Generalization of homological surface codes from graph embeddings on 2D manifolds

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Joint work with **Ted Yoder** (IBM T.J. Watson Research Center)

Talk is based on arXiv paper:

"A graph-based formalism for surface codes and twists", https://arxiv.org/abs/2101.09349

Stabilizer Codes

Describe a code using a stabilizer group $\mathcal{S} \leq \mathcal{P}_N = \{I, X, Y, Z\}^{\otimes N}$.

- E.g. $\mathcal{S} = \langle ZZZZ, XXXX \rangle$, with $\mathcal{C}(\mathcal{S}) = \langle \mathcal{S}, XXII, IXXI, ZZII, IZZI \rangle$.
- $|\mathcal{C}(\mathcal{S})| = 2^{N+2K}$, $D = min\{|p|: p \in \mathcal{C}(\mathcal{S}) - \mathcal{S}\}$.
- D is called the **distance of the stabilizer code**.

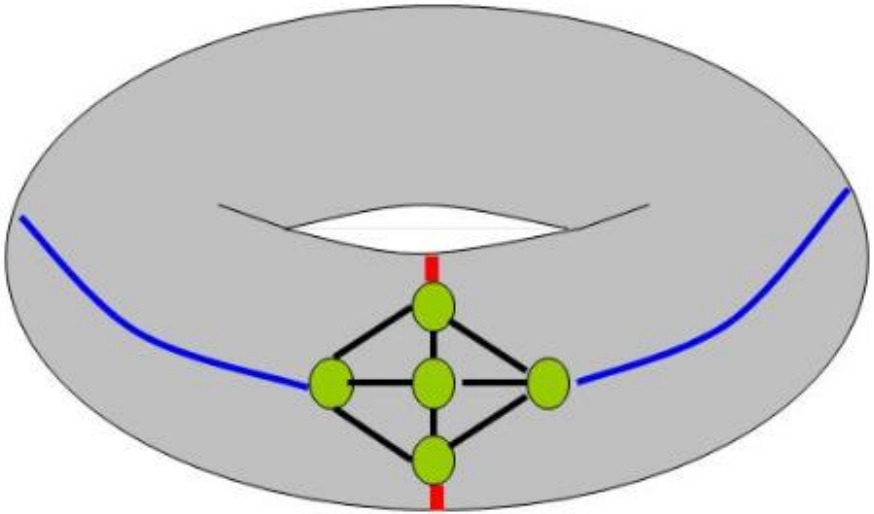
Relations between $\mathcal{S}$, $\mathcal{C}(\mathcal{S})$ and $\mathcal{P}_N$:

- $\mathcal{P}_N$ is the ***N* qubit Pauli group**. $|\mathcal{P}_N| = 4^{N+1}$. If you ignore phases, $|\mathcal{P}_N| = 4^N$.
- $\mathcal{S}$ is the **stabilizer group**. It is a subgroup of $\mathcal{P}_N$ where all elements commute.
- $\mathcal{C}(\mathcal{S})$ is the **centralizer group** of $\mathcal{S}$. It is a subgroup of $\mathcal{P}_N$ and consist of all elements that commute with each element in $\mathcal{S}$. Thus $\mathcal{S}$ is also a subgroup of $\mathcal{C}(\mathcal{S})$.
- $K = N - \dim(\mathcal{S})$.

Graph Embeddings on Manifolds (2- Manifolds, closed)

A **graph embedding** of a graph $G(V, E)$ in a manifold M is a "drawing" of the graph on M such that it has some nice properties:

- Faces are **homeomorphic** to open discs.
- Edges don't intersect except at vertices.



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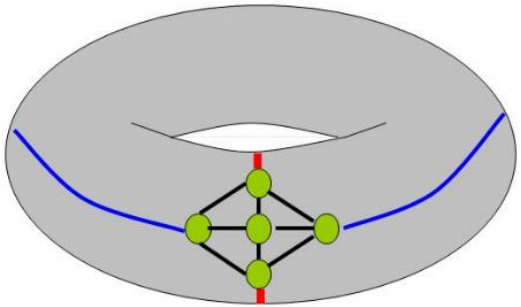

Question: What is the graph on the right?

