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Physics 41N
Mechanics: Insights, Applications and Advances
Lecture 3: Fractals

References:

"Newton Rules Biology: A physical approach to biological problems", by C. J. Pennycuick, Oxford University Press, 1992. This book emphasizes the fractional dimensionality of fractals.

The Wikipedia entry for fractals emphasizes the self-similar nature of fractals.

In the last lecture, we talked about the three fundamental dimensions in terms of which all other dimensions can be expressed in most mechanics problems: length **L**, mass **M**, and time **T**. Let's look at the dimension of length more closely by concentrating on some examples from the real world. Two examples of one-dimensional objects that have a length are a coastline and a border between two countries. It appears that their dimensionality is **L** raised to the first power, $\mathbf{L}^1$. Examples of two-dimensional objects that have an area are the surface of a field of dirt or the surface area of a lung. It appears that their dimensionality is $\mathbf{L}^2$. In this lecture, we'll find that each of these examples cannot be consistently described as having dimensions of length raised to an integer power. We will introduce a new type of dimension with a "fractional" power of length. First, a bit of history...

Lewis Fry Richardson was a scientist interested in the origins of war.[1] Frequently, wars are associated with border disputes, so Richardson was studying various borders in Europe. He happend to notice that the length of the border between Spain and Portugal was given as 987 km in a Spanish encyclopedia and 1214 km (23% longer!) in a Portuguese encyclopedia. Why? Richardson traced the discrepancy to differences in the length of the "scale" that was used to measure the border in the two cases. Since a border can be very irregular (*e.g.,* if it is defined by a river or a coastline), the length we measure will depend on how much detail we account for, and how much detail we just skip over. For example, if you were to measure the length of an

[1] L.F.Richardson, "The problem of contiguity: an appendix of statistics of deadly quarrels," *General Systems Yearbook* 6, 139 (1960).

irregular border on a map, you could set a pair of dividers to some suitable small interval ϵ, and step with it along the border. Suppose you took N steps. Then the length of the border is $N\epsilon$. But if you choose a shorter step size ϵ, you will now include some features in the border that were previously bridged by the larger step size. Your estimate of the total length will be larger than the estimate obtained with the bigger step size. In the case of the border between Spain and Portugal, we might expect that the smaller country (Portugal) will measure the length of its border in more detail than the larger country (Spain). If so, we would expect Portugal to estimate a longer border, which indeed they do. In Figure 1, we see a plot of empirical data on the lengths of borders and coastlines from Richardson's paper. The horizontal axis corresponds to the logarithm of the step size ϵ (in km). The step size ranges from about $10^{1.5}$ km $= 32$ km to 10^3 km $= 1000$ km. The vertical axis corresponds to the logarithm of the resulting estimate of the border length. Note that for each of the five borders the data lie on straight lines with negative slopes on this log-log plot. However, each line has a different slope.

Figure 1 also shows data for a smooth circle. Although the estimated circumference of a circle also depends on step size, it eventually levels off at the true circumference as the step size gets very small. This is not true for the coastlines. As the step size gets smaller, more features of the coastline are included. The lengths of the coastline show no sign of leveling off as the step size gets smaller. Richardson concluded that irregular lines such as borders and coastlines do not have an identifiable length. But could there be a way to express the length of such irregular lines in a unique way?

1 Fractal Geometry

The problem of the dependence of length on measuring scale was taken up by Benoit B. Mandelbrot who then basically founded a new branch of mathematics around 1975. The field is called fractal geometry for reasons that we will see in a moment. Mandelbrot is hailed as the "father of fractals". Mandelbrot addressed the question: if a coastline does not have length, then what property does it have? He sought a single number that expressed how much coastline there is, independent of the step size.

