MS&E 246: Lecture 9
Sequential bargaining

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Nash bargaining solution

Recall Nash’s approach to bargaining:

The planner is *given* the set of achievable payoffs and status quo point.

Implicitly:

The *process* of bargaining does not matter.
Dynamics of bargaining

In this lecture:

We use a dynamic game of perfect information to model the *process* of bargaining.
An interference model

Recall the interference model:

- Two devices
- Device 1 given channel a fraction \( q \) of the time
- For efficiency:
  When device \( n \) has control, it transmits at full power \( P \)
An interference model

• When timesharing is used, the set of Pareto efficient payoffs becomes:
  \[ \{ (\Pi_1, \Pi_2) : \Pi_1 = q R_1, \Pi_2 = (1 - q) R_2 \} \]

• We now assume the devices bargain through a sequence of *alternating offers*. 
Alternating offers

• At time 0:
  • Stage 0A: Device 1 proposes a choice of $q$ (denoted $q_1$)
  • Stage 0B: Device 2 decides to accept or reject device 1’s offer
Two period model

Assumption 1:
If device 2 *rejects* at stage 0B, then predetermined choice $Q \in [0, 1]$ is implemented at time 1.
Discounting

Assumption 2:

Devices care about delay:
Any payoff received by device $i$ at time $k$ is *discounted* by $\delta_{i}^{k}$.

$0 < \delta_{i} < 1$: *discount factor* of device $i$
Game tree

Device 1 makes initial offer

Device 2 accepts or rejects

$\Pi_1 = q_1 R_1$

$\Pi_2 = (1 - q_1) R_2$

$\Pi_1 = \delta_1 Q R_1$

$\Pi_2 = \delta_2 (1 - Q) R_2$
Game tree

• This is a *dynamic game of perfect information*.

• We solve it using *backward induction*.
Backward induction

1. Given $q_1$, at Stage 0B:
   
   - Device 2 *rejects* if:
     $$\delta_2 (1 - Q) > (1 - q_1)$$
Backward induction

1. Given $q_1$, at Stage 0B:

   - Device 2 **rejects** ($s_2(q_1) = R$) if:
     \[ q_1 > 1 - \delta_2 (1 - Q) \]

   - Device 2 **accepts** ($s_2(q_1) = A$) if:
     \[ q_1 < 1 - \delta_2 (1 - Q) \]

   - Device 2 is **indifferent** ($s_2(q_1) \in \{ A, R \}$) if
     \[ q_1 = 1 - \delta_2 (1 - Q) \]
Backward induction

2. At Stage 0A:

- Device 1 maximizes $\Pi_1(q_1, s_2(q_1))$ over offers ($0 \leq q_1 \leq 1$)

- *Claim*: Maximum value of $\Pi_1$ is

\[
(1 - \delta_2 (1 - Q)) R_1
\]
2. At Stage 0A:

- **Claim**: Maximum value of $\Pi_1$ is
  \[
  \Pi_1^{\text{MAX}} = (1 - \delta_2 (1 - Q)) \cdot R_1
  \]

- **Proof**:
  
  (a) Maximum is achievable:
  
  If $q_1$ increases to $1 - \delta_2 (1 - Q)$, then $\Pi_1$ increases to $\Pi_1^{\text{MAX}}$
2. At Stage 0A:

- **Claim**: Maximum value of $\Pi_1$ is

  $$\Pi_1^{\text{MAX}} = (1 - \delta_2 (1 - Q)) R_1$$

- **Proof**:
  
  (b) If $q_1 > 1 - \delta_2 (1 - Q)$, then $\Pi_1 < \Pi_1^{\text{MAX}}$:
  
  Device 2 rejects $\Rightarrow \Pi_1 = \delta_1 Q R_1$
2. At Stage 0A:

- **Claim:** Maximum value of $\Pi_1$ is
  
  $$\Pi_1^{\text{MAX}} = (1 - \delta_2 (1 - Q)) \ R_1$$

- **Proof:**
  
  (b) If $q_1 > 1 - \delta_2 (1 - Q)$, then $\Pi_1 < \Pi_1^{\text{MAX}}$:

