Outline

• Best response and pure strategy
  Nash equilibrium
• Relation to other equilibrium notions
• Examples
• Bertrand competition
Best response set

**Best response set** for player $n$ to $s_{-n}$:

$$R_n(s_{-n}) = \arg \max_{s_n \in S_n} \Pi_n(s_n, s_{-n})$$

[Note: $\arg \max_{x \in X} f(x)$ is the set of $x$ that maximize $f(x)$]
Nash equilibrium

Given: \( N \)-player game

A vector \( s = (s_1, \ldots, s_N) \) is a (pure strategy) Nash equilibrium if:

\[ s_i \in R_i(s_{-i}) \]

for all players \( i \).

Each *individual* plays a best response to the others.
Nash equilibrium

Pure strategy Nash equilibrium is robust to *unilateral deviations*

One of the hardest questions in game theory:

*How do players know to play a Nash equilibrium?*
Example: Prisoner’s dilemma

Recall the routing game:

<table>
<thead>
<tr>
<th></th>
<th>near</th>
<th>far</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCI</td>
<td></td>
<td></td>
</tr>
<tr>
<td>near</td>
<td>(-4, -4)</td>
<td>(-1, -5)</td>
</tr>
<tr>
<td>far</td>
<td>(-5, -1)</td>
<td>(-2, -2)</td>
</tr>
</tbody>
</table>

AT&T
Example: Prisoner’s dilemma

Here \( (\text{near, near}) \) is the unique (pure strategy) NE:

<table>
<thead>
<tr>
<th>MCI</th>
<th>AT&amp;T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>near</td>
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<td>(-4, -4)</td>
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</tbody>
</table>
Summary of relationships

Given a game:

- Any DSE also survives ISD, and is a NE.

(DSE = dominant strategy equilibrium; ISD = iterated strict dominance)
Example: bidding game

Recall the bidding game from lecture 1:

<table>
<thead>
<tr>
<th>Player 1’s bid</th>
<th>$0</th>
<th>$1</th>
<th>$2</th>
<th>$3</th>
<th>$4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>$4.00</td>
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<tr>
<td>$1</td>
<td>$11.00</td>
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<td>$5.67</td>
<td>$5.00</td>
<td>$4.60</td>
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<tr>
<td>$2</td>
<td>$10.00</td>
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<td>$6.00</td>
<td>$5.20</td>
<td>$4.67</td>
</tr>
<tr>
<td>$3</td>
<td>$9.00</td>
<td>$7.00</td>
<td>$5.80</td>
<td>$5.00</td>
<td>$4.43</td>
</tr>
<tr>
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<td>$6.40</td>
<td>$5.33</td>
<td>$4.57</td>
<td>$4.00</td>
</tr>
</tbody>
</table>
**Example: bidding game**

Here \((2,2)\) is the unique (pure strategy) NE:

<table>
<thead>
<tr>
<th>Player 1’s bid</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>$4.00\</td>
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Summary of relationships

Given a game:
- Any DSE also survives ISD, and is a NE.
- If a game is dominance solvable, the resulting strategy vector is a NE.
  
  *Another example of this: the Cournot game.*
- Any NE survives ISD (and is also rationalizable).

(*DSE = dominant strategy equilibrium; ISD = iterated strict dominance*)
Example: Cournot duopoly

Unique NE: \((t/3, t/3)\)

Nash equilibrium = Any point where the best response curves cross each other.
Example: coordination game

Two players trying to *coordinate* their actions:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>L</strong></td>
<td><strong>R</strong></td>
</tr>
<tr>
<td><strong>l</strong></td>
<td>(2,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td><strong>r</strong></td>
<td>(0,0)</td>
<td>(1,2)</td>
</tr>
</tbody>
</table>
Example: coordination game

Best response of player 1:

\[ R_1(L) = \{ l \}, \quad R_1(R) = \{ r \} \]

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<tr>
<td>r</td>
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<td></td>
</tr>
</tbody>
</table>
**Example: coordination game**

Best response of player 2:

\[ R_2(l) = \{ L \}, \ R_2(r) = \{ R \} \]

<table>
<thead>
<tr>
<th>Player 1</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>l</td>
<td>(2,1)</td>
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</tr>
<tr>
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**Example: coordination game**

**Two** Nash equilibria: \((l, L)\) and \((r, R)\).

Moral: NE is not a *unique predictor of play*!

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</table>
**Example: matching pennies**

*No* pure strategy NE for this game

Moral: Pure strategy NE may not exist.

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<tbody>
<tr>
<td></td>
<td><strong>H</strong></td>
<td><strong>T</strong></td>
<td></td>
</tr>
<tr>
<td><strong>H</strong></td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
<td></td>
</tr>
<tr>
<td><strong>T</strong></td>
<td>(-1, 1)</td>
<td>(1, -1)</td>
<td></td>
</tr>
</tbody>
</table>

Player 1

Player 2
Example: Bertrand competition

- In *Cournot* competition, firms choose the *quantity* they will produce.