Think back to the first lecture on dimensional analysis. We discussed why the dimensions on the two sides of an equation must be the same. The answer

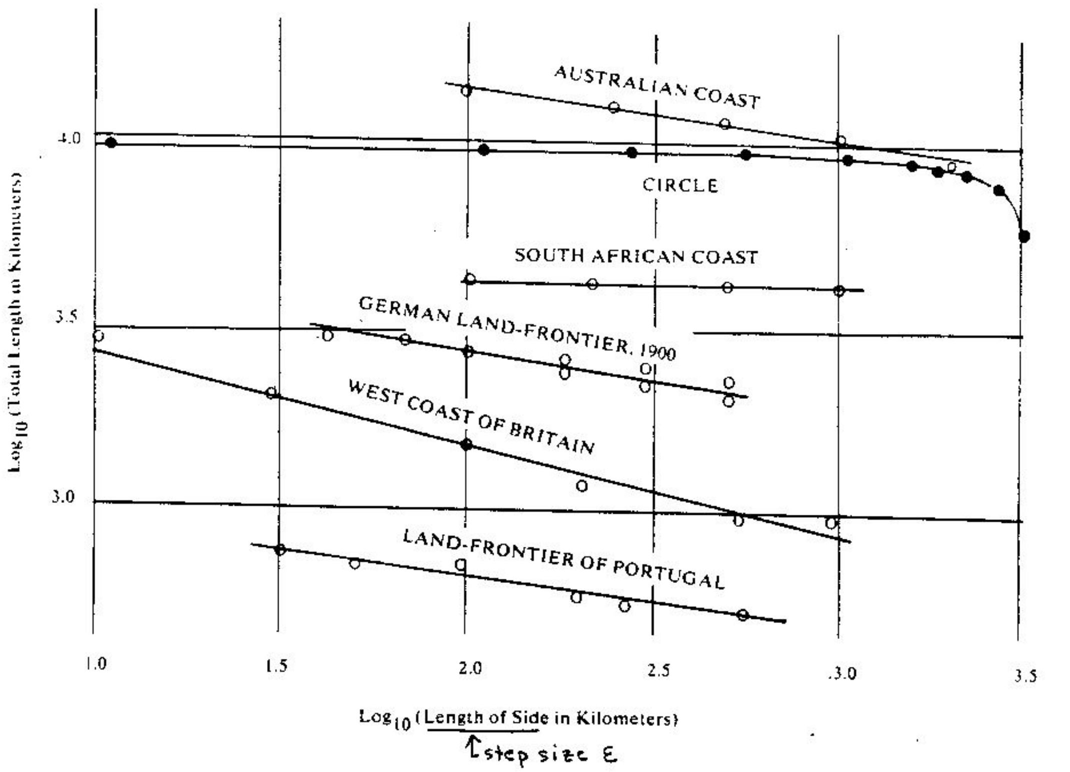

Plate 33 ¤ RICHARDSON'S EMPIRICAL DATA CONCERNING THE RATE OF INCREASE OF COASTLINES' LENGTHS

Figure 1: From *The Fractal Geometry of Nature,* by Benoit B. Mandelbrot (1977). The horizontal axis corresponds to the logarithm of the step size ϵ (in km). The step size ranges from about $10^{1.5}$ km $= 32$ km to 10^3 km $= 1000$ km. The vertical axis corresponds to the logarithm of the resulting estimate of the border length.

was the following: if the dimensions do *not* match, then the equality of the two sides of the equation will depend on the units, or the measuring scale, that is used. Perhaps we are using quantities with the incorrect dimensions for describing irregular lines.

We'll go back to the data on the length of coastlines in Figure 1 but first let's consider what the length of a (smooth) straight line will look like on this plot. If we use a step size ϵ to measure the length of a straight line, the length ℓ is given by

$$\ell = N\epsilon$$

where N is the number of steps. This equation is dimensionally correct since each side has dimensions of $\mathbf{L}^1$. Because it is dimensionally correct, it is independent of the units we use for ϵ. Also, the length of a straight line is independent of the step size ϵ. If we plot ℓ versus ϵ on a log-log plot, what do we get? Since ℓ does not depend on ϵ, we get a line with zero slope as shown in Figure 2.

