Technical Change and Capital from the Point of View of the Dual [1]

In the past few years, it has become increasingly evident that almost every interesting economic problem has an equally interesting problem as its dual. Although formally speaking the dual has exactly the same properties as the primal, often examination of the dual can yield new insights into the problem which were not apparent in the primal. This paper examines the problem of production with durable machines subject to technical change from the point of view of the dual, and presents several new results having to do with technical change and the value of capital goods.

The first section of the paper investigates the problem of the existence of a production function relating output, labour, and some measure of capital. In the dual, the corresponding function, expressing all the properties of the production function, is the factor price function relating the wage to the flow price of machines' services, or rent. Three basic notions of capital theory—deterioration of capital, capital-embodied technical change,[2] and disembodied technical change—are developed within a more general theoretical framework, in which it is shown that the three do not constitute an unambiguous description of reality.

The second section investigates the relation between the price of machines and their rent. The relationship is expressed in the fundamental equation setting the price of a used machine equal to the present value of the rent which it will earn over the remainder of its life. From this equation explicit formulas for both the rent and the price of used machines are derived as functions of the exogenously determined interest rate and price of new machines.

Finally, in the third section, the question of the operational content of the neoclassical theory of technical change and capital is examined. Two theorems on the identification of technical change and deterioration are stated; these show that the theory is in fact meaningful in the sense that it involves refutable hypotheses. While this is a full answer to the critics of neoclassical theory who might claim that it is formally empty, only an empirical application of the methods stated in the theorems can determine if the assumptions of the theory are sufficiently realistic to make it a useful theory.

I. THE VINTAGE MODEL AND THE PARAMETERS OF TECHNICAL CHANGE AND DETERIORATION

In this paper Solow's vintage model of production is assumed to hold [11]. In that model, at any instant in time machines of vintage v have their own production function $f(v, I(v), L(v))$, where $I(v)$ is investment that took place in year v and hence is the present gross capital stock of vintage v, and $L(v)$ is the amount of homogeneous labour applied to this vintage. Total output is the sum of the output on all vintages. If the vintage production function has constant returns to scale, as it is assumed to have throughout this

[1] The author is grateful to Professors Kenneth Arrow, Peter Diamond, Dale Jorgenson, and Robert Solow for their valuable advice. This paper was given at the Winter 1966 meetings of the Econometric Society under the title, " Technical Change, Depreciation, and Obsolescence."

[2] Throughout this paper, technical change is assumed to be capital-augmenting.

paper, its dual relation is Samuelson's factor price function [10] giving the rent $c(v)$ on machines of vintage v as a function of the wage:

$$c(v) = g(v, w). \qquad ...(1)$$

A central problem in vintage models of production concerns the existence of an aggregate capital stock, usually called J, which is one of the arguments of an aggregate production function of the form:

$$F(J, L) = \max \int_{-\infty}^{t} f(v, I(v), L(v))dv,$$

where the maximum is over all labour allocations $L(v)$ such that $\int L(v)dv = L$. Recently a number of authors (see [3] for a bibliography) have shown that a necessary and sufficient condition for the existence of J is that the vintage production function have the very special form:

$$f(v, I(v), L(v)) = F(z(v)I(v), L(v)). \qquad ...(2)$$

Yet another proof of the basic theorem can be obtained very easily by reference to the dual; the proof is sketched here because it helps later to illustrate the close relation between the capital aggregation problem and the problems which arise in calculating the prices associated with used machines:

By Leontief's theorem [8, 9], the existence of a capital aggregate is equivalent to the condition that the ratio of the marginal physical products of machines of different vintages be independent of the amount of labour. In terms of prices, this means that the ratio of the rents for machines of different vintages should be independent of the wage, or that the factor price function should have the form

$$c(v) = z(v)G(w).$$

But if this is true, the production function has the form

$$f(v, I(v), L(v)) = F(z(v)I(v), L(v)),$$

and *vice versa*.[1]

This condition for aggregation is usually described in terms of technical change—in order to form the aggregate J, embodied technical change must be capital-augmenting. More generally, the vintage coefficient $z(v)$ measures all differences in efficiency which distinguish machines of different vintages—that is, both technical change and physical deterioration enter $z(v)$. Only if condition (2) holds is it possible to separate the effects of technical change and deterioration and to speak unambiguously of units of capital services and the relative efficiency of old machines.

