the other hand, for a sufficiently large tax increase, the irreversibility constraint will be effective, as shown in Figure 14. In this case, investment reaches zero before the tax is imposed, and the interest rate drops discontinuously to its blocked value sometime before T. The tax increase occurs while investment is blocked and has no immediate effect -- in a blocked interval capital is a fixed factor so that a tax on it is like a tax on land.

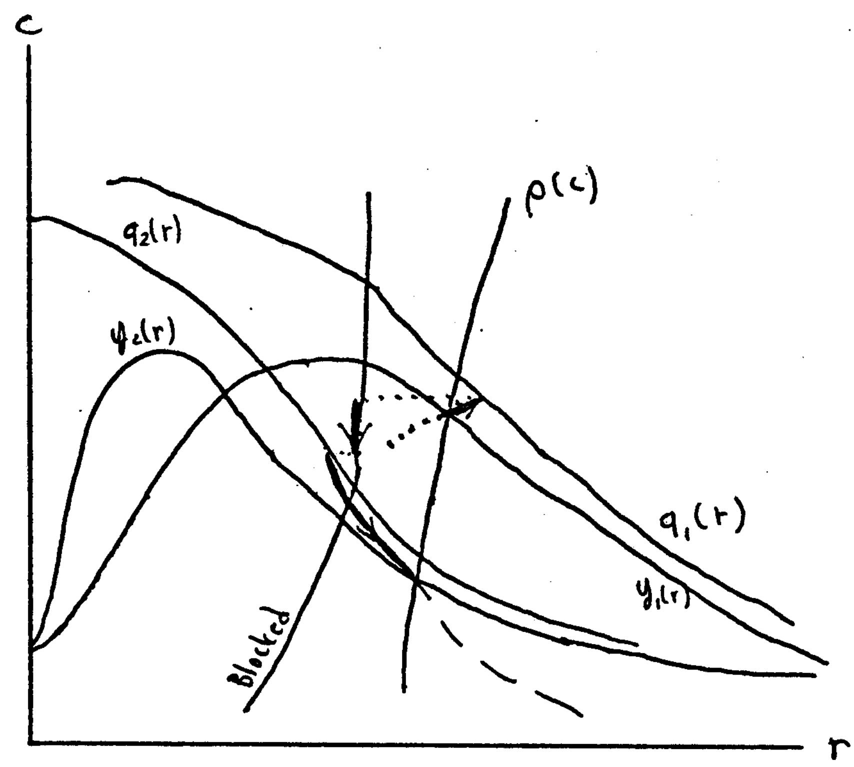

FIGURE 14

Interest tax, irreversible investment

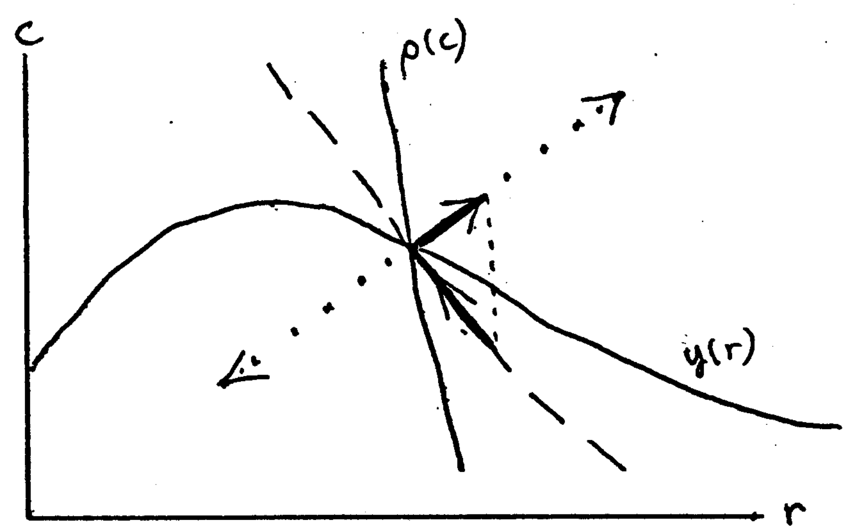

FIGURE 15

Consumption tax increase

Some time after the tax, investment resumes and the interest rate drops again. Finally the economy moves gradually to its new steady state.

c. Consumption tax

A consumption tax has no long-run substitution effect, since as long as the rate is constant, it is equivalent to a fixed levy. However, at the time the tax rate is changed it has a very important substitution effect. With a consumption tax, the consumer's effective discount function is

$$\frac{e^{-R(0,t)}}{1 - \tau(t)} ;$$

the optimal consumption trajectory has marginal utility proportional to this function at each point in time. A discontinuous increase in $\tau(t)$ causes a discontinuous decrease in consumption. The effect of the tax change is the same as that of an infinitely short period of time with an interest rate of minus infinity. Accordingly, the phase diagram will have a discontinuity in the (r,c) trajectory at the moment of the tax increase as shown in Figure 15.

It is possible that the desire to consume in anticipation of the tax increase might be so strong that the economy would enter the blocked state with zero investment. Then the price of capital would begin a downward speculative movement which would be ended just as investment resumed at time T. In this region there would be a discontinuous increase in the price of capital p(t) at time T which would cause a compensating downward jump in the discount function $e^{-R(0,t)}$, since the equation of rent required the present value of capital, $p(t)e^{-R(0,t)}$, be continuous. Then the jump in

$$\frac{e^{-R(0,t)}}{1 - \tau(t)}$$

would be smaller, and in turn, the downward jump in the consumption trajectory would be smaller. Thus the limitation on consumption imposed by the irreversibility of investment causes the price system to cushion the shock in consumption. These events are shown in Figure 16.

d. Investment tax credit

The immediate effect of an investment tax credit is a discontinuous drop in the price of capital from 1 to 1-v, where v denotes the investment credit rate. In a partial equilibrium analysis in which the interest rate is assumed exogenous and finite, a finite capital gain or loss is not consistent with the equation of rent. The result is that investment is blocked for a period sufficiently long for the speculative decline in the price of capital to eliminate the capital loss exactly. For example, if an investment credit were allowed on one kind of machine in an economy with many kinds of capital, no investment in that kind of machine would take place for a few months or years before the credit took effect (or resumed), because it would be profitable to postpone investment until the credit was available. The same is not always true for the economy as a whole. Price adjustments may absorb some of the shock caused by an investment credit. First, a drop in the price of capital is consistent with competitive equilibrium with positive investment if it is accompanied