In Figure 1, we saw that what we define as length ($N\epsilon$) is *not* independent of ϵ for coastlines. That is, our equation $\ell = N\epsilon$ does not appear to describe a fundamental characteristic of irregular lines such as coastlines. Perhaps this should give us a clue (from our discussion of dimensional analysis) that something in our problem is not *dimensionally* correct. We have assumed that we can ascribe to these irregular curves a physical quantity with dimensions of length $\mathbf{L}$. But we found that the length ℓ depends on our step size ϵ in such a way that the dependence is a straight line (with a negative slope) on a log-log plot, indicating a power-law dependence; *i.e.,* ℓ depends on ϵ to some power. So, this gives us a clue that perhaps we can find a measure of an irregular line that is independent of the step size if we consider the step size ϵ raised to some power. Then the dimensionality of our measure of the irregular line will be length raised to some power.

2 Fractal Dimensions

Mandelbrot proposed that the appropriate measure is something that we will call an "extent", given by

$$E = N\epsilon^D,$$

where, as before, N is the number of steps and ϵ is the step size. The dimensionality of the extent E will be $\mathbf{L}^D$. D is not necessarily an integer!

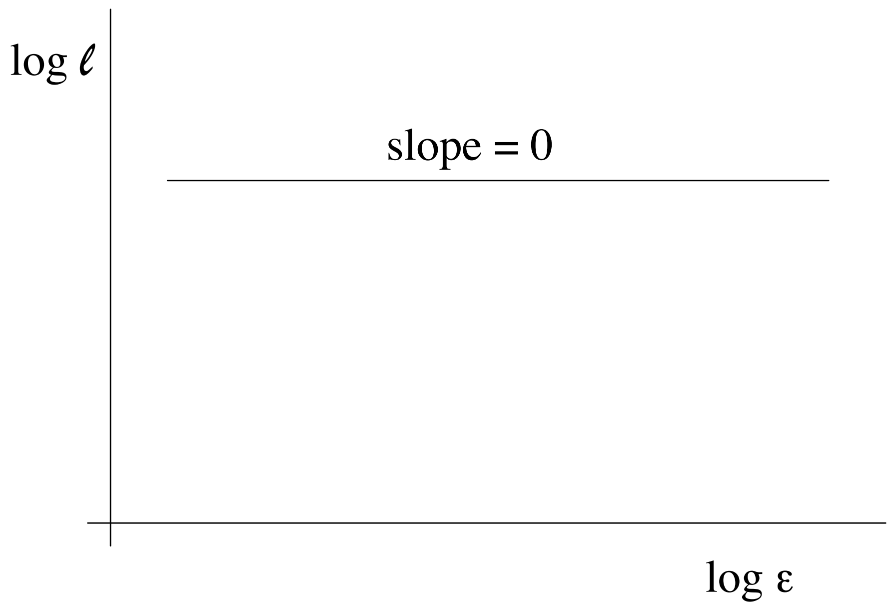

Figure 2: Measured length ℓ of a straight line plotted as a function of step size ϵ on a log-log plot. The curve has zero slope because the measured length of a line does not depend on the step size.

How do we find D? Let's again consider the data in Figure 1. We want to find a measure of length or "extent" that does not depend on the step size ϵ.

Since the length ℓ plotted versus the step size ϵ gives a straight line on a log-log plot, the equation relating ℓ and ϵ can be written

$$\ln \ell = \ln C + k \ln \epsilon$$

where C and k are constants. Since the slope is negative, $k < 0$. Now let's rearrange the above equation so that only the constant term $\ln C$ is on the right-hand side. Then we will have a formula involving ℓ and ϵ that is constant — just what we want!

$$\ln \ell - k \ln \epsilon = \ln C \tag{1}$$

$$\ln \ell \epsilon^{-k} = \ln C \tag{2}$$

$$\ell \epsilon^{-k} = C. \tag{3}$$

But $\ell = N\epsilon$. Therefore,

$$N\epsilon\epsilon^{-k} = C \quad \text{or} \quad N\epsilon^{1-k} = C.$$

This looks just like Mandelbrot's equation $E = N\epsilon^D$ with $D = 1 - k$ and the constant C equal to the extent.