The basic theorem on capital aggregates makes no restriction on the behaviour of the function $z(v)$ over calendar time. From one year to the next, the pattern of efficiency as a function of vintage may change arbitrarily. That is, the efficiency function can be written more generally as $z(t, v)$; this gives the relative efficiency at time t of one machine of vintage v. In most cases, no distinction between technical change and deterioration can be made in this formulation, because z changes completely with time and no regular behaviour which depends only on time or only on age can be found. This formulation is so general as to be almost vacuous—some restriction on the form of $z(t, v)$ is needed.

The simplest restriction which can be made is that $z(t, v)$ is stationary over time—that is, that the efficiency of machines of a particular age is independent of calendar time. This is the case of no technical change; z has the form

$$z(t, v) = \Phi(t-v).$$

[1] The last step follows by defining a new measure of capital $I^*(v) = z(v)I(v)$ with corresponding rent $c^*(v) = c(v)/z(v)$. Then the new factor price function, $G(w)$, is independent of the vintage, v, and so is its uniquely corresponding production function $F(I^*, L)$, which can be written in terms of I as $F(z(v)I(v), L(v))$.

$\Phi(\tau)$ gives the marginal physical product of a machine aged τ years as a fraction of the marginal physical product of a new machine, and thus also gives the corresponding ratio of the rents:

$$\frac{c(t, v)}{c(t, t)} = \Phi(t-v).$$

Under the assumption of no technical change, the formula for J is

$$J = \int_{-\infty}^{t} \Phi(t-v)I(v)dv;$$

this is a conventional formula for calculating capital stock.

Although Φ is often called the depreciation function, in this paper it will be called the deterioration function in order to avoid confusion with the definition of depreciation in terms of the price of used machines which is used in the next section.

A more interesting restriction of the form of $z(t, v)$ can be obtained by relaxing the assumption of stationarity to the extent of allowing changes over time and vintage which are independent of deterioration. Then $z(t, v)$ can be written as the product of a function of time, $d(t)$, a function of vintage, $b(v)$, and the deterioration function $\Phi(t-v)$:

$$z(t, v) = d(t)b(v)\Phi(t-v). \qquad \ldots(3)$$

Both of these new functions have familiar interpretations: $d(t)$ is the index of disembodied technical change and $b(v)$ is the index of embodied technical change. The assumption of independence expressed in the factorization (3) means that neither kind of technical change affects the pattern of deterioration as a machine ages. With this parametrization of technical change, the ratio of the rent on machines of vintage v at time t to the rent on new machines at a base date t_0 is

$$\frac{c(t, v)}{c(t_0, t_0)} = d(t)b(v)\Phi(t-v). \qquad \ldots(4)$$

The corresponding formula for J is

$$J(t) = d(t)\int_{-\infty}^{t} \Phi(t-v)b(v)I(v)dv. \qquad \ldots(5)$$

Examination of the parametrization of technical change, (3), reveals that it has a mathematical property of considerable economic importance—more than one triplet of functions $d(t)$, $b(v)$, and $\Phi(t-v)$ can have the same product $z(t, v)$. This is important because it is $z(t, v)$ which is the basic description of reality; two triplets of technical change and deterioration functions cannot be distinguished by observation or experiment unless they have different products. The parameters of technical change and deterioration which are natural and customary are unfortunately not unique. In short, a problem of identification arises in this model of technical change.

It is easy to characterize the class of triplets of functions of this form which have the same product. Suppose $d(t)$, $b(v)$, and $\Phi(t-v)$ form one triplet. Since all of these functions are required to be positive, any other triplet with the same product can be written $e^{a_1(t)}d(t)$, $e^{a_2(v)}b(v)$ and $e^{a_3(t-v)}\Phi(t-v)$ with the condition that $a_1(t)+a_2(v)+a_3(t-v) = 0$. Then for any period of time of length T,

$$a_1(t+T)-a_1(t) = -a_3(t-v+T)+a_3(t-v).$$

Reducing v by the same T, we have

$$a_2(v)-a_2(v-T) = a_3(t-v+T)-a_3(t-v).$$

Thus,

$$a_1(t+T)-a_1(t) = -(a_2(v)-a_2(v-T)), \qquad \ldots(6)$$

or the difference in $a_1(t)$ is independent of t and the difference in $a_2(v)$ is independent of v. The only functions with this property are linear:

$$a_1(t) = \alpha_1+\beta_1 t \text{ and } a_2(v) = \alpha_2+\beta_2 v.$$

α_1 and α_2 may be set equal to zero without loss of generality—this amounts to setting all equivalent indexes equal at time zero. Substitution in (6) shows that $\beta_1 = -\beta_2 = \beta$. β can be any number, positive or negative. Finally.