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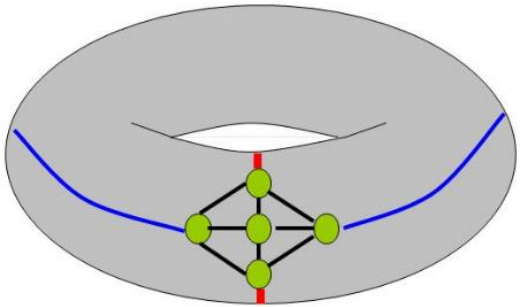

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The graph is K_5.

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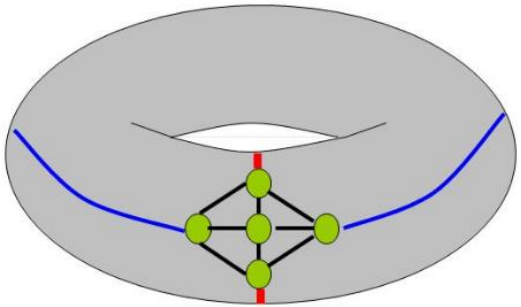

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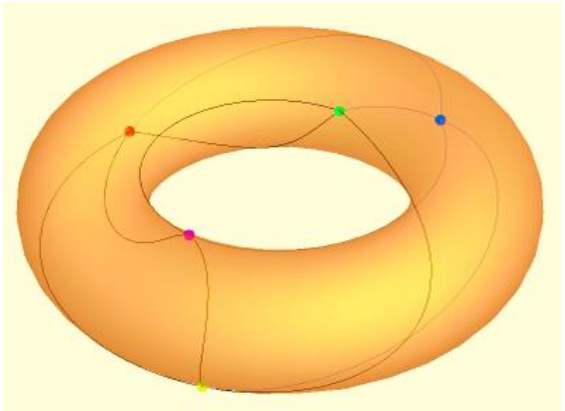
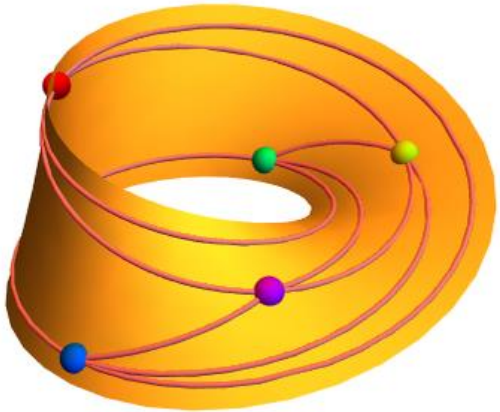
Manifolds for us will be 2-manifolds (surfaces), which are closed (meaning does not have a boundary and are compact). Can be orientable or non-orientable.

- Sphere, Torus are orientable manifolds.
- Real projective plane is a non-orientable manifold (these can be difficult to imagine if you have not encountered them before).

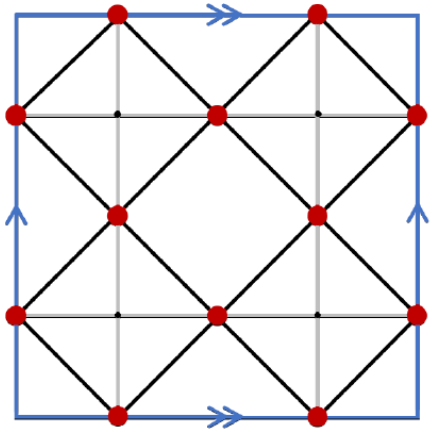
Examples of graph embeddings



Torus



Mobius band
(just for illustration,
not a closed manifold)



A different way to
represent a **Torus**

Source: Wolfram

Rotation systems (for orientable manifolds)

-- A rotation system is a combinatorial description of a graph embedding

-- It consists of

1) A set of half-edges H
2) Permutation $\nu: H \to H$
3) Permutation $\varepsilon: H \to H$ s.t.