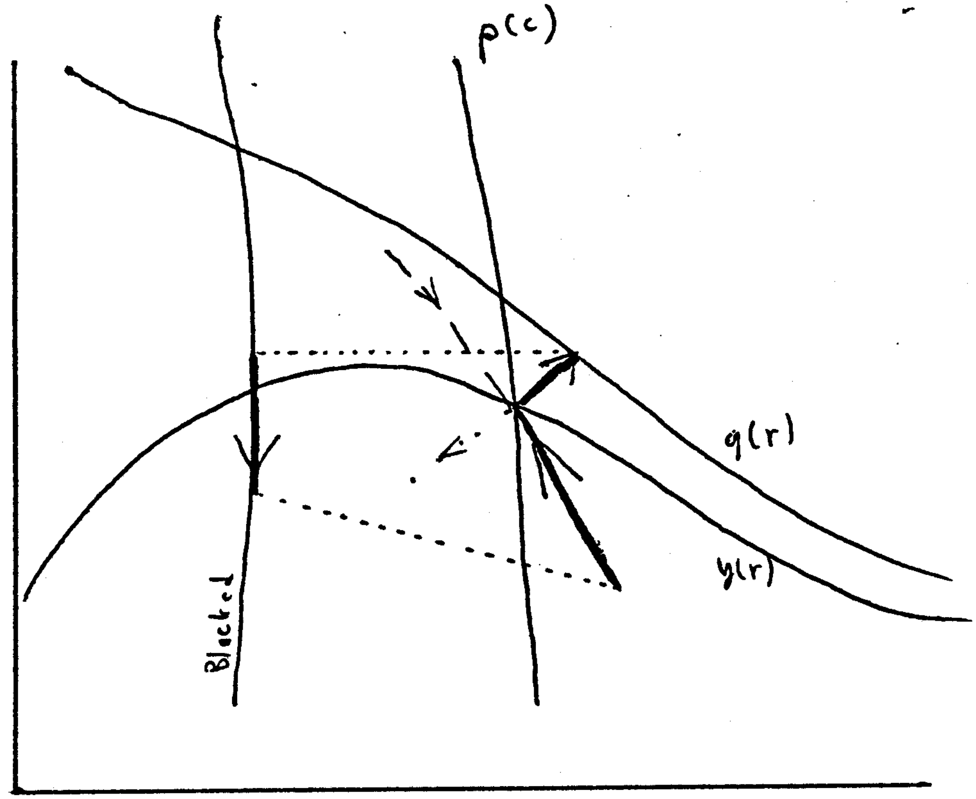

FIGURE 16

Consumption tax with irreversible investment

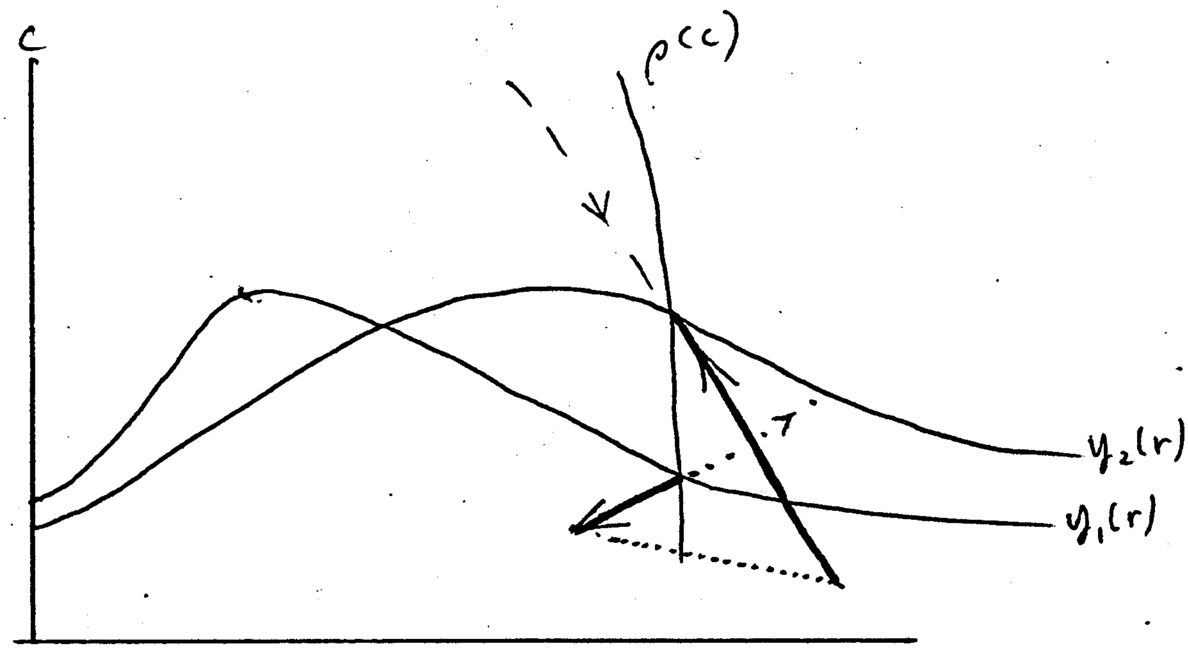

FIGURE 17

Investment tax credit without blocking

by an upward jump in the discount function (that is, a negative impulse in the interest rate). Second, the relation between the capital stock and the interest rate is altered by the investment credit in the way given in equation (6) -- this causes an upward jump in the interest rate. Figure 17 shows a competitive equilibrium trajectory which does not have a blocked segment. In this case investment begins before the tax credit is allowed, in anticipation of the larger capital stock which will be optimal after time T. It is also possible that the competitive equilibrium trajectory might follow the upper emerging arm of the pre-tax system, depending on the relative magnitudes of the jumps in consumption and the interest rate. The second kind of trajectory may have a blocked segment whose properties are similar to those discussed in connection with the consumption tax.

3. Conclusions

Once again I emphasize that I do not consider these results directly relevant to the kind of analysis of fiscal problems required, say, by the Treasury Department. The accomplishment of this essay, I think, is to introduce to fiscal economies the notion that the participants in an economy are concerned about the future; the defect of my method is that I am required to assume that everyone in the economy knows the future with certainty. Clearly the next step is a model in which while people care about the future, they do not know exactly what will happen to them then. But progress in this direction is unlikely to come very easily.

In order to make the point of this essay most forcefully it would be useful to compare the behavior of the economy described here to that of a similar economy which did not use information about the future in determining today's equilibrium. A natural initial choice for comparison would be Solow's one-sector model of economic growth, which assumes the same technology coupled with a simple Keynesian savings hypothesis. For the case of a fixed levy the comparison is instructive -- in Solow's model capital and consumption would remain at their steady-state values right up to the time of the tax. The decline in private GNP caused by the tax would be absorbed by investment and consumption in proportion to the savings ratio. With less investment the economy would decline further to a new steady state. In the model of this essay, investment rises before the tax so that the whole decline in private GNP is absorbed by it.

Unfortunately the comparison is less interesting for the more complicated differential effects discussed later in the essay. The difficulty is that there are no differential effects at all in an economy with a constant savings ratio; the competitive equilibrium is always given by

$$\frac{dk}{dt} = sf(k) - (\delta + n)k \quad ,$$

and none of these variables can be affected by a program of taxes and equal transfer payments. Tax rates are reflected only in the relation between the capital stock and the interest rate, but the interest rate does not affect the real quantities c and k. Dynamic models with constant savings ratios are unable to distinguish between alternative fiscal policies.