$$\begin{aligned} a_3(t-v) &= (a_1(t)+a_2(v)) \\ &= -\beta \,.\, (t-v). \end{aligned}$$

Thus all members of a class of equivalent triplets have the form $e^{\beta t}d(t)$, $e^{-\beta v}b(v)$ and $e^{-\beta(t-v)}\Phi(t-v)$, where $d(t)$, $b(v)$ and $\Phi(t-v)$ form an arbitrary member of the class. This means that given rates of embodied and disembodied technical change and a given deterioration function cannot be distinguished from a lower rate of embodied technical change, a higher rate of disembodied change, and a higher rate of deterioration.

In order to escape this ambiguity, some normalization rule must be adopted so that each $z(t, v)$ can be represented by at most one triplet $d(t)$, $b(v)$ and $\Phi(t-v)$. With such a rule, the notions of embodied technical change, disembodied technical change, and deterioration each have a well-defined meaning and each can be measured from suitable observations. Without it, serious confusion can develop in both theoretical and empirical work.

A normalization rule may be applied to any one of the three functions $d(t)$, $b(v)$ and $\Phi(t-v)$. For a period of observation $t_0 \leqq t \leqq t_1$, convenient rules are provided by setting net technical change of one kind equal to zero over the period—either $b(t_1) = b(t_0)$ or $d(t_1) = d(t_0)$. Alternatively, it is possible to require that no net deterioration take place over the first τ_0 years of the life of a machine: $\Phi(\tau_0) = \Phi(0) = 1$, although this is a less natural normalization. Each of these rules makes it clear that disembodied technical change, embodied technical change, and deterioration are three distinct phenomena only to the extent that they all are different from exponential functions. For example, in his well-known applications of the embodied technical change model [11 and 12], Solow assumes

$$d(t) = 1$$

$$b(v) = e^{bv}$$

$$\Phi(t-v) = e^{-\delta(t-v)}.$$

This corresponds to the normalization rule requiring that no disembodied change take place over the period. The alternative form of the model for no embodied change is

$$d(t) = e^{bt}$$

$$b(v) = 1$$

$$\Phi(t-v) = e^{-(\delta+b)(t-v)}.$$

Curiously enough, all of the empirical work in Solow's papers could have been carried out without any reference to embodied technical change. This is a consequence of his choice of functional forms for technical change and deterioration—it is incorrect to conclude that embodied and disembodied technical change can never be distinguished. Only the exponential parts of these functions cannot be distinguished.

II. THE PRICE OF MACHINES AND THE PRICE OF MACHINES' SERVICES

The goal of this section is to derive an expression for the price of a used machine, $p(\tau, t)$, as a function of the age of the machine and calendar time. From this, the rent will be calculated by applying the equation of yield. The analysis is carried out first for the case of no technical change and then is generalized to take account of technical change.

Suppose that an interest rate $r(t)$ prevails, so receipts in year u are discounted to year t by the ratio

$$e^{-R(t,u)} = e^{-\int_t^u r(s)ds}.$$

Now let $c_0(t)$ be the rent on a new machine of vintage t at time t. In the absence of technical change, suppose that

$$c(t, v) = \Phi(t-v)c_0(t).$$

From the proof of the basic theorem on capital aggregates, one can see that this will hold if and only if the conditions for forming a capital aggregate hold. This assumption results in a crucial simplification of the theory of the determination of the price $p(\tau, t)$. It is important to emphasize that the assumption that there is a general rent for capital, independent of vintage, is exactly as restrictive as the assumption that there is an aggregate capital stock.

The price of machines, $p(\tau, t)$, is subject to the boundary condition that the price of machines of age zero be equal to the exogenously determined supply price: $p(0, t) = p_0(t)$. This assumption is extremely important in the derivation of the price function $p(\tau, t)$—its consequences will be discussed later. It is worth pointing out that the fact that the rent for an old machine has a fixed relation to the rent for a new one does not imply that the price of an old machine has a fixed relation to the price of a new one.

The fundamental behavioural assumption which determines the price of used machines is that firms buy and sell machines (invest and disinvest) so as to maximize the present value of the firm. From this assumption Jorgenson [6] and Arrow [1] have derived a theory of optimal investment by applying the calculus of variations. The rent appears as a byproduct dual variable in their formulation; the price function could be deduced from the rent, but this has never been done except in the very simple case of exponential deterioration.