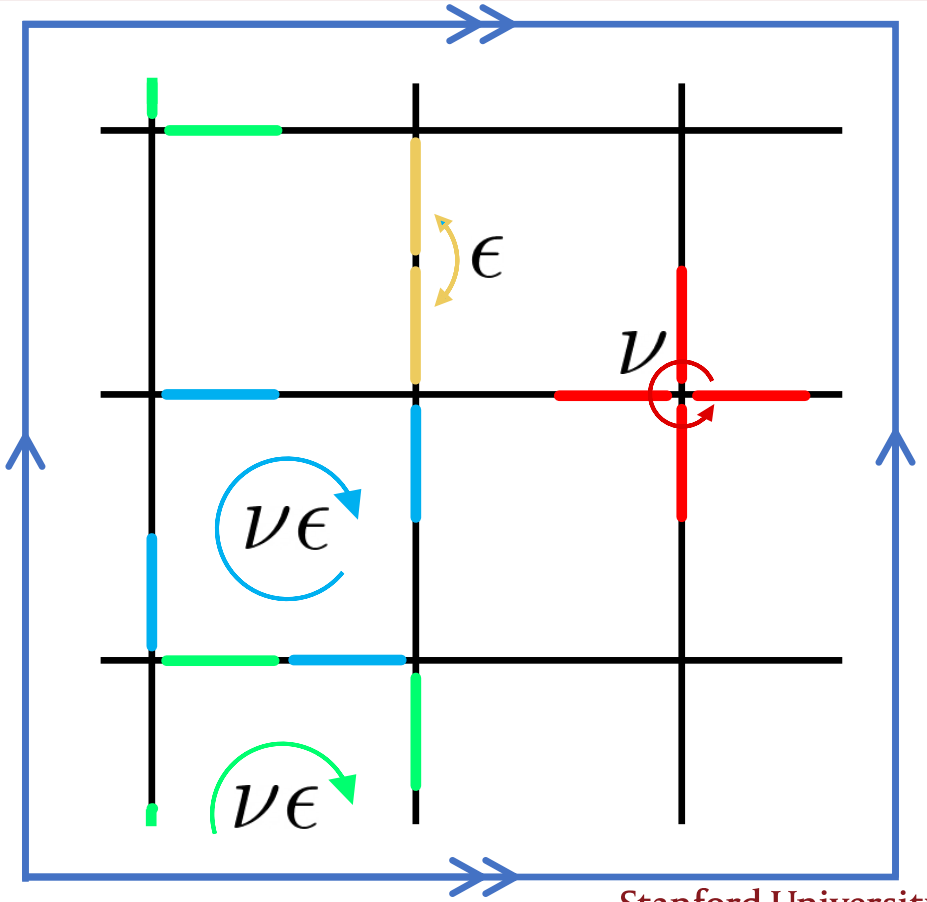
$\varepsilon \circ \varepsilon = id$ and $\forall x, \varepsilon(x) \neq x$.

Vertices: orbits of ν

Edges: orbits of ε

Faces: orbits of $\nu \circ \varepsilon$

Genus: $g = 1 - \frac{1}{2}(|V| - |E| + |F|)$



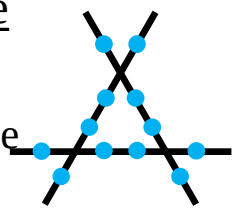
Outline of the construction

Embedded graph

Embed a graph in a 2d-manifold

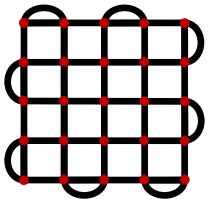
Majorana surface code

Assign Majoranas to half-edges & odd-degree vertices



Qubit surface code

Associate qubits with subsets of Majoranas



Cyclically anticommuting sets of Paulis

Also included

1) Formula for $K =$ (# encoded qubits)

2) Examples!
   1) Cyclic toric codes
   2) Stellated codes g>0
   3) Planar-packing code

3) Bounds on $D =$ (code distance)

Majorana codes

Majorana operators: $\{\gamma_i\}_i$ s.t. $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$