- In *Bertrand* competition, firms choose the *prices* they will charge.
Bertrand competition: model

- Two firms
- Each firm $i$ chooses a price $p_i \geq 0$
- Each unit produced incurs a cost $c \geq 0$
- Consumers only buy from the producer offering the lowest price
- Demand is $D > 0$
Bertrand competition: model

- Two firms
- Each firm $i$ chooses a price $p_i$
- Profit of firm $i$:
  \[ \Pi_i(p_1, p_2) = (p_i - c)D_i(p_1, p_2) \]
  where
  \[ D_i(p_1, p_2) = \begin{cases} 
  0, & \text{if } p_i > p_{-i} \\
  D, & \text{if } p_i < p_{-i} \\
  \frac{1}{2} D, & \text{if } p_i = p_{-i} 
  \end{cases} \]
Bertrand competition: analysis

Suppose firm 2 sets a price = $p_2 < c$. What is the *best response set* of firm 1?

Firm 1 wants to price higher than $p_2$. 

\[ R_1(p_2) = (p_2, \infty) \]
Bertrand competition: analysis

Suppose firm 2 sets a price = $p_2 > c$. What is the best response set of firm 1?

Firm 1 wants to price slightly lower than $p_2$ ... but there is no best response!

$$R_1 (p_2) = \emptyset$$
Bertrand competition: analysis

Suppose firm 2 sets a price = $p_2 = c$. What is the best response set of firm 1?

Firm 1 wants to price at or higher than $c$.

$$R_1 (p_2) = [c, \infty)$$
Best response of firm 1:
Bertrand competition: analysis

Best response of firm 2:

![Diagram showing the best response of firm 2 with the area $R_2(p_1)$ shaded. The axes are labeled $p_1$, $p_2$, and $c$.]
Bertrand competition: analysis

Where do they “cross”?

\[ R_2(p_1) \]

\[ R_1(p_2) \]
Bertrand competition: analysis

Thus the unique NE is where \( p_1 = c, \ p_2 = c. \)
Bertrand competition

Straightforward to show:

The same result holds if demand depends on price, i.e., if the demand at price $p$ is $D(p) > 0$.

Proof technique:

1. Show $p_i < c$ is never played in a NE.
2. Show if $c < p_1 < p_2$, then firm 2 prefers to lower $p_2$.
3. Show if $c < p_1 = p_2$, then firm 2 prefers to lower $p_2$. 
Bertrand competition

What happens if $c_1 < c_2$?

No pure NE exists; however, an $\varepsilon$-NE exists:

Each player is happy as long as they are within $\varepsilon$ of their optimal payoff.

$\varepsilon$-NE: $p_2 = c_2$, $p_1 = c_2 - \delta$

(where $\delta$ is infinitesimal)
Bertrand vs. Cournot

Assume demand is $D(p) = a - p$.

*Interpretation:* $D(p)$ denotes the total number of consumers willing to pay *at least* $p$ for the good.

Then the *inverse demand* is

$$P(Q) = a - Q.$$ 

This is the market-clearing price at which $Q$ total units of supply would be sold.
Bertrand vs. Cournot

Assume demand is \( D(p) = a - p \).
Then the inverse demand is
\[
P(Q) = a - Q.
\]
Assume \( c < a \).
Bertrand eq.: \( p_1 = p_2 = c \)
Cournot eq: \( q_1 = q_2 = (a - c)/3 \)
\[\Rightarrow \text{ Cournot price} = a/3 + 2c/3 > c\]
Bertrand vs. Cournot

Consumer surplus

Cournot total profits (Producer surplus)

Bertrand eq. (perfectly competitive)

Cournot eq.

\[ P(Q) \]

\[ a \]

\[ p \]

\[ c \]
Bertrand vs. Cournot

Consumer surplus

\[ P(Q) \]

Cournot total profits (Producer surplus)

Deadweight loss:
Consumers are willing to pay, and firms could have made a profit by selling...
Bertrand vs. Cournot

• Cournot eq. price > Bertrand eq. price
• Bertrand price = marginal cost of production
• In Cournot eq., there is positive deadweight loss.

This is because firms have *market power*: *they anticipate their effect on prices.*
Questions to think about

• Can a *weakly dominated* strategy be played in a Nash equilibrium?
• Can a *strictly dominated* strategy be played in a Nash equilibrium?
• Why is any NE rationalizable?
• What are real-world examples of Bertrand competition? Cournot competition?
Finding NE is typically a matter of checking the definition.

Two basic approaches...
Finding NE: Approach 1

First approach to finding NE:

(1) Compute the complete best response mapping for each player.
(2) Find where they intersect each other (graphically or otherwise).
Second approach to finding NE:

Fix a strategy vector \((s_1, \ldots, s_N)\).
Check if any player has a profitable deviation.
If so, it cannot be a NE.
If not, it is an NE.