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Chapter 4

SPECULATION IN ASSETS WHICH LAST FOREVER

Assets which last forever present a special problem to the economy in determining their prices. The problem arises because there may be competitive prices which are uniquely associated with infinite time in the sense that they cannot be found as the limit as the time horizon goes to infinity of the equilibrium prices in an economy with a finite horizon. For example, suppose that the yield from tulip bulbs is zero. Then in any economy with a finite horizon, no matter how far distant, the price of tulip bulbs must be zero. On the other hand, if the price is always expected to rise just enough to meet the interest cost of holding bulbs with a positive price, the price can rise in a self-fulfilling, never-ending upward speculative boom. Nothing in the ordinary notions about the equilibrium in an asset market can rule out this behavior. For tulip bulbs, this observation is hardly more than a curiosity. Recently, however, it has been suggested that there may be speculative booms in reproducible physical assets which may affect the allocation of investment in these assets. The basic result of this study is a Non-Speculation Theorem which shows that this is not the case for at least one kind of economy with heterogeneous capital goods. If we take as the basic equilibrium condition in the market for these assets the requirement that there be no profit in holding a unit of an asset for any period of length T, we obtain the following

<u>Zero Profit Condition</u>:*

(1) $$p(t) = \int_0^T e^{-R(t,t+\tau)} y(t+\tau) d\tau + e^{-R(t,t+T)} p(t+T) \quad ,$$

for all t and all T > 0; p(t) is the price of the asset and y(t) is its yield.

Let

(2) $$\bar{p}(t) = \int_0^\infty e^{-R(t,t+\tau)} y(t+\tau) d\tau \quad ,$$

if this integral converges.

Nothing so far guarantees that the price in a market ruled by (1) is that given in (2); that is, p(t) is not necessarily equal to $\bar{p}(t)$. This gives rise to the following

<u>Definition of Speculation</u>

Speculation is said to exist whenever $p(t) \neq \bar{p}(t)$; the speculative component of the price is

(3) $$s(t) = p(t) - \bar{p}(t) \quad .$$

The speculative component of an asset's price is the part of the price which can be explained by the expectation of future price increases; its form is given in the

<u>Characterization of Speculation</u>

The only speculative component consistent with the zero-profit condition (1) is

---

*Alternatively and equivalently, we could posit the equation of yield: $y(t) = r(t)p(t) - \dot{p}(t)$.

(4) $$s(t) = s_o e^{R(0,t)} ,$$

where s_o is any constant, positive or negative.

Proof:

From (1), (2), and (3),

$$c(0) = \int_o^t e^{-R(0,\tau)} y(\tau) d\tau + e^{-R(0,t)} p(t)$$

$$- \int_o^\infty e^{-R(0,\tau)} y(\tau) d$$

$$= -e^{-R(0,t)} \bar{p}(t) + e^{-R(0,t)} p(t)$$

$$= e^{-R(0,t)} s(t) .$$

Let $s_o = s(0)$; then

$$s(t) = s_o e^{R(0,t)} .$$

The speculative component either rises or falls as the inverse of the discounting function.

The speculative component is further restricted by the

No-Negative-Speculation Theorem

If $p(0) < \bar{p}(0)$ then there is a T for which $p(t) < 0$ for $t > T$.

<u>proof</u>:

From (1),

$$p(t) = e^{R(0,t)}[p(0) - \int_0^t e^{-R(0,\tau)}y(\tau)d\tau] \quad .$$

There are two cases to consider. First, if $\bar{p}(0)$ is finite, there exists a T such that

$$\int_0^t e^{-R(0,\tau)}y(\tau)d\tau > p(0)$$

for all $t > T$. But then since $e^{R(0,t)}$ is positive, $p(t) < 0$ for all $t > T$.

If $\bar{p}(0)$ is infinitely large, then the inequality $p(0) < \bar{p}(0)$ means that $p(0)$ is finite. But

$$\int_0^t e^{-R(0,\tau)}y(\tau)d\tau$$

can be made arbitrarily large: there exists a T for any $p(0)$ such that

$$\int_0^t e^{-R(0,\tau)}y(\tau)d\tau > p(0)$$

for all $t > T$. But then $p(t) < 0$ for these t's.

This theorem has two important implications. First, if negative asset prices are ruled out by free disposability, it shows that the speculative component of a price cannot be negative. Second, it shows that if there is a market-clearing price $p(0)$ then the integral for $\bar{p}(0)$ must converge. The convergence of this integral is the basis of the proof of the Non-Speculation Theorem. In preparation for it, I demonstrate a

Lemma on Prices and Rents

Consider a profit-maximizing economy with m factors of production whose supply is fixed and n factors which can be produced. Gross output of the reproducible factors is given by a set of constant returns to scale production functions

$$f^i(x_1^i, \ldots, x_{m+1}^i, \ldots, x_{m+n}^i) \quad \text{for } i = m+1, \ldots, m+n \; .$$

Each factor has a rent or wage q_i, $i=1,\ldots,m+n$, and a price p_i. Then the prices of the reproducible factors are limited by the rents of all factors in the following way: There exists a constant, v, such that

$$p_i \leq v \sum_{j=1}^{m+n} q_j \; .$$

Proof:

Consider setting each factor input in each industry at the unit level. Then since the price p_i is always less than or equal to the cost of production, and the cost of production at the profit maximizing factor ratios is less than or equal to the cost at the unit level

$$p_i \leq \frac{\sum_{j=1}^{m+n} q_j}{f^i(1, \ldots, 1)}$$

The required v is given by

$$v = \frac{1}{\min_i f^i(1, \ldots, 1)}$$

Non-Speculation Theorem

With the technology of the previous lemma and with no deterioration of reproducible assets, there is no speculation in reproducible assets; that is, $s_{oi} = 0$ for $i = m+1, \ldots, m+n$.

Proof:

Since $\overline{p}_i(0)$ is finite for all assets, there exists a t for any $s_{oi} > 0$ such that

$$e^{-R(0,t)} \sum_{j=1}^{m+n} q_j(t) < \frac{s_{oi}}{2v} .$$

Then by the lemma,

$$e^{-R(0,t)} p(t) < \frac{s_{oi}}{2} .$$

Recall that

$$s_{oi} = e^{-R(0,t)} p_i(t) - e^{-R(0,t)} \overline{p}_i(t) \quad ,$$

so

$$s_{oi} \leq e^{-R(0,t)} p(t) \quad .$$

Thus,

$$s_{oi} < \frac{s_{oi}}{2} \quad ,$$

a contradiction, so $s_{oi} \leq 0$. But from the No-Negative-Speculation theorem, $s_{oi} \geq 0$. Therefore $s_{oi} = 0$ and there is no speculation in reproducible assets.

The assumption that all factors of production are treated as assets is critically important in the previous theorem. If there is labor it must be supplied by slaves. If there is a factor such as free labor which is not required to have a finite asset price, then speculation in reproducible assets can take place. What is required to allow speculation is that the marginal product of capital become small relative to the wage, so that $e^{-R(0,t)}w(t)$ does not go to zero ($w(t)$ is the wage). Then the present value of the price, $e^{-R(0,t)}p_i(t)$ need not go to zero and p_i may have a speculative component.

An example of the possibility is the simple economy with labor and capital producing a single output with constant factor proportions. If the economy is in a steady state with redundant capital, then the interest rate will be zero and $\bar{p}_k$ will also be zero. But it is perfectly consistent and reasonable for p_k to equal 1 -- capital provides a way to store wealth. This steady state might be generated by a Diamond-Cass -- Yaari model; it is worth noting that this equilibrium is inefficient in the sense of Phelps and Koopmans. Phelps-Koopmans efficiency itself is not quite strong enough to prevent speculation, but a stronger, related efficiency property is sufficient. It guarantees that the present value of the wage is finite. I am currently investigating this topic.