  But note that: $\delta_1 Q + \delta_2 (1 - Q) < 1$
2. At Stage 0A:

- **Claim**: Maximum value of $\Pi_1$ is

$$\Pi_1^{\text{MAX}} = (1 - \delta_2 (1 - Q)) R_1$$

- **Proof**:

  (b) If $q_1 > 1 - \delta_2 (1 - Q)$, then $\Pi_1 < \Pi_1^{\text{MAX}}$:

  But note that: $\delta_1 Q < 1 - \delta_2 (1 - Q)$
2. At Stage 0A:

- **Claim:** Maximum value of $\Pi_1$ is

  $$\Pi_1^{\text{MAX}} = (1 - \delta_2 (1 - Q)) R_1$$

- **Proof:**

  (b) If $q_1 > 1 - \delta_2 (1 - Q)$, then $\Pi_1 < \Pi_1^{\text{MAX}}$:

  So

  $$\delta_1 Q R_1 < (1 - \delta_2 (1 - Q)) R_1$$
Backward induction

2. At Stage 0A:

- Best responses for device 1:
  All choices of $q_1$ that achieve $\Pi_1^{MAX}$

  The only possibility:

  $$q_1^* = 1 - \delta_2 \left( 1 - Q \right)$$
Backward induction

2. At Stage 0A:

- If $s_2(q_1^*) = \text{reject}$,
  
  *no best response exists for device 1!*

- If $s_2(q_1^*) = \text{accept}$,
  
  *best response for device 1 is* $q_1 = q_1^*$
Unique SPNE

What is the unique SPNE?

• Must give \textit{strategies} for both players!
**Unique SPNE**

What is the unique SPNE?

- **Device 1:**
  At Stage 0A, offer \( q_1 = q_1^* \)

- **Device 2:**
  At Stage 0B,
  *accept* if \( q_1 \leq q_1^* \), *reject* if \( q_1 > q_1^* \)
Payoffs at unique SPNE

• So the offer of device 1 is *accepted immediately* by device 2.

• Device 1 gets:  \[ \Pi_1 = (1 - \delta_2(1 - Q)) \ R_1 \]

• Device 2 gets:  \[ \Pi_2 = \delta_2(1 - q_0) \ R_2 \]
Infinite horizon

More realistic model:

Devices alternate offers indefinitely.

For simplicity: assume $\delta_1 = \delta_2 = \delta$
Finite horizon

Device 1    Device 2

0A        0B        1

$q_{10}$    REJECT

ACCEPT

$\Pi_1 = q_{10}R_1$

$\Pi_2 = (1 - q_{10})R_2$
Infinite horizon

\[ \Pi_1 = q_{10} R_1 \]
\[ \Pi_2 = (1 - q_{10}) R_2 \]

\[ \Pi_1 = \delta q_{21} R_1 \]
\[ \Pi_2 = \delta (1 - q_{21}) R_2 \]
Infinite horizon: formal model

- Device 1 offers $q_{1k}$ at stage $kA$, for $k$ even
- Device 2 offers $q_{2k}$ at stage $kA$, for $k$ odd

- Device 2 accepts/rejects stage $kA$ offer at stage $kB$, for $k$ even
- Device 1 accepts/rejects stage $kA$ offer at stage $kB$, for $k$ odd
Infinite horizon: formal model

- Payoffs:

\[ \Pi_1 = \Pi_2 = 0 \text{ if no offer ever accepted} \]

(similar to status quo in NBS)
Infinite horizon: formal model

- Payoffs:
  If offer made at stage $kA$ by player $i$ accepted at stage $kB$:

  $\Pi_1 = \delta^k q_{ik} R_1$

  $\Pi_2 = \delta^k (1 - q_{ik}) R_2$
Infinite horizon

- Can’t use backward induction!