Let's try applying this to the example of measuring the length of a straight line (Figure 2). In that case, the slope k is equal to 0 and we get $D = 1 - k = 1$. We conclude that a smooth curve has dimension $\mathbf{L}^D = \mathbf{L}^1$ (length to the first power), as expected. However, for coastlines, the slope is some negative number ($k < 0$). Therefore, $D = 1 - k$ is greater than 1 for coastlines; also, the more rugged the coastline, the larger the value of D. Typical values for D for coastlines range from 1.1 for a relatively smooth one to 1.5 for a rough one. This fractional dimension D is called the *fractal dimension* of the line. For irregular lines, the fractal dimension is always between 1 and 2.

3 Self Similarity

What distinguishes a fractal curve (with $1 < D < 2$) from a smooth curve (with $D = 1$)? One feature is the *self-similarity* that often appears in fractal objects. In Figure 3 below, we show a simple example of a fractal curve (called a Koch island) that exhibits self-similarity. Consider the construction of the curve. Start with an equilateral triangle (A). Then cut out the middle one-third of each side, replacing it with two sides of a smaller equilateral triangle, whose sides are one-third as long as the original sides, giving the six-pointed (12-sided) star shown in B. The middle one-third of each of the 12 sides is then cut out and replaced by two sides of a still smaller equilateral triangle (C), and so on indefinitely. Try plotting the perimeter of the Koch island for a couple of different step sizes ϵ to show that the fractal dimension of this curve is $\ln 4/\ln 3 \approx 1.26$. Any Koch island, no matter how big it is, has the same fractal dimension ($D = 1.26$). However, it is the extent, defined as $E = N\epsilon^{1.26}$, that distinguishes a big Koch Island from a small one. For any particular island, E does not change if the step length is changed.

The Koch island is an example of a fractal object that is *self-similar* because if you take a small segment of the circumference and magnify it, it looks the same irrespective of the magnification. It is this self-similarity that leads to the normally defined length becoming infinite as ϵ goes to zero.

In real physical or biological objects, self-similarity can only be maintained over a finite range of scales. With mathematical objects, such as the Koch island, the self-similarity can be continued indefinitely.

4 Random and Nonrandom Fractals

Another difference between real and mathematical fractal objects is their randomness. Mathematically defined fractals usually (but not always) have no randomness in them. For example, the Koch island will look exactly the same at all scales. Physical fractal objects, on the other hand, almost always have an element of randomness. Examples are coastlines, networks of neurons in the retina, and networks of blood vessels.

5 Interpretation of Fractional Dimensions

Most likely you have never encountered an object that has dimensions of length to a fractional power. We are very familiar with objects whose size has dimensions of length to the first power (for example, lines and smooth curves). We are also familiar with objects whose size has dimensions of length to the second power (for example, surface areas and cross-sectional areas). But what is the meaning of a fractional dimension – a length raised