Chapter 5

TECHNICAL CHANGE AND CAPITAL FROM THE POINT OF VIEW OF THE DUAL

In the past few years, it has become increasingly evident that almost every interesting economic problem has an equally interesting problem as its dual. Although formally speaking the dual has exactly the same properties as the primal, often examination of the dual can yield new insights into the problem which were not apparent in the primal. This paper examines the problem of production with durable machines subject to technical change from the point of view of the dual, and presents several new results having to do with technical change and the value of capital goods.

The first section of the paper investigates the problem of the existence of a production function relating output, labor, and some measure of capital. In the dual, the corresponding function, expressing all the properties of the production function, is the factor price function relating the wage to the flow price of machines' services, or rent. Three basic notions of capital theory -- deterioration of capital, capital-embodied technical change,[1] and disembodied technical change -- are developed within a more general theoretical framework, in which it is shown that the three do not constitute an unambiguous description of reality.

The second section investigates the relation between the price of machines and their rent. The relationship is expressed in the fundamental equation setting the price of a used machine equal to the present value of the rent which it will earn over the remainder of its life. From this equation explicit formulas for both the rent and the price of used machines

[1]Throughout this paper, technical change is assumed to be capital-augmenting.

are derived as functions of the exogenously determined interest rate and price of new machines.

Finally, in the third section, the question of the operational content of the neoclassical theory of technical change and capital is examined. Two theorems on the identification of technical change and deterioration are stated; these show that the theory is in fact meaningful in the sense that it involves refutable hypotheses. While this is a full answer to the critics of neoclassical theory who might claim that it is formally empty, only an empirical application of the methods stated in the theorems can determine if the assumptions of the theory are sufficiently realistic to make it a useful theory.

1. The vintage model and the parameters of technical change and deterioration

In this paper Solow's vintage model of production is assumed to hold (11). In that model, at any instant in time machines of vintage v have their own production function f(v,I(v),L(v)), where I(v) is investment that took place in year v and hence is the present gross capital stock of vintage v, and L(v) is the amount of homogeneous labor applied to this vintage. Total output is the sum of the output on all vintages. If the vintage production function has constant returns to scale, as it is assumed to have throughout this paper, its dual relation is Samuelson's factor price function (10) giving the rent c(v) on machines of vintage v as a function of the wage:

$$c(v) = g(v,w) \tag{1}$$

A central problem in vintage models of production concerns the existence of an aggregate capital stock, usually called J, which is one of the arguments of an aggregate production function of the form:

$$F(J,L) = \max \int_{-\infty}^{t} f(v,I(v),L(v))dv \quad ,$$

where the maximum is over all labor allocations L(v) such that $\int L(v)dv = L$. Recently a number of authors (see (3) for a bibliography) have shown that a necessary and sufficient condition for the existence of J is that the vintage production function have the very special form:

$$f(v,I(v),L(v)) = F(z(v)I(v),L(v)) \quad . \tag{2}$$

Yet another proof of the basic theorem can be obtained very easily by reference to the dual; the proof is sketched here because it helps later to illustrate the close relation between the capital aggregation problem and the problems which arise in calculating the prices associated with used machines:

By Leontief's theorem (8,9), the existence of a capital aggregate is equivalent to the condition that the ratio of the marginal physical products of machines of different vintages be independent of the amount of labor. In terms of prices, this means that the ratio of the rents for machines of different vintages should be independent of the wage, or that the factor price function should have the form

$$c(v) = z(v)G(w) \quad .$$

But if this is true, the production function has the form

$$f(v,I(v),L(v)) = F(z(v)I(v),L(v)) \quad ,$$

and *vice versa*.[2]

This condition for aggregation is usually described in terms of technical change -- in order to form the aggregate J, embodied technical change must be capital-augmenting. More generally, the vintage coefficient z(v) measures all differences in efficiency which distinguish machines of different vintages -- that is, both technical change and physical deterioration enter z(v). Only if condition (2) holds is it possible to separate the effects of technical change and deterioration and to speak unambiguously of units of capital services and the relative efficiency of old machines.

The basic theorem on capital aggregates makes no restriction on the behavior of the function z(v) over calendar time. From one year to the next, the pattern of efficiency as a function of vintage may change arbitrarily. That is, the efficiency function can be written more generally as z(t,v); this gives the relative efficiency at time t of one machine of vintage v. In most cases, no distinction between technical change and deterioration can be made in this formulation, because z changes completely with time and no regular behavior which depends only on time or only on age can be found. This formulation is so general as to be almost vacuous -- some restriction on the form of z(t,v) is needed.

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[2]The last step follows by defining a new measure of capital $I^*(v) = z(v)I(v)$ with corresponding rent $c^*(v) = c(v)/z(v)$. Then the new factor price function, G(w), is independent of the vintage, v, and so is its uniquely corresponding production function $F(I^*,L)$, which can be written in terms of I as $F(z(v)I(v),L(v))$.

The simplest restriction which can be made is that $z(t,v)$ is stationary over time -- that is, that the efficiency of machines of a particular age is independent of calendar time. This is the case of no technical change; z has the form

$$z(t,v) = \Phi(t-v) \quad .$$

$\Phi(\tau)$ gives the marginal physical product of a machine aged τ years as a fraction of the marginal physical product of a new machine, and thus also gives the corresponding ratio of the rents:

$$\frac{c(t,v)}{c(t,t)} = \Phi(t-v)$$

Under the assumption of no technical change, the formula for J is

$$J = \int_{-\infty}^{t} \Phi(t-v) I(v) dv \quad ;$$

this is a conventional formula for calculating capital stock.

Although Φ is often called the depreciation function, in this paper it will be called the deterioration function in order to avoid confusion with the definition of depreciation in terms of the price of used machines which is used in the next section.

A more interesting restriction of the form of $z(t,v)$ can be obtained by relaxing the assumption of stationarity to the extent of allowing changes over time and vintage which are independent of deterioration. Then $z(t,v)$ can be written as the product of a function of time, $d(t)$, a function of vintage, $b(v)$, and the deterioration function $\Phi(t-v)$:

(3) $$z(t,v) = d(t)b(v)\Phi(t-v) \quad .$$

Both of these new functions have familiar interpretations: $d(t)$ is the index of disembodied technical change and $b(v)$ is the index of embodied technical change. The assumption of independence expressed in the factorization (3) means that neither kind of technical change affects the pattern of deterioration as a machine ages. With this parametrization of technical change, the ratio of the rent on machines of vintage v at time t to the rent on new machines at a base date t_o is

$$(4) \qquad \frac{c(t,v)}{c(t_o,t_o)} = d(t)b(v)\Phi(t-v) \quad .$$

The corresponding formula for J is

$$(5) \qquad J(t) = d(t)\int_{-\infty}^{t} \Phi(t-v)b(v)I(v)dv \quad .$$

Examination of the parametrization of technical change, (3), reveals that it has a mathematical property of considerable economic importance -- more than one triplet of functions $d(t)$, $b(v)$, and $\Phi(t-v)$ can have the same product $z(t,v)$. This is important because it is $z(t,v)$ which is the basic description of reality; two triplets of technical change and deterioration functions cannot be distinguished by observation or experiment unless they have different products. The parameters of technical change and deterioration which are natural and customary are unfortunately not unique. In short, a problem of identification arises in this model of technical change.