The dual of the assumption of maximization of present value can be expressed in the form of the following fundamental equation governing the price and rent:

$$p(\tau, t) = \int_0^T e^{-R(t,t+u)}c_0(t+u)\Phi(\tau+u)du + e^{-R(t,t+T)}p(\tau+T, t+T), \qquad \ldots(7)$$

for all t, all $\tau \geqq 0$ and all $T \geqq 0$. In words, the price at time t of a machine aged τ years is equal to the present value of the rent it will earn over the period from t to $t+T$ plus the discounted resale value at the end of the period. This form of the fundamental equation is essentially the same as the one proposed by Hotelling [5] except that resale is allowed at any time and machines are assumed to have no scrap value. Hotelling differentiated an equation like (7) to obtain an expression for $c_0(t)$ in terms of p and its derivatives. In the present case, we can differentiate with respect to T to obtain

$$\begin{aligned} 0 = {} & e^{-R(t,t+T)}c_0(t+T)\Phi(\tau+T) \\ & -r(t+T)e^{-R(t,t+T)}p(\tau+T, t+T) \\ & +e^{-R(t,t+T)}\left(\frac{\partial p}{\partial \tau} + \frac{\partial p}{\partial t}\right). \end{aligned}$$

Without loss of generality, we set $T = 0$, or

$$\Phi(\tau)c_0(t) = r(t)p(\tau, t) - \frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial t}. \qquad \ldots(8)$$

This is the well-known equation of yield for the case of physical capital—the rent for a machine aged τ years is equal to the interest cost $r(t)p(\tau, t)$ plus depreciation, $-\frac{\partial p}{\partial \tau}$, less capital gains, $\frac{\partial p}{\partial t}$. For new machines, substitute the boundary condition, so that

$$c_0(t) = r(t)p_0(t) - \left.\frac{\partial p}{\partial \tau}\right|_{\tau=0} - \dot{p}_0(t). \qquad \text{...(9)}$$

Unfortunately the depreciation term $-\left.\frac{\partial p}{\partial \tau}\right|_{\tau=0}$ in (9) involves the yet-undetermined function $p(\tau, t)$, so this approach is not directly useful in finding explicit formulas for $p(\tau, t)$ or $c_0(t)$.

In fact, in order to find such formulas, it is necessary to make an additional assumption about the behaviour of the price:

$$\lim_{T\to\infty} e^{-R(t,t+T)}p(T, t+T) = 0. \qquad \text{...(10)}$$

That is, the present value of the future price must tend to zero as the future resale date tends to infinity. This assumption rules out speculation in old machines.[1] Under this assumption the possibility of resale can be ignored by taking (7) to the limit as $T\to\infty$:

$$p(\tau, t) = \int_0^\infty e^{-R(t,t+u)}c_0(t+u)\Phi(\tau+u)du. \qquad \text{...(11)}$$

In order to solve for $p(\tau, t)$ as a function of known variables, first define the mortality function for machines:

$$\phi(\tau) = -\frac{d}{d\tau}\Phi(\tau); \qquad \text{...(12)}$$

$\phi(\tau)$ is the rate at which the machine is wearing away at age τ. Second define the replacement density function for used machines: [2]

$$\psi(\tau, s) = \sum_{j=0}^{\infty} v_j(\tau, s) \qquad \text{...(13)}$$

where

$$v_0(\tau, s) = \phi(s+\tau)$$

and

$$v_j(\tau, s) = \int_0^s v_{j-1}(\tau, z)\phi(s-z)dz.$$

$\psi(\tau, s)$ is the rate at which new machines must be bought to maintain a stock equivalent to one machine which was aged τ years when it was bought s years ago.

Now by substituting the boundary condition in (11), we have

$$p_0(t) = \int_0^\infty e^{-R(t,t+u)}c_0(t+u)\Phi(u)du. \qquad \text{...(14)}$$

By multiplying the two sides of (14) by $\psi(\tau, s)$, taking the present value of both sides and then adding (11), we obtain the following interesting alternative form of the fundamental equation:

[1] This condition arises as the transversality condition in the corresponding optimal investment problem. For a discussion of the circumstances under which it holds in a competitive economy, see [4].

[2] The importance of the replacement density function in this application was suggested by Arrow's paper [1].

$$p(\tau, t)+\int_0^\infty e^{-R(t,t+s)}\psi(\tau, s)p_0(t+s)ds = \Phi(\tau)\int_0^\infty e^{-R(t,t+u)}c_0(t+u)du. \qquad \ldots(15)$$

(See the appendix for the details of this step.)