- Use *stationarity*:

  Subgame rooted at 1A is *the same as the original game*, with roles of 1 and 2 reversed.
Define $V$ and $v$:

$V R_1 = \text{highest time 0 payoff to device 1 among all SPNE}$

$v R_1 = \text{lowest time 0 payoff to device 1 among all SPNE}$
Then if device 2 rejects at 0B:

\[ V R_2 = \text{highest time 1 payoff to device 2 among all SPNE} \]

\[ v R_2 = \text{lowest time 1 payoff to device 2 among all SPNE} \]
SPNE: Two inequalities

\[ v R_1 \geq (1 - \delta V) R_1 \]

At Stage 0B:
Device 2 will accept any \( q_{10} < 1 - \delta V \)

So at Stage 0A:
Device 1 must earn at least \( (1 - \delta V) R_1 \)
SPNE: Two inequalities

- \( V R_1 \leq (1 - \delta v) R_1 \)

If offer \( q_{10} \) is *accepted* at stage 0B, device 2 must get a timeshare of at least \( \delta v \)

\[ \Rightarrow \quad q_{10} \leq 1 - \delta v \]
SPNE: Two inequalities

- \( VR_1 \leq (1 - \delta v) R_1 \)

If offer \( q_{10} \) is rejected at stage 0B, device 1 earns at most \( \delta (1 - v) R_1 \) since device 2 earns at least \( \delta v R_2 \)

\[ \Rightarrow \Pi_1 \leq \delta (1 - v) R_1 \leq (1 - \delta v) R_1 \]
Combining inequalities

- \( v \leq V \)
- \( v \geq 1 - \delta V \)
- \( V \leq 1 - \delta v \)
Combining inequalities

- $v \leq V$
- $v + \delta V \geq 1$
- $V + \delta v \leq 1$

So:

$$V + \delta v \leq v + \delta V$$

$$\Rightarrow (1 - \delta) V \leq (1 - \delta) v$$

$$\Rightarrow V = v$$
Unique SPNE

- So $V = 1 - \delta V \Rightarrow$
  \[ V = \frac{1}{1 + \delta} \]

- SPNE strategies for device 1:
  At Stage $kA$, $k$ even:
  Offer $q_{1k} = 1 - \delta V$

  At Stage $kB$, $k$ odd:
  Accept if $q_{2k} \geq \delta V$
Unique SPNE

• So $V = 1 - \delta V \Rightarrow$

$$V = \frac{1}{1 + \delta}$$

• SPNE strategies for device 2:
  At Stage $kA$, $k$ odd:
  Offer $q_{2k} = \delta V$
  At Stage $kB$, $k$ even:
  Accept if $q_{1k} \leq 1 - \delta V$
Unique SPNE: Payoffs

Stage 0A offer by device 1 is accepted in Stage 0B by device 2.

\[ \Pi_{1}^{\text{SPNE}} = \frac{R_1}{1 + \delta}, \quad \Pi_{2}^{\text{SPNE}} = \frac{\delta R_2}{1 + \delta} \]
Infinite horizon: Discussion

- **Outcome is efficient:**
  No “lost utility” due to discounting

- **Stationary SPNE strategies:**
  Actions do not depend on time \( k \)

- **First mover advantage:**
  \[ \Pi_1^{SPNE} > \Pi_2^{SPNE} \]
Shortening time periods

*Shorten* each time step to length $t < 1$ ...

... Same as changing discount factor to $\delta^t$

\[ \Pi_1^{SPNE} = \frac{R_1}{1 + \delta^t}, \quad \Pi_2^{SPNE} = \frac{\delta^t R_2}{1 + \delta^t} \]

As $t \to 0$, note that $\Pi_i^{SPNE} \to R_i/2$.

*Nash bargaining solution!*
In general

If $\delta_1 \neq \delta_2$:

Find SPNE using two period model:
Note that $Q$ must be SPNE payoff when device 2 offers first

Can show (for an appropriate limit) that 
*weighted* NBS obtained as $t \to 0$:
More patient player weighted higher
Summary

• Alternating offers: finite horizon
  Backward induction solution
• Alternating offers: infinite horizon
  Unique SPNE
  Relation to Nash bargaining solution