It is easy to characterize the class of triplets of functions of this form which have the same product. Suppose $d(t)$, $b(v)$, and $\Phi(t-v)$ form one triplet. Since all of these functions are required to be positive, any other triplet with the same product can be written $e^{a_1(t)}d(t)$, $e^{a_2(v)}b(v)$

and $e^{a_3(t-v)}\Phi(t-v)$ with the condition that $a_1(t) + a_2(v) + a_3(t-v) = 0$. Then for any period of time of length T,

$$a_1(t+T) - a_1(t) = -a_3(t-v+T) + a_3(t-v) .$$

Reducing v by the same T, we have

$$a_2(v) - a_2(v-T) = a_3(t-v+T) - a_3(t-v) .$$

Thus,

$$a_1(t+T) - a_1(t) = -(a_2(v) - a_2(v-T)) , \tag{6}$$

or the difference in $a_1(t)$ is independent of t and the difference in $a_2(v)$ is independent of v. The only functions with this property are linear:

$$a_1(t) = \alpha_1 + \beta_1 t \text{ and } a_2(v) = \alpha_2 + \beta_2 v .$$

α_1 and α_2 may be set equal to zero without loss of generality -- this amounts to setting all equivalent indexes equal at time zero. Substitution in (6) shows that $\beta_1 = -\beta_2 = \beta$. β can be any number, positive or negative. Finally,

$$a_3(t-v) = -(a_1(t) + a_2(v))$$

$$= -\beta\cdot(t-v) .$$

Thus all members of a class of equivalent triplets have the form $e^{\beta t}d(t)$, $e^{-\beta v}b(v)$ and $e^{-\beta(t-v)}\Phi(t-v)$, where d(t), b(v) and $\Phi(t-v)$ form an arbitrary member of the class. This means that given rates of embodied and disembodied technical change and a given deterioration function cannot be distinguished from a lower rate of embodied technical change, a higher rate of disembodied change, and a higher rate of deterioration.

In order to escape this ambiguity, some normalization rule must be adopted so that each $z(t,v)$ can be represented by at most one triplet $d(t)$, $b(v)$ and $\Phi(t-v)$. With such a rule, the notions of embodied technical change, disembodied technical change, and deterioration each have a well-defined meaning and each can be measured from suitable observations. Without it, serious confusion can develop in both theoretical and empirical work.

A normalization rule may be applied to any one of the three functions $d(t)$, $b(v)$ and $\Phi(t-v)$. For a period of observation $t_o \leq t \leq t_1$, convenient rules are provided by setting net technical change of one kind equal to zero over the period -- either $b(t_1) = b(t_o)$ or $d(t_1) = d(t_o)$. Alternatively, it is possible to require that no net deterioration take place over the first τ_o years of the life of a machine: $\Phi(\tau_o) = \Phi(0) = 1$, although this is a less natural normalization. Each of these rules makes it clear that disembodied technical change, embodied technical change, and deterioration are three distinct phenomena only to the extent that they all are different from exponential functions. For example, in his well-known applications of the embodied technical change model (11 and 12), Solow assumes

$$d(t) = 1$$
$$b(v) = e^{bv}$$
$$\Phi(t-v) = e^{-\delta(t-v)} \quad .$$

This corresponds to the normalization rule requiring that no disembodied change take place over the period. The alternative form of the model for no embodied change is

$$d(t) = e^{bt}$$

$$b(v) = 1$$

$$\Phi(t-v) = e^{-(\delta+b)(t-v)} \quad .$$

Curiously enough, all of the empirical work in Solow's papers could have been carried out without any reference to embodied technical change. This is a consequence of his choice of functional forms for technical change and deterioration -- it is incorrect to conclude that embodied and disembodied technical change can never be distinguished. Only the exponential parts of these functions cannot be distinguished.

2. The price of machines and the price of machines' services

The goal of this section is to derive an expression for the price of a used machine, $p(\tau,t)$, as a function of the age of the machine and calendar time. From this, the rent will be calculated by applying the equation of yield. The analysis is carried out first for the case of no technical change and then is generalized to take account of technical change.

Suppose that an interest rate $r(t)$ prevails, so receipts in year u are discounted to year t by the ratio

$$e^{-R(t,u)} = e^{-\int_t^u r(s)ds} \quad .$$

Now let $c_o(t)$ be the rent on a new machine of vintage t at time t. In the absence of technical change, suppose that

$$c(t,v) = \Phi(t-v)c_o(t) \quad .$$

From the proof of the basic theorem on capital aggregates, one can see that this will hold if and only if the conditions for forming a capital aggregate hold. This assumption results in a crucial simplification of the theory of the determination of the price $p(\Upsilon,t)$. It is important to emphasize that the assumption that there is a general rent for capital, independent of vintage, is exactly as restrictive as the assumption that there is an aggregate capital stock.

The price of machines, $p(\Upsilon,t)$, is subject to the boundary condition that the price of machines of age zero be equal to the exogenously determined supply price: $p(0,t) = p_o(t)$. This assumption is extremely important in the derivation of the price function $p(\Upsilon,t)$ -- its consequences will be discussed later. It is worth pointing out that the fact that the rent for an old machine has a fixed relation to the rent for a new one does not imply that the price of an old machine has a fixed relation to the price of a new one.

The fundamental behavioral assumption which determines the price of used machines is that firms buy and sell machines (invest and disinvest) so as to maximize the present value of the firm. From this assumption Jorgenson (6) and Arrow (1) have derived a theory of optimal investment by applying the calculus of variations. The rent appears as a byproduct dual variable in their formulation; the price function could be deduced from the rent, but this has never been done except in the very simple case of exponential deterioration.

The dual of the assumption of maximization of present value can be expressed in the form of the following fundamental equation governing the price and rent:

$$(7) \qquad p(\tau,t) = \int_0^T e^{-R(t,t+u)} c_o(t+u)\bar{\Phi}(\tau+u)\,du$$

$$+ e^{-R(t,t+T)} p(\tau+T,t+T) \qquad ,$$

for all t, all $\tau \geq 0$, and all $T \geq 0$.

In words, the price at time t of a machine aged τ years is equal to the present value of the rent it will earn over the period from t to t+T plus the discounted resale value at the end of the period. This form of the fundamental equation is essentially the same as the one proposed by Hotelling (5) except that resale is allowed at any time and machines are assumed to have no scrap value. Hotelling differentiated an equation like (7) to obtain an expression for $c_o(t)$ in terms of p and its derivatives. In the present case, we can differentiate with respect to T to obtain

$$0 = e^{-R(t,t+T)} c_o(t+T)\bar{\Phi}(\tau+T)$$

$$- r(t+T)e^{-R(t,t+T)} p(\tau+T,t+T)$$

$$+ e^{-R(t,t+T)}\left(\frac{\partial p}{\partial \tau} + \frac{\partial p}{\partial t}\right) \quad .$$

Without loss of generality, we set T = 0, or

$$(8) \qquad \bar{\Phi}(\tau)c_o(t) = r(t)p(\tau,t) - \frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial t} \quad .$$

This is the well-known equation of yield for the case of physical capital -- the rent for a machine aged τ years is equal to the interest cost $r(t)p(\tau,t)$ plus depreciation, $-\frac{\partial p}{\partial \tau}$, less capital gains, $\frac{\partial p}{\partial t}$. For new machines, substitute the boundary condition, so that

$$(9) \qquad c_o(t) = r(t)p_o(t) - \left.\frac{\partial p}{\partial \tau}\right|_{\tau=o} - \dot{p}_o(t)$$

Unfortunately the depreciation term $-\left.\frac{\partial p}{\partial \tau}\right|_{\tau=o}$ in (9) involves the yet-undetermined function $p(\tau,t)$, so this approach is not directly useful in finding explicit formulas for $p(\tau,t)$ or $c_o(t)$.