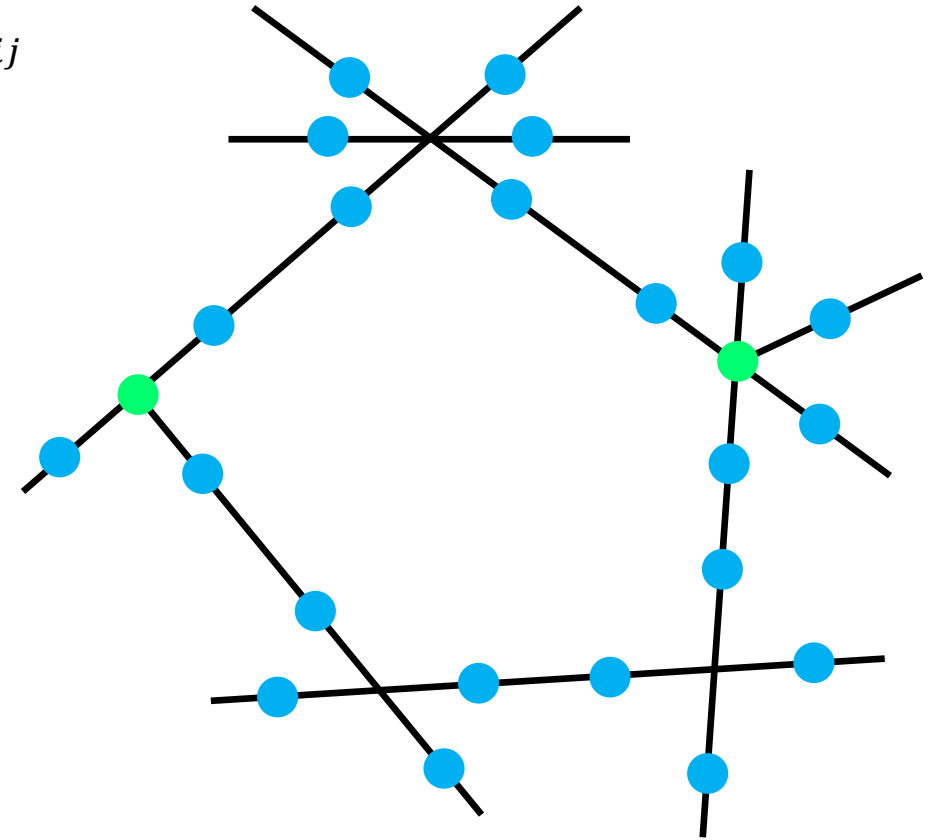
The full group: $\mathcal{M} = \langle \gamma_i \rangle$

Stabilizer group $\mathcal{S} \leq \mathcal{M}$ where

1) $-I \notin \mathcal{S}$
2) each $s \in \mathcal{S}$ has even weight

Given embedded graph, associate a Majorana to

1) Each half-edge
2) Each odd-degree vertex



Similar to [Wen, ‘03], [Kitaev, ‘03], [Bravyi et al. ’17]

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Given embedded graph, associate a Majorana to

