Note on a Proposition of Hall (2005)

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This note provides a proof for the proposition on p. 55 of Hall (2005). The original proof contains a mistake, which was pointed out to me by Luis Zermeno Valles during a lecture.

Consider the discrete time search and matching model of the labor market in Hall (2005). In particular, consider the following linear system of equations, which corresponds to (12) in Hall (2005),

\[ Y_s = w - \lambda + \beta (1 - \phi_s - \delta) \sum_{s'} \pi_{ss'} Y_{s'}, \]  

(1)

where \( Y_s \) represents the difference between the value of employment and the value of unemployment, conditional on the aggregate state \( s \), \( \pi_{ss'} \) are transition probabilities for the aggregate state of the economy which follows a discrete Markov process, \( \delta \) is the probability of separation for employed workers and \( \phi_s \) the probability of finding a job for unemployed workers.

Since \( \beta (1 - \phi_s - \delta) \in (-1, 1) \) the right hand side of (1) defines a contraction mapping, so there exists a unique vector \( \{Y_s\} \) that satisfies (1). The following proposition characterizes the sign of this vector.

**Proposition 1** (Hall, 2005) If \( w - \lambda > 0 \) then \( Y_s > 0 \) for all \( s \).

Hall (2005) represents the problem in matrix form as

\[ \mathbf{Y} = \mathbf{b} + \mathbf{A} \mathbf{Y}, \]

where

\[ \mathbf{A} = \beta \begin{bmatrix} (1 - \phi_1 - \delta) \pi_1 \\ \vdots \\ (1 - \phi_S - \delta) \pi_S \end{bmatrix}, \quad \mathbf{b} = (w - \lambda) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \]

and claims that all the terms of the following expression are non-negative

\[ \mathbf{Y} = \mathbf{b} + \mathbf{A} \mathbf{b} + \mathbf{A}^2 \mathbf{b} + \mathbf{A}^3 \mathbf{b} + \ldots \]

Unfortunately, this argument does not work, because \( 1 - \phi_s - \delta \) can be negative for some \( s \). The following proof provides an alternative argument that works.
Proof. Pick the states $s$ and $\bar{s}$ so that $s \in \arg \min_s Y_s$ and $\bar{s} \in \arg \max_s Y_s$ and define $\underline{Y} = Y_{s, \underline{\phi}}$, $\overline{Y} = Y_{\bar{s}, \overline{\phi}}$ and $\overline{\phi} = \phi_{\bar{s}}$. Then consider the following three cases.

First, suppose $1 - \underline{\phi} - \delta \geq 0$. Then, by definition,
\[ \underline{Y} \geq w - \lambda + \beta \left(1 - \underline{\phi} - \delta\right) \underline{Y}, \]
which immediately implies
\[ \underline{Y} \geq \frac{w - \lambda}{1 - \beta \left(1 - \underline{\phi} - \delta\right)} > 0. \]

Second, suppose $1 - \underline{\phi} - \delta < 0$ and $1 - \overline{\phi} - \delta \geq 0$ then
\[ \underline{Y} \geq w - \lambda + \beta \left(1 - \underline{\phi} - \delta\right) \overline{Y}, \tag{2} \]
and
\[ \overline{Y} \leq w - \lambda + \beta \left(1 - \overline{\phi} - \delta\right) \overline{Y}. \]
The last inequality implies
\[ \overline{Y} \leq \frac{w - \lambda}{1 - \beta \left(1 - \overline{\phi} - \delta\right)}, \]
which can be substituted in (2) to get
\[ \underline{Y} \geq \frac{w - \lambda}{1 - \beta \left(1 - \overline{\phi} - \delta\right)} \left(1 - \frac{\beta \delta}{1 - \beta \left(1 - \delta\right)}\right) (w - \lambda) > 0. \]

Finally, suppose $1 - \overline{\phi} - \delta < 0$ and $1 - \overline{\phi} - \delta < 0$. Then
\[ \overline{Y} \leq w - \lambda + \beta \left(1 - \overline{\phi} - \delta\right) \overline{Y}, \]
which can be substituted in (2) to get
\[ \underline{Y} \geq \frac{w - \lambda + \beta \left(1 - \overline{\phi} - \delta\right) \left(w - \lambda + \beta \left(1 - \overline{\phi} - \delta\right) \overline{Y}\right)}{1 - \beta^2 \left(1 - \overline{\phi} - \delta\right) \left(1 - \overline{\phi} - \delta\right)} (w - \lambda) > 0. \]