In fact, in order to find such formulas, it is necessary to make an additional assumption about the behavior of the price:

$$(10) \qquad \lim_{T\to\infty} e^{-R(t,t+T)}p(T,t+T) = 0 \quad .$$

That is, the present value of the future price must tend to zero as the future resale date tends to infinity. This assumption rules out speculation in old machines.[3] Under this assumption the possibility of resale can be ignored by taking (7) to the limit as $T\to\infty$:

$$(11) \qquad p(\tau,t) = \int_o^{\infty} e^{-R(t,t+u)}c_o(t+u)\Phi(\tau+u)\,du \quad .$$

In order to solve for $p(\tau,t)$ as a function of known variables, first define the mortality function for machines:

$$(12) \qquad \phi(\tau) = -\frac{d}{d\tau}\Phi(\tau) \quad ;$$

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[3]This condition arises as the transversality condition in the corresponding optimal investment problem. For a discussion of the circumstances under which it holds in a competitive economy, see (4).

$\phi(\tau)$ is the rate at which the machine is wearing away at age τ. Second define the replacement density function for used machines:[4]

$$(13) \qquad \Psi(\tau,s) = \sum_{j=0}^{\infty} v_j(\tau,s)$$

where

$$v_o(\tau,s) = \phi(s+\tau)$$

and

$$v_j(\tau,s) = \int_0^s v_{j-1}(\tau,z)\phi(s-z)dz \quad .$$

$\Psi(\tau,s)$ is the rate at which new machines must be bought to maintain a stock equivalent to one machine which was aged τ years when it was bought s years ago.

Now by substituting the boundary condition in (11), we have

$$(14) \qquad p_o(t) = \int_0^{\infty} e^{-R(t,t+u)} c_o(t+u)\Phi(u)du \quad .$$

By multiplying the two sides of (14) by $\Psi(\tau,s)$, taking the present value of both sides and then adding (11), we obtain the following interesting alternative form of the fundamental equation:

$$(15) \qquad p(\tau,t) + \int_0^{\infty} e^{-R(t,t+s)} \Psi(\tau,s)p_o(t+s)ds$$

$$= \Phi(\tau) \int_0^{\infty} e^{-R(t,t+u)} c_o(t+u)du \quad .$$

(See the appendix for the details of this step.)

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[4]The importance of the replacement density function in this application was suggested by Arrow's paper, (1).

That is, the cost of a machine today plus the present value of the cost of maintaining a stock equivalent to that machine forever is equal to the present value of the rent from a stock equivalent to that machine. Their common value is the market value of a contract to provide $\Phi(\tau)$ units of machine services forever.

Substituting the boundary condition in (15), we find

$$(16)\qquad p_o(t) + \int_o^\infty e^{-R(t,t+s)}\psi(0,s)p_o(t+s)ds = \int_o^\infty e^{-R(t,t+u)}c_o(t+u)du \quad .$$

By multiplying (16) by $\Phi(\tau)$ and subtracting it from (15), we obtain the solution

$$(17)\qquad p(\tau,t) = \Phi(\tau)p_o(t) - \int_o^\infty e^{-R(t,t+s)}[\psi(\tau,s)-\Phi(\tau)\psi(0,s)]p_o(t+s$$

The solution shows that the price of a used machine is the fraction of the services of a new machine which the old one provides times the price of a new machine less an adjustment for the extent to which buying a used machine rather than part of a new machine will require more replacement expenditures in the future. It is possible to show that the second term is identically zero if and only if the deterioration function is exponential: $\Phi(\tau) = e^{-\delta\tau}$. Only in this special case is the price of machines today independent of the future values of the interest rate and the price of new machines.

By differentiating (17) with respect to τ and integrating by parts, we obtain the following formula for depreciation:

$$(18)\quad -\frac{\partial p}{\partial \tau} = \int_0^\infty e^{-R(t,t+s)}\psi(\tau,s)[r(t+s)p_o(t+s) - \dot{p}_o(t+s)]ds \ .$$

Similarly, capital gains are given by

$$(19)\quad \frac{\partial p}{\partial t} = \Phi(\tau)\dot{p}_o(t) - r(t)[\Phi(\tau)p_o(t) - p(\tau,t)]$$

$$+ \int_0^\infty e^{-R(t,t+s)}[\psi(\tau,s)-\Phi(\tau)\psi(0,s)][r(t+s)p_o(t+s) - \dot{p}_o(t+s)]ds \ .$$

From (8), rent is given by

$$c_o(t) = \frac{1}{\Phi(\tau)}\left[r(t)p(\tau,t) - \frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial t}\right] .$$

Substituting (18) and (19) and simplifying, we have

$$(20)\quad c_o(t) = r(t)p_o(t) + \int_0^\infty e^{-R(t,t+s)}\psi(0,s)[r(t+s)p_o(t+s) - \dot{p}_o(t+s)]ds$$

$$- \dot{p}_o(t) \quad ;$$

$c_o(t)$ is independent of τ, as required. By comparing (20) to (9), we can interpret the terms; the first is interest cost on new machines, the second is depreciation on new machines, and the third is capital gains on new machines.

Introducing technical change complicates the analysis only slightly. A machine aged τ years is equivalent to $\Phi(\tau)\frac{b(t-\tau)}{b(t)}$ new machines at time t. The fundamental equation governing the market is

$$(21)\qquad p(\tau,t) = \int_0^\infty e^{-R(t,t+u)} c_o(t+u)\bar{\Phi}(\tau+u)\frac{b(t-\tau)}{b(t+u)}du \quad .$$

From this equation, the ambiguity in the parametrization of technical change may be demonstrated again, although formally speaking this is redundant. More than one pair of functions $\bar{\Phi}(\tau)$ and $b(t)$ lead to the same value for $\bar{\Phi}(\tau+u)\frac{b(t-\tau)}{b(t+u)}$. By very much the same argument as in section 2, the ambiguity can be seen to take the form of an exponential function: $e^{-\beta\tau}\bar{\Phi}(\tau)$ and $e^{\beta t}b(t)$ cannot be distinguished from $\bar{\Phi}(\tau)$ and $b(t)$.