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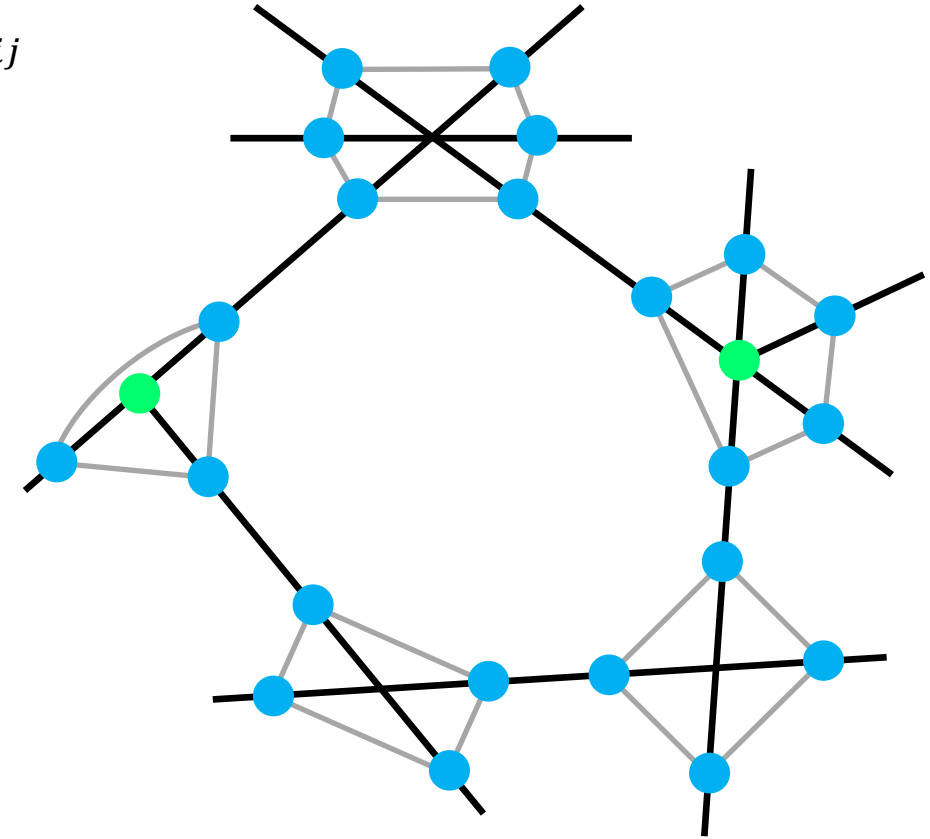
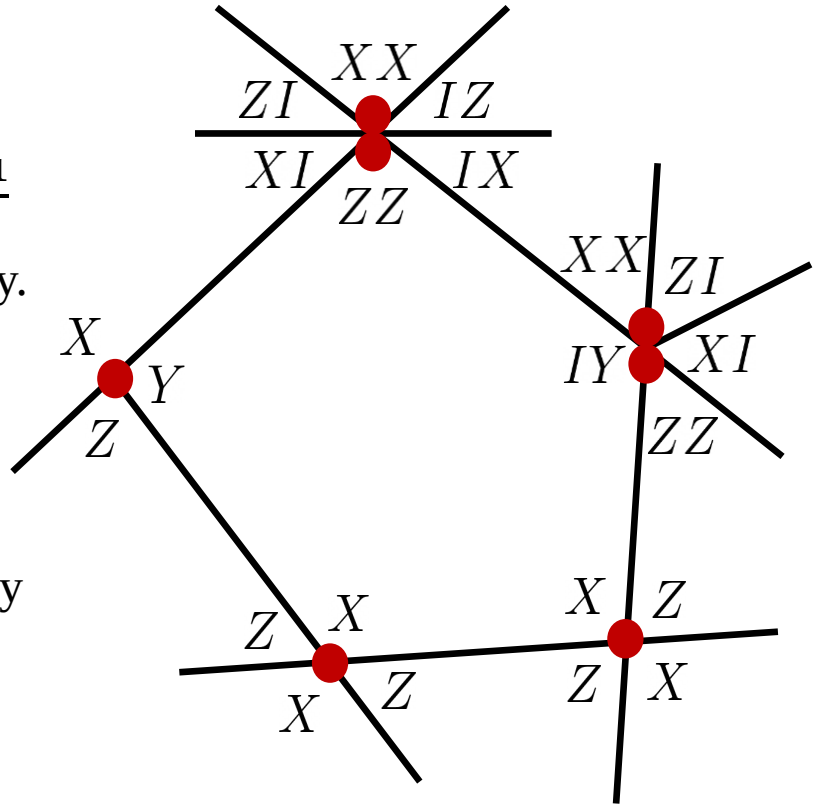
2) Each odd-degree vertex
3) A stabilizer to each vertex and face



Equivalent qubit surface codes

Stabilizer codes from graph embedding:

- Write a cyclically anticommuting list of Paulis around each vertex.
- Number of qubits to place at a vertex v is $\frac{\deg(v)-1}{2}$ or $\frac{\deg(v)-2}{2}$ if $\deg(v) \geq 3$ odd or even respectively.
- The **tensor product of Paulis within a face** is a stabilizer.



