Course overview

(Mainly) noncooperative game theory.

**Noncooperative:**
Focus on individual players’ incentives
(note these might lead to cooperation!)

**Game theory:**
Analyzing the behavior of rational, self interested players
What’s in a game?

1. **Players**: Who?
2. **Strategies**: What actions are available?
3. **Rules**: How? When? What do they know?
4. **Outcomes**: What results?
5. **Payoffs**:
   How do players evaluate outcomes of the game?
Example: Chess

1. **Players**: Chess masters
2. **Strategies**: Moving a piece
3. **Rules**: How pieces are moved/removed
4. **Outcomes**: Victory or defeat
5. **Payoffs**:
   - Thrill of victory,
   - Agony of defeat
Rationality

Players are *rational* and *self-interested*:

They will *always* choose actions that maximize their payoffs, given everything they know.
Static games

We first focus on *static games*.

(one-shot games, simultaneous-move games)

For any such game, the rules say:

*All players must simultaneously pick a strategy.*

This immediately determines an outcome, and hence their payoff.
Knowledge

• All players know the *structure of the game*: players, strategies, rules, outcomes, payoffs
Common knowledge

- All players know the structure of the game
  - All players know all players know the structure
    - All players know all players know all players know the structure
      and so on... ⇒

We say: the structure is *common knowledge*. This is called *complete information*. 
PART I: Static games of complete information
Representation

- $N$: # of players
- $S_n$: strategies available to player $n$
- Outcomes:
  Composite strategy vectors
- $\Pi_n(s_1, \ldots, s_N)$:
  payoff to player $n$ when player $i$ plays strategy $s_i$, $i = 1, \ldots, N$
Example: A routing game

**MCI and AT&T:**

A Chicago customer of MCI wants to send 1 MB to an SF customer of AT&T.

A LA customer of AT&T wants to send 1 MB to an NY customer of MCI.

Providers minimize their own cost.

**Key:** MCI and AT&T only exchange traffic ("peer") in NY and SF.
Example: A routing game

Costs (per MB): Long links = 2; Short links = 1
Example: A routing game

Players: MCI and AT&T \( (N = 2) \)

Strategies: Choice of traffic exit
\[ S_1 = S_2 = \{ \text{nearest exit, furthest exit} \} \]

Payoffs:
Both choose furthest exit: \( \Pi_{MCI} = \Pi_{AT&T} = -2 \)
Both choose nearest exit: \( \Pi_{MCI} = \Pi_{AT&T} = -4 \)
MCI chooses near, AT&T chooses far:
\[ \Pi_{MCI} = -1, \; \Pi_{AT&T} = -5 \]
Example: A routing game

Games with $N = 2$, $S_n$ finite for each $n$ are called *bimatrix games*. 

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<th>AT&amp;T</th>
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<tr>
<td>near</td>
<td>(-4,-4)</td>
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<td>(-5,-1)</td>
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MCI

AT&T

far

near

(-2,-2)
Example: Matching pennies

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<tr>
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<tr>
<td><strong>Player 1</strong></td>
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<tr>
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<td>(1,-1)</td>
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This is a zero-sum matrix game.
Dominance

\( s_n \in S_n \) is a (weakly) dominated strategy if

there exists \( s_n^* \in S_n \) such that

\[ \Pi_n(s_n^*, s_{-n}) \geq \Pi_n(s_n, s_{-n}), \]

for any choice of \( s_{-n} \), with

strict ineq. for at least one choice of \( s_{-n} \).

If the ineq. is always strict, then \( s_n \) is a strictly dominated strategy.
Dominance

\[ s_n^* \in S_n \text{ is a } \text{weak dominant strategy} \text{ if} \]
\[ s_n^* \text{ weakly dominates all other } s_n \in S_n. \]

\[ s_n^* \in S_n \text{ is a } \text{strict dominant strategy} \text{ if} \]
\[ s_n^* \text{ strictly dominates all other } s_n \in S_n. \]

(Note: dominant strategies are unique!)
Dominant strategy equilibrium

\[ s \in S_1 \times \cdots \times S_N \text{ is a strict (or weak) dominant strategy equilibrium} \]

if \( s_n \) is a strict (or weak) dominant strategy for each \( n \).
## Back to the routing game

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Nearest exit is strict dominant strategy for MCI.
Back to the routing game

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Nearest exit is strict dominant strategy for AT&T.
Back to the routing game

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<tr>
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Both choosing nearest exit is a strict dominant strategy equilibrium.
Example: Second price auction

- $N$ bidders
- Strategies: $S_n = [0, \infty); \ s_n = \text{“bid”}$
- Rules & outcomes:
  High bidder wins, pays second highest bid
- Payoffs:
  - Zero if a player loses
  - If player $n$ wins and pays $t_n$, then
    $$\Pi_n = v_n - t_n$$
  - $v_n$ : valuation of player $n$
Example: Second price auction

- **Claim**: *Truthful bidding* ($s_n = v_n$) is a weak dominant strategy for player $n$.

- **Proof**:
  
  If player $n$ considers a bid $> v_n$:
  
  Payoff may be lower when $n$ wins, and the same (zero) when $n$ loses.
Example: Second price auction

- **Claim:** *Truthful bidding* \( (s_n = v_n) \) is a weak dominant strategy for player \( n \).