That is, the cost of a machine today plus the present value of the cost of maintaining a stock equivalent to that machine forever is equal to the present value of the rent from a stock equivalent to that machine. Their common value is the market value of a contract to provide $\Phi(\tau)$ units of machine services forever.

Substituting the boundary condition in (15), we find

$$p_0(t)+\int_0^\infty e^{-R(t,t+s)}\psi(0, s)p_0(t+s)ds = \int_0^\infty e^{-R(t,t+u)}c_0(t+u)du. \qquad \ldots(16)$$

By multiplying (16) by $\Phi(\tau)$ and subtracting it from (15), we obtain the solution

$$p(\tau, t) = \Phi(\tau)p_0(t)-\int_0^\infty e^{-R(t,t+s)}[\psi(\tau, s)-\Phi(\tau)\psi(0, s)]p_0(t+s)ds. \qquad \ldots(17)$$

The solution shows that the price of a used machine is the fraction of the services of a new machine which the old one provides times the price of a new machine less an adjustment for the extent to which buying a used machine rather than part of a new machine will require more replacement expenditures in the future. It is possible to show that the second term is identically zero if and only if the deterioration function is exponential: $\Phi(\tau) = e^{-\delta\tau}$. Only in this special case is the price of machines today independent of the future values of the interest rate and the price of new machines.

By differentiating (17) with respect to τ and integrating by parts, we obtain the following formula for depreciation:

$$-\frac{\partial p}{\partial \tau} = \int_0^\infty e^{-R(t,t+s)}\psi(\tau, s)[r(t+s)p_0(t+s)-\dot{p}_0(t+s)]ds. \qquad \ldots(18)$$

Similarly, capital gains are given by

$$\frac{\partial p}{\partial t} = \Phi(\tau)\dot{p}_0(t)-r(t)[\Phi(\tau)p_0(t)-p(\tau, t)]$$
$$+\int_0^\infty e^{-R(t,t+s)}[\psi(\tau, s)-\Phi(\tau)\psi(0, s)][r(t+s)p_0(t+s)-\dot{p}_0(t+s)]ds. \qquad \ldots(19)$$

From (8), rent is given by

$$c_0(t) = \frac{1}{\Phi(\tau)}\left[r(t)p(\tau, t)-\frac{\partial p}{\partial \tau}-\frac{\partial p}{\partial t}\right].$$

Substituting (18) and (19) and simplifying, we have

$$c_0(t) = r(t)p_0(t)+\int_0^\infty e^{-R(t,t+s)}\psi(0, s)[r(t+s)p_0(t+s)-\dot{p}_0(t+s)]ds-\dot{p}_0(t); \qquad \ldots(20)$$

$c_0(t)$ is independent of τ, as required. By comparing (20) with (9), we can interpret the terms; the first is interest cost on new machines, the second is depreciation on new machines, and the third is capital gains on new machines.

Introducing technical change complicates the analysis only slightly. A machine aged τ years is equivalent to $\Phi(\tau)\dfrac{b(t-\tau)}{b(t)}$ new machines at time t. The fundamental equation governing the market is

$$p(\tau, t) = \int_0^\infty e^{-R(t,t+u)}c_0(t+u)\Phi(\tau+u)\frac{b(t-\tau)}{b(t+u)}du. \qquad \ldots(21)$$

From this equation, the ambiguity in the parametrization of technical change may be demonstrated again, although formally speaking this is redundant. More than one pair of functions $\Phi(\tau)$ and $b(t)$ lead to the same value for $\Phi(\tau+u)\dfrac{b(t-\tau)}{b(t+u)}$. By very much the same argument as in section 2, the ambiguity can be seen to take the form of an exponential function: $e^{-\beta\tau}\Phi(\tau)$ and $e^{\beta t}b(t)$ cannot be distinguished from $\Phi(\tau)$ and $b(t)$.

Equation (21) can be written as

$$\frac{p(\tau,\, t)}{b(t-\tau)}=\int_0^\infty e^{-R(t,\,t+u)}\frac{c_0(t+u)}{b(t+u)}\Phi(\tau+u)du.$$

This puts it in exactly the form of equation (11); the solution is obtained by substituting for the appropriate variables in (17) and rearranging:

$$p(\tau,\, t)=\Phi(\tau)\frac{b(t-\tau)}{b(t)}p_0(t)-\int_0^\infty e^{-R(t,\,t+s)}[\psi(\tau,\, s)-\Phi(\tau)\psi(0,\, s)]\frac{b(t-\tau)}{b(t+s)}p_0(t+s)ds. \quad \ldots(22)$$