Cyclically anticommuting lists (CAL):

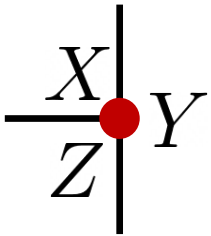
An ordered list of Paulis $\{p_0, p_1, \ldots, p_{l-1}\}$ is cyclically anticommuting if

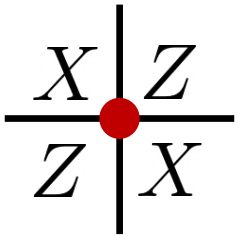
1) $\{p_i, p_{i+1\ (mod\ l)}\} = 0$
2) $[p_i, p_j] = 0$ when $j \neq i \pm 1\ (mod\ l)$

Eg. $\{X, Y, Z\}$ and $\{X, Z, X, Z\}$ are CALs

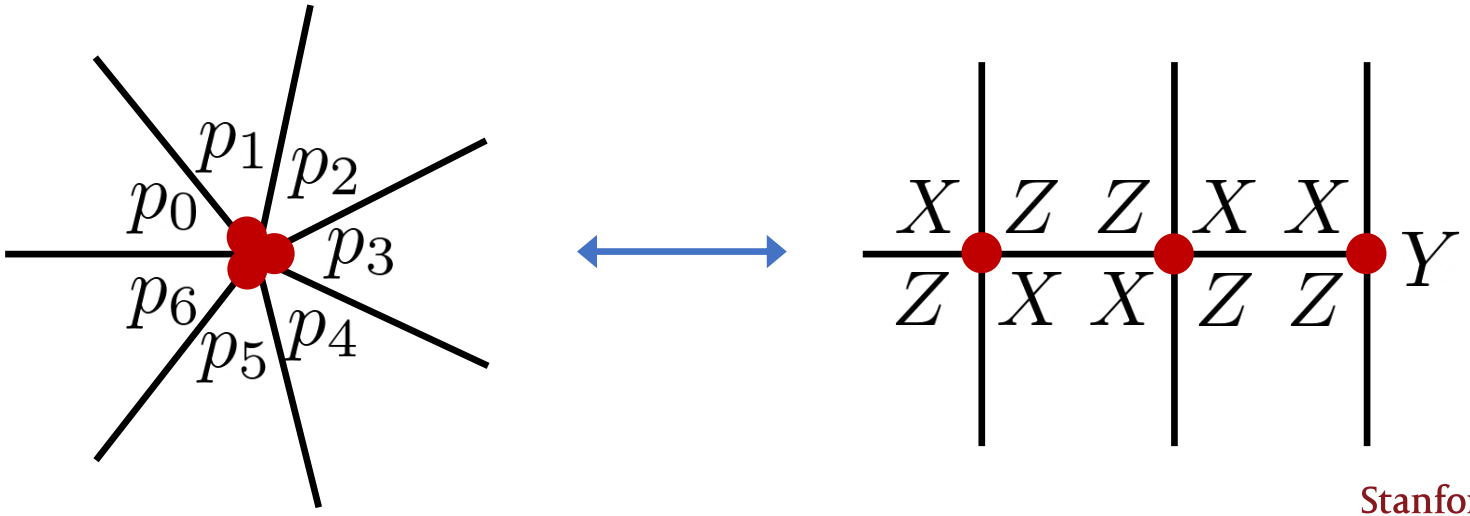
Reduction to graphs with degree between 3 and 4