Equation (21) can be written as

$$\frac{p(\tau,t)}{b(t-\tau)} = \int_0^\infty e^{-R(t,t+u)} \frac{c_o(t+u)}{b(t+u)}\bar{\Phi}(\tau+u)du \quad .$$

This puts it in exactly the form of equation (11); the solution is obtained by substituting for the appropriate variables in (17) and rearranging:

$$(22)\qquad p(\tau,t) = \bar{\Phi}(\tau)\frac{b(t-\tau)}{b(t)}p_o(t)$$

$$- \int_0^\infty e^{-R(t,t+s)}[\Psi(\tau,s) - \bar{\Phi}(\tau)\Psi(0,s)]\frac{b(t-\tau)}{b(t+s)}p_o(t+s)ds \quad .$$

Similarly, the rent is

$$(23)\qquad c_o(t) = r(t)p_o(t) + p_o(t)\frac{\dot{b}(t)}{b(t)}$$

$$+ \int_0^\infty e^{-R(t,t+s)}\Psi(0,s)\frac{b(t)}{b(t+s)}[r(t+s)p_o(t+s) - \dot{p}_o(t+s) + \frac{\dot{b}(t+s)}{b(t+s)}p_o(t+s)ds$$

$$- \dot{p}_o(t) \quad .$$

The term $p_o(t)\frac{\dot{b}(t)}{b(t)}$ which is added to the depreciation component of the rent may be interpreted as the cost of obsolescence, but because of the identification problem, $\frac{\dot{b}(t)}{b(t)}$ is known only down to the arbitrary additive constant β. That is, for exactly the same reason that embodied technical change and deterioration cannot be distinguished unambiguously, the part of the cost of machines' services associated with aging cannot be separated unambiguously into depreciation and obsolescence. Thus, businessmen are correct in entering depreciation plus obsolescence in their income statements, without attempting to separate the two.

Some comment on the boundary condition equating the supply price of machines and the price of machines of age zero is now in order. This boundary condition ensures that the prices and rent for machines are independent of the circumstances in which they are used in production -- the markets for machines and machines' services are driven by the supply price and the interest rate. This assumption corresponds to the assumption of no negative gross investment which Arrow requires in his treatment of the optimal investment problem. On the other hand, if no new machines are sold during an interval of time, as would occur if the supply price were much higher than usual during the interval, the boundary condition does not hold and the markets work in reverse -- machines are in inelastic supply and their rent is determined by the demand for the output produced with them.[5] This point has considerable importance in empirical work on investment; it suggests that the rent should be calculated from data from markets for used machines, rather than from the prices quoted by the sellers of new machines.

[5] In a more recent paper, (2), Arrow presents an algorithm for determining exactly when the boundary condition does not hold.

3. Measuring technical change and deterioration

Some of the critics of the theory of embodied technical change (notably Jorgenson (7)) have argued that the theory has no operational content because technical change cannot be measured from the data usually available to econometricians, or because the hypotheses of the theory are irrefutable with the data available. This section will show that if data on the prices of used machines and the interest rate are available, then the index of embodied technical change and the deterioration function can in fact be calculated from these data. Further, with sufficient data, they can be calculated in more than one way; if the results disagree, at least one of the hypotheses of the theory is refuted -- technical change is not capital augmenting or is not independent of deterioration, the market is not competitive, or foresight is not perfect. In other words, calculation of the index of technical change and the deterioration function does not always exhaust the degrees of freedom of the data, so the calculation can provide a test of the hypotheses underlying the theory. These results are stated formally in:

Identification theorem I:

Given any price function $p(\tau, t)$ and the interest rate $r(t)$ for a period of time $0 \leq t \leq t_o$ and a range of ages $0 \leq \tau \leq t_o$, there is at most one deterioration function $\Phi(\tau)$ and one index of embodied technical change $b(t)$ consistent with the price function and meeting the normalization $\Phi(0) = 1$ and $b(0) = b(t_o) = 1$.

proof:

Estimates can be derived by applying the equation of yield:

$$r(t)p(\tau,t) - \frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial t} = \frac{b(t-\tau)}{b(t)} \Phi(t)c_o(t) \quad .$$

For convenience, we define the capital loss function

$$L(\tau,t) = -\frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial t} \quad .$$

Dividing each side by its own value with $\tau = 0$, we get an equation in relative efficiencies:

$$\frac{r(t)p(\tau,t) + L(\tau,t)}{r(t)p_o(t) + L(0,t)} \quad \frac{b(t-\tau)}{b(t)} \Phi(\tau) \quad .$$

Call the log of the left side $y(\tau,t)$ and let $b^*(t) = \log b(t)$ and $\Phi^*(\tau) = \log \Phi(\tau)$. Then

$$y(\tau,t) = b^*(t-\tau) + \Phi^*(\tau) - b^*(t).$$

The proof is constructive and consists in stating a formula which can be used to calculate a new value of b^* and a new value of Φ^*, given two values of b, $b(t_1)$ and $b(t_2)$. We obtain the formula by solving the equation of yield for two points:

$$y(\tau,t_2) = b^*(t_1+\tau) + \Phi^*(\tau) - b^*(t_2)$$

$$y(\tau,t_1+\tau) = b^*(t_1) + \Phi^*(\tau) - b^*(t_1+\tau) \quad ,$$

where $\tau = \frac{t_2-t_1}{2}$.

The solution is

$$\Phi^*(\tau) = \frac{1}{2}[y(\tau,t_1+\tau) + y(\tau,t_2) - b^*(t_1) + b^*(t_2)]$$

$$b^*(t_1+\tau) = y(\tau,t_2) - \Phi^*(\tau) + b^*(t_2) \quad .$$

Since no division is involved, the new values are always unique. The process can be started by setting $t_1 = 0$ and $t_2 = t_o$. After the first step, the new values of Φ^* are each calculated more than once from independent data, so the hypotheses of the theory are checked at each step. Values of $\Phi^*(\tau)$ for $\tau > \frac{t_o}{2}$ can be calculated from another application of the equation of yield:

$$\Phi^*(t_2-t_1) = y(t_2-t_1,t_2) - b^*(t_1) + b^*(t_2) \quad .$$

An econometric problem of equal or perhaps greater importance is to calculate the index of technical change, the deterioration function. and the interest rate from data on prices alone. This is a difficult undertaking, particularly since the estimation equations are intrinsically nonlinear. Before embarking on a study of this kind, we should verify that the problem is identified, at least in the weak sense that the estimation equations are not identically singular for all price data. Identical singularity would arise if there were a formal difficulty in the estimation problem, as there would be, for example, in estimating technical change without introducing the appropriate normalization. The following theorem states that if an estimation procedure fails, it is because of the particular characteristics of the price data and not because of an inherent singularity, with one important exception:

Identification theorem II:

Estimation of the interest rate, the index of embodied technical change, and the deterioration function is identically singular for all prices if and only if both technical change and deterioration are exponential functions.

<u>proof</u>:

The proof consists in showing that the analogue for this problem of the interpolation method of the proof of identification theorem I is not identically singular except in the exponential case. More precisely, we show that the Jacobian matrix of the set of nonlinear equations for the problem is not identically singular.