Similarly, the rent is

$$c_0(t)=r(t)p_0(t)+p_0(t)\frac{\dot{b}(t)}{b(t)}$$

$$+\int_0^\infty e^{-R(t,\,t+s)}\psi(0,\, s)\frac{b(t)}{b(t+s)}[r(t+s)p_0(t+s)-\dot{p}_0(t+s)+\frac{\dot{b}(t+s)}{b(t+s)}p_0(t+s)]ds-\dot{p}_0(t). \ldots(23)$$

The term $p_0(t)\dfrac{\dot{b}(t)}{b(t)}$ which is added to the depreciation component of the rent may be interpreted as the cost of obsolescence, but because of the identification problem, $\dfrac{\dot{b}(t)}{b(t)}$ is known only down to the arbitrary additive constant β. That is, for exactly the same reason that embodied technical change and deterioration cannot be distinguished unambiguously, the part of the cost of machines' services associated with aging cannot be separated unambiguously into depreciation and obsolesence. Thus, businessmen are correct in entering depreciation plus obsolescence in their income statements, without attempting to separate the two.

Some comment on the boundary condition equating the supply price of machines and the price of machines of age zero is now in order. This boundary condition ensures that the prices and rent for machines are independent of the circumstances in which they are used in production—the markets for machines and machines' services are driven by the supply price and the interest rate. This assumption corresponds to the assumption of no negative gross investment which Arrow requires in his treatment of the optimal investment problem. On the other hand, if no new machines are sold during an interval of time, as would occur if the supply price were much higher than usual during the interval, the boundary condition does not hold and the markets work in reverse—machines are in inelastic supply and their rent is determined by the demand for the output produced with them.[1] This point has considerable importance in empirical work on investment; it suggests that the rent should be calculated from data from markets for used machines, rather than from the prices quoted by the sellers of new machines.

III. MEASURING TECHNICAL CHANGE AND DETERIORATION

Some of the critics of the theory of embodied technical change (notably Jorgenson [7]) have argued that the theory has no operational content because technical change cannot

[1] In a more recent paper [2], Arrow presents an algorithm for determining exactly when the boundary condition does not hold.

be measured from the data usually available to econometricians, or because the hypotheses of the theory are irrefutable with the data available. This section will show that if data on the prices of used machines and the interest rate are available, then the index of embodied technical change and the deterioration function can in fact be calculated from these data. Further, with sufficient data, they can be calculated in more than one way; if the results disagree, at least one of the hypotheses of the theory is refuted—technical change is not capital augmenting or is not independent of deterioration, the market is not competitive, or foresight is not perfect. In other words, calculation of the index of technical change and the deterioration function does not always exhaust the degrees of freedom of the data, so the calculation can provide a test of the hypotheses underlying the theory. These results are stated formally in:

Identification Theorem I. Given any price function $p(\tau, t)$ and the interest rate $r(t)$ for a period of time $0 \leqq t \leqq t_0$ and a range of ages $0 \leqq \tau \leqq t_0$, there is at most one deterioration function $\Phi(\tau)$ and one index of embodied technical change $b(t)$ consistent with the price function and meeting the normalization $\Phi(0) = 1$ and $b(0) = b(t_0) = 1$.

Proof. Estimates can be derived by applying the equation of yield:

$$r(t)p(\tau, t) - \frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial t} = \frac{b(t-\tau)}{b(t)} \Phi(\tau) c_0(t).$$

For convenience, we define the capital loss function

$$L(\tau, t) = -\frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial t}.$$

Dividing each side by its own value with $\tau = 0$, we get an equation in relative efficiencies:

$$\frac{r(t)p(\tau, t) + L(\tau, t)}{r(t)p_0(t) + L(0, t)} = \frac{b(t-\tau)}{b(t)} \Phi(\tau).$$

Call the log of the left side $y(\tau, t)$ and let $b^*(t) = \log b(t)$ and $\Phi^*(\tau) = \log \Phi(\tau)$. Then

$$y(\tau, t) = b^*(t-\tau) + \Phi^*(\tau) - b^*(t).$$

The proof is constructive and consists in stating a formula which can be used to calculate a new value of b^* and a new value of Φ^*, given two values of b, $b(t_1)$ and $b(t_2)$. We obtain the formula by solving the equation of yield for two points:

$$\begin{aligned} y(\tau, t_2) &= b^*(t_1+\tau) + \Phi^*(\tau) - b^*(t_2) \\ y(\tau, t_1+\tau) &= b^*(t_1) + \Phi^*(\tau) - b^*(t_1+\tau), \end{aligned}$$

where $\tau = \dfrac{t_2 - t_1}{2}$.