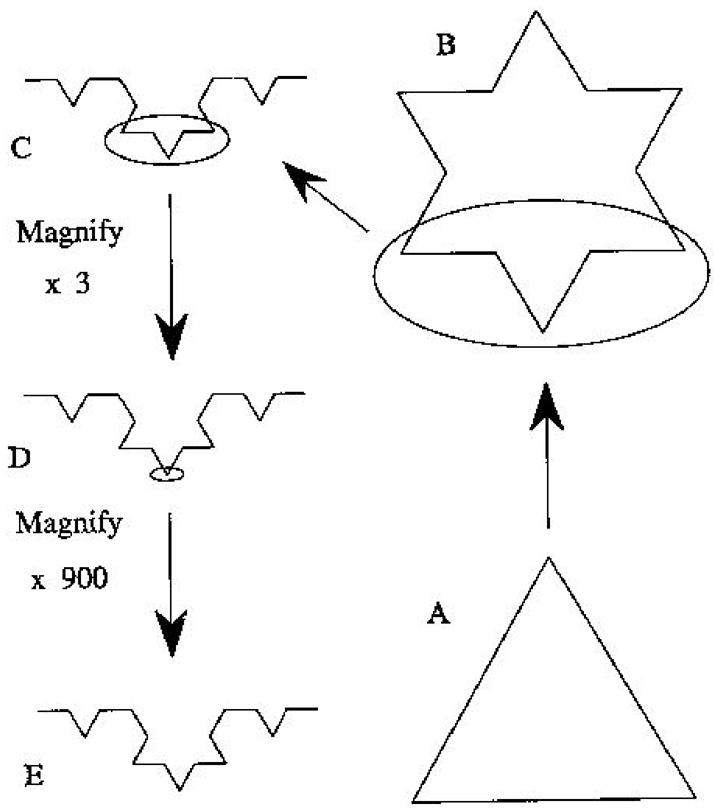

Figure 3: A simple fractal object: the Koch Island. The Koch island is an example of a fractal object that is *self-similar* because if you take a small segment of the circumference and magnify it, it looks the same irrespective of the magnification. By plotting the perimeter of the Koch island for a couple of different step sizes ϵ, one can show that the fractal dimension of this curve is $\ln 4/\ln 3 \approx 1.26$.

to a fractional power ($1 < D < 2$). Fractional dimensions can be used to measure the extent of irregular lines such as coastlines. An irregular line can be thought of as having a physical character intermediate between 'length' and 'area'. A line so infinitely convoluted that it leaves no empty space at all would have dimension 2, and would be the same as an area.

6 Comparisons of Quantities Involving Different Fractal Dimensions

Suppose you are a biologist studying the nesting habits of birds, and are asked to compare the density of osprey nesting on power poles along a road and the density of pheasants nesting in a corn field. You would have difficulty making this comparison because the density of osprey would be expressed in birds per unit length while the density of pheasants would be expressed in birds per unit area. We cannot compare these densities directly because they have different dimensions. A reasonable thing to do might be to raise the number of pheasants per unit area to the power of one-half. Then the density of pheasants will be expressed with dimensions of birds per unit length. The meaning of this quantity is that its inverse is the average spacing between pheasants. This can be directly compared to the average spacing between osprey, which is just one over the number of osprey nests per unit length of powerline.

The same problems arise when trying to make comparisons involving different fractal dimensions. I will use the example of the density of bald eagles along the coastlines of two Aleutian Islands, Adak and Amchitka, shown in Figure 4.[2] Some bald eagles build their nests on coastal cliffs. Before hearing this lecture, you would probably have considered a natural way to express the density of eagle nests along the coastline to be nests per unit length of coastline. But you now know that there is no unique measure of length of coastline. The meaningful measure of a coastline is fractal extent. So, we must determine the fractal dimension of the coastline, D, and then calculate the density of nests as the number of nests per extent E, where extent has dimensions of length to the power D.

[2] C.J.Pennycuick and N.C.Kline, "Units of measurement for fractal extent, applied to the coastal distribution of bald eagle nests in the Aleutian Islands, Alaska," Oecologia (Berlin) 68, 254 (1986).