A curious counterexample with state-dependent separation

The result in the proposition seems intuitive: if wages are above the utility flow from unemployment, workers always prefer being employed to be unemployed. The reason why the result is non obvious has to do with the possibility that $1 - \delta > \phi_{s}$, that is, the probability of shifting to employment tomorrow may be higher from unemployment than it is from employment. This mechanism can
actually reverse the result in the proposition if we allow separation rates to be state dependent, \( \delta_s \). Suppose there are two states \( s \in \{1, 2\} \). Then (1) becomes

\[
Y_1 = 1 + \beta (1 - \delta_1 - \phi_1) (\pi_{11} Y_1 + \pi_{12} Y_2) \\
Y_2 = 1 + \beta (1 - \delta_2 - \phi_2) (\pi_{21} Y_1 + \pi_{22} Y_2)
\]

or, in matrix form,

\[
\begin{bmatrix}
1 - \beta (1 - \delta_1 - \phi_1) \pi_{11} & -\beta (1 - \delta_1 - \phi_1) \pi_{12} \\
-\beta (1 - \delta_2 - \phi_2) \pi_{21} & 1 - \beta (1 - \delta_2 - \phi_2) \pi_{22}
\end{bmatrix}
Y = (w - b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

The solution is

\[
Y = \frac{w - b}{\Delta} \begin{bmatrix}
1 - \beta (1 - \delta_2 - \phi_2) \pi_{22} + \beta (1 - \delta_1 - \phi_1) \pi_{12} \\
1 - \beta (1 - \delta_1 - \phi_1) \pi_{11} + \beta (1 - \delta_2 - \phi_2) \pi_{22}
\end{bmatrix},
\]

where \( \Delta \) is the determinant

\[
\Delta = (1 - \beta (1 - \delta_1 - \phi_1) \pi_{11}) (1 - \beta (1 - \delta_2 - \phi_2) \pi_{22}) - \beta^2 \pi_{12} \pi_{21} (1 - \delta_1 - \phi_1) (1 - \delta_2 - \phi_2).
\]

Suppose \( \pi_{22} = 1, \pi_{12} = 1, \phi_1 = 1 \) and \( \phi_2 = 0 \). Then

\[
Y_1 = \frac{1 - \beta (1 - \delta_2) - \beta \delta_1}{1 - \beta (1 - \delta_2)} (w - b)
\]

which is negative iff

\[
1 - \delta_2 + \delta_1 > 1/\beta,
\]

that is, if the probability of losing a job in the state 2 is sufficiently low relative to the probability of losing it in state 1. In this example, it is better to be unemployed than to be employed in state 1, because: (i) today the probability of finding a job is high and next period we will switch for sure to state 2 where the probability of finding a job is zero and (ii) separations rates are high today, so it is more likely to be employed tomorrow for an unemployed than for an employed today.

It is clear, by a continuity argument, that we do not need state 2 to be an absorbing state to get this result.

**Continuous time**

None of the problems above arise if the same model is setup in continuous time. Letting \( V_s \) denote the value of an employed worker and \( U_s \) that of an unemployed worker, we have

\[
rV_s = w + \delta_s (U_s - V_s) + \alpha \sum_{s'} \pi_{ss'} (V_{s'} - V_s),
\]

\[
rU_s = \lambda + \phi_s (V_s - U_s) + \alpha \sum_{s'} \pi_{ss'} (U_{s'} - U_s),
\]
where $\alpha$ is the Poisson probability of a change of state. We then have

$$Y_s = \frac{w - \lambda}{r + \delta_s + \phi_s} + \frac{\alpha}{\alpha + r + \delta_s + \phi_s} \sum_{s'} \pi_{ss'} Y_{s'},$$

and we can apply Hall’s argument to show that $Y_s > 0$ for all $s$ if $w - \lambda > 0$, given that

$$\frac{\alpha}{\alpha + r + \delta_s + \phi_s} > 0.$$ 

The crucial thing is that in a continuous time setting the probability of being employed at time $t''$ conditional on being employed at some time $t' < t''$ is always larger than the probability of being employed at time $t''$ conditional on being unemployed at the same $t'$. 