The solution is

$$\begin{aligned} \Phi^*(\tau) &= \tfrac{1}{2}[y(\tau, t_1+\tau) + y(\tau, t_2) - b^*(t_1) + b^*(t_2)] \\ b^*(t_1+\tau) &= y(\tau, t_2) - \Phi^*(\tau) + b^*(t_2). \end{aligned}$$

Since no division is involved, the new values are always unique. The process can be started by setting $t_1 = 0$ and $t_2 = t_0$. After the first step, the new values of Φ^* are each calculated more than once from independent data, so the hypotheses of the theory are checked at each step. Values of $\Phi^*(\tau)$ for $\tau > \dfrac{t_0}{2}$ can be calculated from another application of the equation of yield:

$$\Phi^*(t_2 - t_1) = y(t_2 - t_1, t_2) - b^*(t_1) + b^*(t_2).$$

An econometric problem of equal or perhaps greater importance is to calculate the index of technical change, the deterioration function, and the interest rate from data on prices alone. This is a difficult undertaking, particularly since the estimation equations are intrinsically nonlinear. Before embarking on a study of this kind, we should verify that the problem is identified, at least in the weak sense that the estimation equations are not identically singular for all price data. Identical singularity would arise if there were a formal difficulty in the estimation problem, as there would be, for example, in estimating technical change without introducing the appropriate normalization. The following theorem states that if an estimation procedure fails, it is because of the particular characteristics of the price data and not because of an inherent singularity, with one important exception:

Identification Theorem II. Estimation of the interest rate, the index of embodied technical change, and the deterioration function is identically singular for all prices if and only if both technical change and deterioration are exponential functions.

Proof. The proof consists in showing that the analogue for this problem of the interpolation method of the proof of identification theorem I is not identically singular except in the exponential case. More precisely, we show that the Jacobian matrix of the set of nonlinear equations for the problem is not identically singular.

The simplest possible interpolation method is to calculate 3 values of the interest rate, 4 values of the deterioration function, and 5 values of the index of technical change from 12 observations on $p(\tau, t)$, starting from two known values of the index, $b(t_1)$ and $b(t_2)$. Let $\tau_1 = \frac{1}{2}(t_2-t_1)$, $\tau_2 = t_2-t_1$, $\tau_3 = \frac{3}{2}(t_2-t_1)$ and $\tau_4 = 2(t_2-t_1)$. Then the unknowns in the problem are $r(t_1)$, $r(t_1+\tau_1)$, $r(t_2)$, $\Phi(\tau_i)$, $i = 1, \ldots, 4$, $b(t_1-\tau_1)$, $i = 1, \ldots, 4$, and $b(t_1+\tau_1)$. The system of equations is given by

$$\frac{r(t_j)p(\tau_i, t_j)+L(\tau_i, t_j)}{r(t_j)p(0, t_j)+L(0, t_j)} = \frac{b(t_j-\tau_i)}{b(t_j)}\Phi(\tau_i),$$

$$i = 1, \ldots, 4, j = 1, 2, 3 \text{ and } t_3 = \frac{t_1+t_2}{2}.$$

Taking logs as before, we have

$$\log\left[r(t_j)p(\tau_i, t_j)+L(\tau_i, t_j)\right]-\log\left[r(t_j)p(0, t_j)+L(0, t_j)\right] = b^*(t_j-\tau_i)+\Phi^*(\tau_i)-b^*(t_j).$$

The Jacobian matrix for this system has 12 rows corresponding to the 12 observations on $p(\tau, t)$ and 12 columns, one for each unknown. Each row has exactly one term of the form

$$\frac{p(\tau_i, t_j)}{r(t_j)p(\tau_i, t_j)+L(\tau_i, t_j)} - \frac{p(0, t_j)}{r(t_j)p(0, t_j)+L(0, t_j)},$$

and two zeros in the first three columns; all of the elements of the last 9 columns are zeros, +1's, or −1's. Since the last 9 columns are known to be linearly independent from identification theorem I, any identical linear dependence must arise from the condition:

$$\frac{p(\tau, t)}{r(t)p(\tau, t)+L(\tau, t)} - \frac{p(0, t)}{r(t)p(0, t)+L(0, t)} = k$$

for all τ. Now consider a sequence of τ's converging to zero. From the continuity of $p(\tau, t)$ and $L(\tau, t)$ (both of which may be represented as integrals), we can see that $k = 0$; substituting this value, we find that the terms involving $r(t)$ cancel, leaving

$$\frac{p(\tau, t)}{L(\tau, t)} = \frac{p(0, t)}{L(0, t)} \text{ for all } \tau, t. \qquad \ldots(24)$$