The simplest possible interpolation method is to calculate 3 values of the interest rate, 4 values of the deterioration function, and 5 values of the index of technical change from 12 observations on $p(\tau,t)$, starting from two known values of the index, $b(t_1)$ and $b(t_2)$. Let $\tau_1 = \frac{1}{2}(t_2-t_1)$, $\tau_2 = t_2-t_1$, $\tau_3 = \frac{3}{2}(t_2-t_1)$ and $\tau_4 = 2(t_2-t_1)$. Then the unknowns in the problem are $r(t_2)$, $r(t_1+\tau_1)$, $r(t_2)$, $\bar{\Phi}(\tau_i)$, $i=1,\ldots,4$, $b(t_1-\tau_i)$, $i=1,\ldots,4$, and $b(t_1+\tau_1)$. The system of equations is given by

$$\frac{r(t_j)p(\tau_i,t_j) + L(\tau_i,t_j)}{r(t_j)p(0,t_j) + L(0,t_j)} = \frac{b(t_j-\tau_i)}{b(t_j)}\,\bar{\Phi}(\tau_i) \quad ,$$

$i=1,\ldots,4$, $j=1,2,3$, and $t_3 = \frac{t_1+t_2}{2}$.

Taking logs as before, we have

$$\log[r(t_j)p(\tau_i,t_j) + L(\tau_i,t_j)] -$$

$$\log[r(t_j)p(0,t_j) + L(0,t_j)] =$$

$$b^*(t_j-\tau_i) + \bar{\Phi}^*(\tau_i) - b^*(t_j)$$

The Jacobian matrix for this system has 12 rows corresponding to the 12 observations on $p(\tau,t)$ and 12 columns, one for each unknown. Each row has exactly one term of the form

$$\frac{p(\tau_i,t_j)}{r(t_j)p(\tau_i,t_j)+L(\tau_i,t_j)} - \frac{p(0,t_j)}{r(t_j)p(0,t_j)+L(0,t_j)} ,$$

and two zeroes in the first three columns; all of the elements of the last 9 columns are zeroes, +1's, or -1's. Since the last 9 columns are known to be linearly independent from identification theorem I, any identical linear dependence must arise from the condition:

$$\frac{p(\tau,t)}{r(t)p(\tau,t)+L(\tau,t)} - \frac{p(0,t)}{r(t)p(0,t)+L(0,t)} = k$$

for all τ,t. Now consider a sequence of τ's converging to zero. From the continuity of $p(\tau,t)$ and $L(\tau,t)$ (both of which may be represented as integrals), we can see that $k = 0$; substituting this value, we find that the terms involving $r(t)$ cancel, leaving

$$(24) \qquad \frac{p(\tau,t)}{L(\tau,t)} = \frac{p(0,t)}{L(0,t)} \qquad \text{for all } \tau,t.$$

From equations (18) and 19),

$$L(\tau,t) = \bar{\Phi}(\tau)[L(0,t)+r(t)p_o(t)] - r(t)p(\tau,t).$$

Substituting in (24) and cross-multiplying, we have

$$p(\tau,t)L(0,t) = p_o(t)\bar{\Phi}(\tau)[L(0,t)+r(t)p_o(t)]$$

$$-p_o(t)r(t)p(\tau,t),$$

or

$$p(\tau,t) = p_o(t)\bar{\Phi}(\tau) .$$

From the discussion following equation (17), and from equation (22), we can see that this is true if and only if both

$$\bar{\Phi}(\tau) = e^{-\delta\tau}$$

for some constant, δ , and

$$b(t) = 1 .$$

Appendix

We seek to show that the fundamental equation,

$$(11) \qquad p(\tau,t) = \int_0^\infty e^{-R(t,t+u)} c_o(t+u)\bar{\Phi}(\tau+u)\,du$$

and its corollary,

$$(14) \qquad p_o(t) = \int_0^\infty e^{-R(t,t+u)} c_o(t+u)\bar{\Phi}(u)\,du$$

imply the alternative form,

$$(15) \qquad p(\tau,t) + \int_0^\infty e^{-R(t,t+s)} \psi(\tau,s) p_o(t+s)\,ds =$$

$$\bar{\Phi}(\tau) \int_0^\infty e^{-R(t,t+u)} c_o(t+u)\,du \quad .$$

We begin by multiplying both sides of equation (14) by $e^{-R(t,t+s)}\psi(\tau,s)$ and integrating to get

$$(A1) \qquad \int_0^\infty e^{-R(t,t+s)} \psi(\tau,s) p_o(t+s)\,ds =$$

$$\int_0^\infty e^{-R(t,t+s)} \psi(\tau,s) \int_0^\infty e^{-R(t+s,t+s+u)} c_o(t+s+u)\bar{\Phi}(u)\,du\,ds \quad .$$

Next, on the right hand side, we change the limit of integration and combine the discount terms to get

(A2) $$\text{RHS} = \int_0^\infty \int_s^\infty e^{-R(t,t+u)} \bar{\Phi}(u-s) \Psi(\tau,s) c_o(t+u)\,du\,ds.$$

Now by changing the order of integration, we obtain a new form for equation (A1):

(A3) $$\int_0^\infty e^{-R(t,t+s)} \Psi(\tau,s) p_o(t+s)\,ds =$$

$$\int_0^\infty e^{-R(t,t+u)} c_o(t+u) \int_0^u \bar{\Phi}(u-s) \Psi(\tau,s)\,ds\,du \quad .$$

Next we add equation (11) to equation (A3) to obtain

(A4) $$p(\tau,t) + \int_0^\infty e^{-R(t,t+s)} \Psi(\tau,s) p_o(t+s)\,ds =$$

$$\int_0^\infty e^{-R(t,t+u)} c_o(t+u) [\bar{\Phi}(\tau+u) + \int_0^u \Psi(\tau,s)\bar{\Phi}(u-s)\,ds]\,du \quad .$$

Now

(A5) $$\int_0^u \Psi(\tau,s)\bar{\Phi}(u-s)\,ds = \int_0^u [\Psi(\tau,s) - \int_0^s \Psi(\tau,s)\phi(s-z)\,dz]\,ds \;;$$

this can be shown by differentiating both sides with respect to u and observing that when u = 0 both sides are equal. Further,

(A6) $$\int_0^s \Psi(\tau,z)\phi(s-z)\,dz = \int_0^s \sum_{j=o}^{\infty} v_j(\tau,z)\phi(s-z)\,dz$$

$$= \sum_{j=o}^{\infty} v_{j+1}(\tau,s)$$

$$= \Psi(\tau,s) - \phi(s+\tau) \quad .$$

Thus,

(A7) $$\int_0^u \Psi(\tau,s)\overline{\Phi}(u-s)\,ds = \int_0^u \phi(s+\tau)\,ds = \overline{\Phi}(\tau) - \overline{\Phi}(\tau+u) .$$

By substituting (A7) in (A4), we obtain the desired result:

(15) $$p(\tau,t) + \int_0^\infty e^{-R(t,t+s)}\,\psi(\tau,s)\,p_o(t+s)\,ds = \overline{\Phi}(\tau)\int_0^\infty e^{-R(t,t+u)}\,c_o(t+u)\,du .$$

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BIOGRAPHICAL NOTE

Robert E. Hall was born August 13, 1943 in Palo Alto, California. He graduated from University High School, Los Angeles, California, in 1961 and entered the University of California, Berkeley, where he was elected to Phi Beta Kappa in 1963 and received the degree of Bachelor of Arts in 1964.