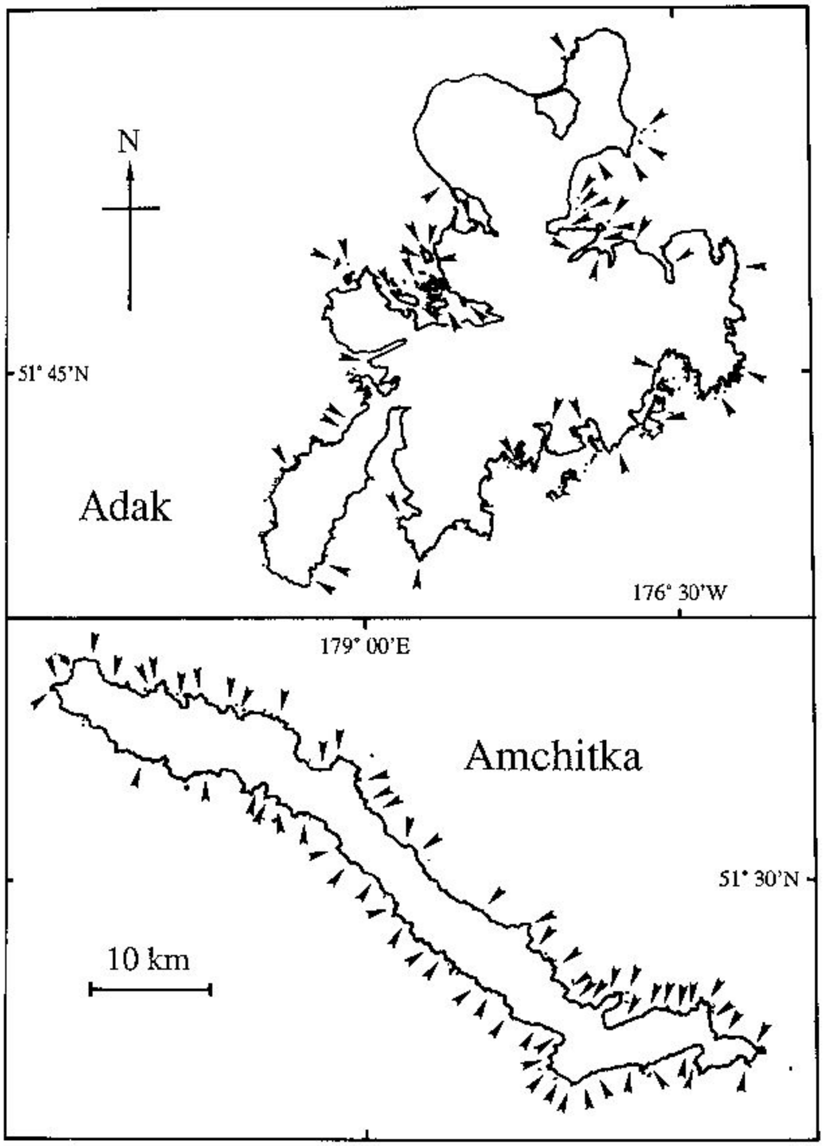

Fig 5.4 Bald eagle nests on Amchitka and Adak Islands in the Aleutians. The dimension of Amchitka's coastline (1·66) is much higher than Adak's (1·20) because the coast is broken into gullies and stacks on a scale too small to show on the map (from Pennycuick and Kline, 1986).

Figure 4: From C.J.Pennycuick and N.C.Kline, "Units of measurement for fractal extent, applied to the coastal distribution of bald eagle nests in the Aleutian Islands, Alaska," Oecologia (Berlin) 68, 254 (1986).

The fractal dimension, extent and number of eagle's nests were determined for the two Alueutian Islands shown in Figure 4. The fractal dimension of the island of Adak is 1.20. The island of Amchitka has the rather high fractal dimension of 1.66 because of the large number of offshore islets that are too small to appear on this map. Adak had 47 nests and an extent $E = 1.08 \times 10^6$ m$^{1.20}$; Amchitka had 66 nest and an extent $E = 1.51 \times 10^7$ m$^{1.66}$. We cannot compare the extents of Adak and Amchitka directly, anymore than we can compare an area directly with a length, because they have different dimensions. The ratio of the extents is not a pure number, but has dimensions $\mathbf{L}^{1.20-1.66}$. In order to make a comparison between the eagle density of the two islands, we must reduce them to the same dimension so that the ratio is a pure number. You might get a hint of how to do this from the example of the densities of ospreys along power lines and pheasants in fields discussed above. In order to make the comparison, we expressed both densities in terms of the average spacing between nests. The spacing has units of length. For the ospreys, the spacing is just the length of powerline divided by the number of nests. For the pheasants, it is the square-root of the area of the field divided by the number of nests. In both cases, the spacing S is given by