- **Proof:**
  - If player \( n \) considers a bid \( < v_n \):
    - Payoff will be same when \( n \) wins, but may be worse when \( n \) loses.
Example: Second price auction

• We conclude:

Truthful bidding is a (weak) dominant strategy equilibrium for the second price auction.
Example: Matching pennies

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No dominant/dominated strategy exists!
Moral: *Dominant strategy eq. may not exist.*
Iterated strict dominance

Given a game:

• Construct a new game by *removing* a strictly dominated strategy from one of the strategy spaces $S_n$.

• Repeat this procedure until no strictly dominated strategies remain.

If this results in a unique strategy profile, the game is called *dominance solvable*.
Iterated strict dominance

- Note that the bidding game in Lecture 1 was dominance solvable.

- There the unique resulting strategy profile was (6,6).
### Example

<table>
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<tr>
<th>Player 1</th>
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<th>Player 2</th>
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<tbody>
<tr>
<td></td>
<td>Left</td>
<td>Middle</td>
</tr>
<tr>
<td><strong>Up</strong></td>
<td>(1,0)</td>
<td>(1,2)</td>
</tr>
<tr>
<td><strong>Down</strong></td>
<td>(0,3)</td>
<td>(0,1)</td>
</tr>
</tbody>
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Example

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## Example

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</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>Left: (1,0)</td>
</tr>
<tr>
<td>Down</td>
<td>Left: (0,3)</td>
</tr>
</tbody>
</table>
Example

Thus the game is dominance solvable.
Example: Cournot duopoly

- Two firms ($N = 2$)
- *Cournot competition*: each firm chooses a quantity $s_n \geq 0$
- Cost of producing $s_n$: $c_s n$
- *Demand curve*:
  \[
  \text{Price} = P(s_1 + s_2) = a - b \ (s_1 + s_2)
  \]
- Payoffs:
  \[
  \text{Profit} = \Pi_n(s_1, s_2) = P(s_1 + s_2) \ s_n - c \ s_n
  \]
Example: Cournot duopoly

• Claim:
  The Cournot duopoly is dominance solvable.

• Proof technique:
  First construct the best response for each player.
Best response

*Best response set* for player $n$ to $s_{-n}$:

$$R_n(s_{-n}) = \arg \max_{s_n \in S_n} \Pi_n(s_n, s_{-n})$$

[Note: $\arg \max_{x \in X} f(x)$ is the set of $x$ that maximize $f(x)$]
Example: Cournot duopoly

Calculating the best response given $s_n$:

$$\max_{s_n \geq 0} [(a - bs_n - bs_{-n})s_n - cs_n] \implies$$

Differentiate and solve:

$$a - c - bs_{-n} - 2bs_n = 0$$

So:

$$R_n(s_{-n}) = \left[ \frac{a - c}{2b} - \frac{s_{-n}}{2} \right]^+$$
Example: Cournot duopoly

For simplicity, let \( t = (a - c)/b \)
Example: Cournot duopoly

Step 1: Remove strictly dominated $s_1$.

All $s_1 > t/2$ are strictly dominated by $s_1 = t/2$
Example: Cournot duopoly

Step 2: Remove strictly dominated $s_2$.

All $s_2 > t/2$ are strictly dominated by $s_2 = t/2$...
Example: Cournot duopoly

Step 2: Remove strictly dominated $s_2$.

All $s_2 > t/2$ are strictly dominated by $s_2 = t/2$...

...and all $s_2 < t/4$ are strictly dominated by $s_2 = t/4$. 
Example: Cournot duopoly

Step 3: Remove strictly dominated $s_1$. 

$$R_1(s_2)$$

$$R_2(s_1)$$
Example: Cournot duopoly

Step 4: Remove strictly dominated $s_2$. 

\[
R_1(s_2) \\
R_2(s_1)
\]
Example: Cournot duopoly

The process converges to the intersection point: \( s_1 = t/3, \ s_2 = t/3 \)

<table>
<thead>
<tr>
<th>Step #</th>
<th>Undominated ( s_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0, ( t/2 )]</td>
</tr>
<tr>
<td>3</td>
<td>[( t/4 ), ( 3t/8 )]</td>
</tr>
<tr>
<td>5</td>
<td>[( 5t/16 ), ( 11t/32 )]</td>
</tr>
<tr>
<td>7</td>
<td>[( 21t/64 ), ( 43t/128 )]</td>
</tr>
</tbody>
</table>
Example: Cournot duopoly

Lower bound =

\[ t \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{2k} = t/3. \]

Upper bound =

\[ t \left( 1 - \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{2k-1} \right) = t/3. \]
Dominance solvability: comments

- Order of elimination doesn’t matter
- Just as most games don’t have DSE, most games are not dominance solvable
Rationalizable strategies

Given a game:

- For each player $n$, remove strategies from each $S'_n$ that are not best responses for any choice of other players’ strategies.
- Repeat this procedure.

Strategies that survive this process are called *rationalizable strategies*. 
In a two player game, a strategy $s_1$ is rationalizable for player 1 if there exists a chain of justification

$$s_1 \rightarrow s_2 \rightarrow s_1' \rightarrow s_2' \rightarrow \ldots \rightarrow s_1$$

where each is a best response to the one before.
Rationalizable strategies

• If $s_n$ is rationalizable, it also survives iterated strict dominance. (Why?)

$\Rightarrow$ For a dominance solvable game, there is a unique rationalizable strategy, and it is the one given by iterated strict dominance.
Rationalizability: example

Note that $M$ is not *rationalizable*, but it survives iterated strict dominance.

<table>
<thead>
<tr>
<th></th>
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<td>Player 1</td>
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Rationalizable strategies

Note for later:

When “mixed” strategies are allowed, rationalizability = iterated strict dominance for two player games.