From equations (18) and (19),

$$L(\tau, t) = \Phi(\tau)[L(0, t)+r(t)p_0(t)]-r(t)p(\tau, t).$$

Substituting in (24) and cross-multiplying, we have

$$p(\tau, t)L(0, t) = p_0(t)\Phi(\tau)[L(0, t)+r(t)p_0(t)]-p_0(t)r(t)p(\tau, t),$$

or

$$p(\tau, t) = p_0(t)\Phi(\tau).$$

From the discussion following equation (17), and from equation (22), we can see that this is true if and only if both

$$\Phi(\tau) = e^{-\delta\tau}.$$

for some constant, δ, and

$$b(t) = 1.$$

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APPENDIX

We seek to show that the fundamental equation,

$$p(\tau, t) = \int_0^\infty e^{-R(t, t+u)}c_0(t+u)\Phi(\tau+u)du \qquad ...(11)$$

and its corollary,

$$p_0(t) = \int_0^\infty e^{-R(t, t+u)}c_0(t+u)\Phi(u)du \qquad ...(14)$$

imply the alternative form,

$$p(\tau, t)+\int_0^\infty e^{-R(t, t+s)}\psi(\tau, s)p_0(t+s)ds = \Phi(\tau)\int_0^\infty e^{-R(t, t+u)}c_0(t+u)du. \qquad ...(15)$$

We begin by multiplying both sides of equation (14) by $e^{-R(t, t+s)}\psi(\tau, s)$ and integrating to get

$$\int_0^\infty e^{-R(t, t+s)}\psi(\tau, s)p_0(t+s)ds$$
$$= \int_0^\infty e^{-R(t, t+s)}\psi(\tau, s)\int_0^\infty e^{-R(t+s, t+s+u)}c_0(t+s+u)\Phi(u)duds. \qquad ...(A1)$$

Next, on the right-hand side, we change the limit of integration and combine the discount terms to get

$$RHS = \int_0^\infty\int_s^\infty e^{-R(t, t+u)}\Phi(u-s)\psi(\tau, s)c_0(t+u)duds. \qquad ...(A2)$$

Now by changing the order of integration, we obtain a new form for equation ($A1$):

$$\int_0^\infty e^{-R(t, t+s)}\psi(\tau, s)p_0(t+s)ds = \int_0^\infty e^{-R(t, t+u)}c_0(t+u)\int_0^u \Phi(u-s)\psi(\tau, s)dsdu. \qquad ...(A3)$$

Next we add equation (11) to equation ($A3$) to obtain

$$p(\tau, t)+\int_0^\infty e^{-R(t, t+s)}\psi(\tau, s)p_0(t+s)ds$$
$$= \int_0^\infty e^{-R(t, t+u)}c_0(t+u)[\Phi(\tau+u)+\int_0^u \psi(\tau, s)\Phi(u-s)ds]du. \qquad ...(A4)$$

Now

$$\int_0^u \psi(\tau, s)\Phi(u-s)ds = \int_0^u [\psi(\tau, s) - \int_0^s \psi(\tau, z)\phi(s-z)dz]ds; \qquad \ldots(A5)$$

This can be shown by differentiating both sides with respect to u and observing that when $u = 0$ both sides are equal. Further,

$$\begin{aligned} \int_0^s \psi(\tau, z)\phi(s-z)dz &= \int_0^s \sum_{j=0}^{\infty} v_j(\tau, z)\phi(s-z)dz \\ &= \sum_{j=0}^{\infty} v_{j+1}(\tau, s) \\ &= \psi(\tau, s) - \phi(s+\tau). \end{aligned} \qquad \ldots(A6)$$

Thus,

$$\int_0^u \psi(\tau, s)\Phi(u-s)ds = \int_0^u \phi(s+\tau)ds = \Phi(\tau) - \Phi(\tau+u). \qquad \ldots(A7)$$

By substituting ($A7$) in ($A4$), we obtain the desired result:

$$p(\tau, t) + \int_0^{\infty} e^{-R(t, t+s)}\psi(\tau, s)p_0(t+s)ds = \Phi(\tau)\int_0^{\infty} e^{-R(t, t+u)}c_0(t+u)du. \qquad \ldots(15)$$

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