CME 305: Discrete Mathematics and Algorithms
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Midterm Practice Problems – 02/14/17

Probabilistic Method

Let X be a non-negative random variable with expectation $\mathbb{E}[X] = \mu \geq 0$ and variance σ^2. Prove that for all $\lambda > 0$,

$$\Pr(X \geq \lambda) \leq \frac{\lambda\mu + \sigma^2}{\lambda^2 + \sigma^2}.$$

Let F be a finite collection of binary strings of length at most k and assume no member of F is a prefix of another one. Let N_i be the number of strings of length i in F. Prove that

$$\sum_{i=1}^{k} \frac{N_i}{2^i} \leq 1.$$

Hint: consider generating a random binary string of length k.

Imagine n cars, each of which travels at a different speed. Initially, the cars are queued in uniform random order at the start of a semi-infinite, one-lane highway. Each car drives at the minimum of its maximum speed and the speed at which the car in front of it is driving. The cars will form "clumps". What is the expected number of clumps?

Flow

At lunchtime it's crucial for people to get to the food trucks as quickly as possible. The building is represented by a graph $G = (V, E)$ where each room, landing, or other location is represented by a vertex and each corridor or stairway is represented by an edge. Each corridor has associated capacity c, meaning that at most c people can pass through the corridor at once. Traversing a corridor from one end to the other takes one timestep and people can decide to stay in a room for the entire timestep.

Suppose all people are initially in a single room s, and that the building has a single exit t. Give a polynomial time algorithm to find the fastest way to get everyone out of the building.

Trees

Let T be a tree. Let $\Delta(T) = d$, i.e. max-degree is d. Show that T has at least d leaves.

A minimum bottleneck spanning tree of a weighted graph G is a spanning tree of G that minimizes the maximum weight of any edge in the spanning tree. Prove that a minimum spanning tree is also minimum bottleneck spanning tree, but that the converse is not true.

Basic Graph Theory and Definitions

In a village there are three schools with n students in each of them. Every student from any of the schools is on speaking terms with at least $n + 1$ students from the other two schools. Show that we can find three students, no two from the same school, who are on speaking terms with each other.

A connected k-regular graph on n nodes is one in which all vertices have degree k. Prove that the diameter of a connected k-regular graph is $O(n/k)$.

Show that every graph on m edges has a subgraph on at least $m/2$ edges which is bipartite.

Prove that a graph with $2n$ nodes and minimum degree n must be connected.