$$S = (\frac{E}{n})^{1/D}$$

where the extent E is the length of the power line or the area of the field, n is the number of osprey or pheasant nests, and D is the dimensionality of the problem ($D = 1$ for osprey nests, $D = 2$ for pheasants). Note that raising an extent E of dimension $\mathbf{L}^D$ to the power $1/D$ produces a length (dimension $\mathbf{L}^1$).

So, let's apply the same algorithm to compare the spacing of eagle's nests along the coasts of the two islands. For Adak, $S = (1.08 \times 10^6\,\mathrm{m}^{1.20}/47)^{1/1.20} =$ 4.31 km; for Amchitka, $S = (1.51 \times 10^7\,\mathrm{m}^{1.66}/66)^{1/1.66} = 1.69$ km. Therefore, the spacing is greater for Adak than for Amchitka. Amchitka has a higher density of eagle's nests.

One might wonder whether the difference in the spacing is actually statistically significant, given the relatively small number of eagle's nests. Whenever one is counting the number of 'events' of some sort, the statistical uncertainty in the number n is just $\sqrt{n}$. The meaning of the statistical uncertainty is that if the experiment were repeated many times, we expect that 68% of the time the number of events observed will lie between $n - \sqrt{n}$ and $n + \sqrt{n}$.

The best way to propagate this uncertainty through the equation for the spacing S is to actually calculate S for the values $n - \sqrt{n}$ and $n + \sqrt{n}$. For Adak, n is 47 and $\sqrt{n} = 6.9$. So we calculate S for $(47 + 6.9)$ eagles and $(47 - 6.9)$ eagles. We get S equal to 3.84 km and 4.92 km. Therefore, we would say that for Adak, the spacing is $(4.31^{+0.61}_{-0.47})$ km. Similarly, for Amchitka, n is 66 and $\sqrt{n} = 8.1$. So we calculate S for $(66 + 8.1)$ eagles and $(66 - 8.1)$ eagles. We get S equal to 1.58 km and 1.83 km. Therefore, we would say that for Amchitka, the spacing is $(1.69^{+0.14}_{-0.11})$ km. We conclude that the spacing S is significantly greater for Adak than for Amchitka.

7 Fractal Surfaces

In the earlier sections we discussed the impossibility of defining a 'length' (of dimension 1) for irregular curves such as a coastline. Now imagine trying to measure the area of an irregular surface, such as mountainous terrain, by 'tiling' the surface with flat tiles, all having the same area, and adding up the total area of the tiles. If we now do the measurement again using smaller tiles, we will follow the contours of smaller features that were previously bridged over by the larger tiles. Hence we will get a larger area. We conclude that the 'area' of an irregular surface is an undefined quantity, just as 'length' is an undefined quantity for an irregular line. An irregular surface is intermediate in character between a surface and a volume. (If the surface is convoluted enough, it can fill up space.) Therefore, irregular surfaces are described by fractal dimensions between 2 and 3. Absorptive surfaces, such as in respiratory or digestive organs, are normally fractal in character because they tend to be self-similar. Mandelbrot cites apparently conflicting measurements of the alveolar 'area' of the human lung. If estimated from light microscopy, it comes out to be about 80 m^2; with electron microscopy at higher resolution, the area is 140 m^2. These and other measurements made over a wide range of scales can be reconciled if the surface area of the lung is not an area, but a fractal surface with dimension 2.17. This means that it is wrong to measure and compare lung surface 'area' in square meters, and impossible to get a consistent result by so doing. Instead, we should compare 'extent', which can be measured in units of $m^{2.17}$.