Clearly, for $l = 3, 4$





So for $l = 7$



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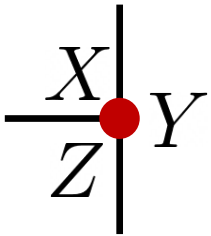
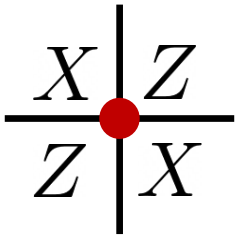
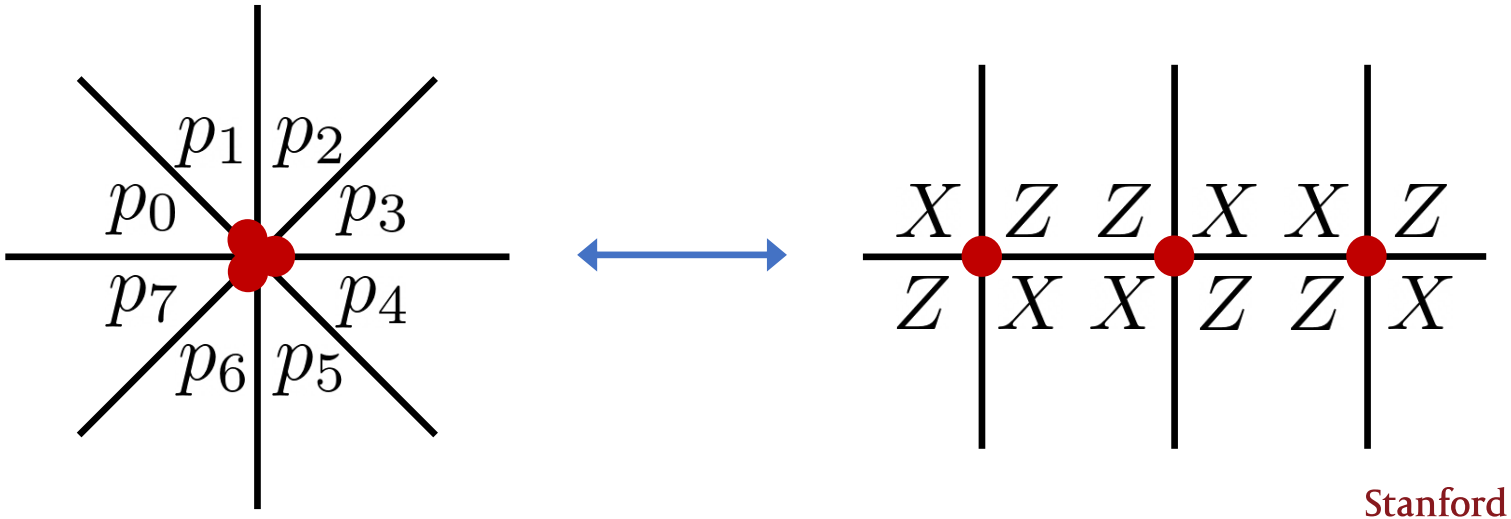




So for $l = 8$



Number of encoded qubits

We now have a set of Paulis defined by each face of the graph embedding.

- It turns out that because of the CAL property, this set of Paulis **commute** with each other.
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How many qubits does this code encode?

Theorem: A surface code on a genus g manifold with M odd degree vertices encodes K logical qubits given by

$$K = \left\{ \begin{array}{ll} 2g & , \ orientable \\ g & , \ non\text{-}orientable \end{array} \right\} + \left\{ \begin{array}{ll} 0 & , \ checkerboardable \\ (M-2)/2 & , \ not \ checkerboardable \end{array} \right\}$$

Here g is the orientable / non-orientable genus of the manifold M.

A graph is **checkerboardable** if its faces can be two-colored, with adjacent faces colored differently.

Checkerboarding examples

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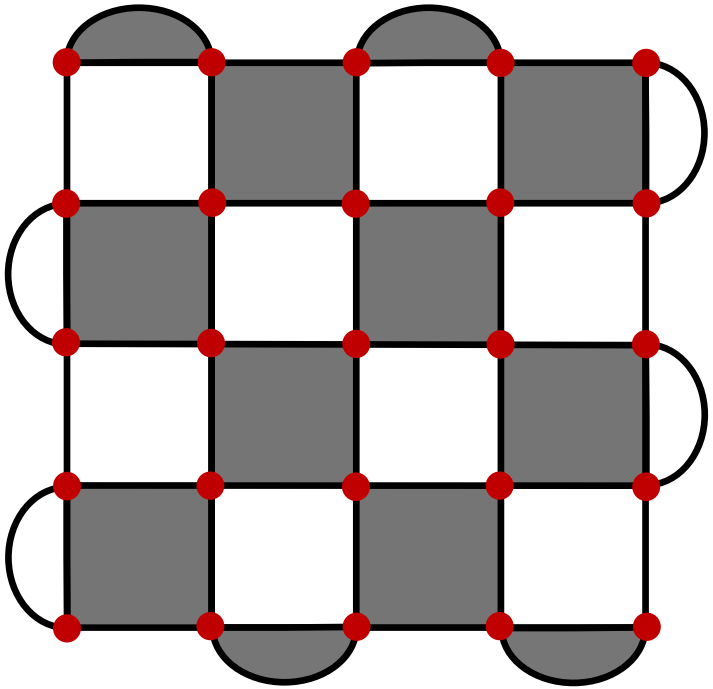
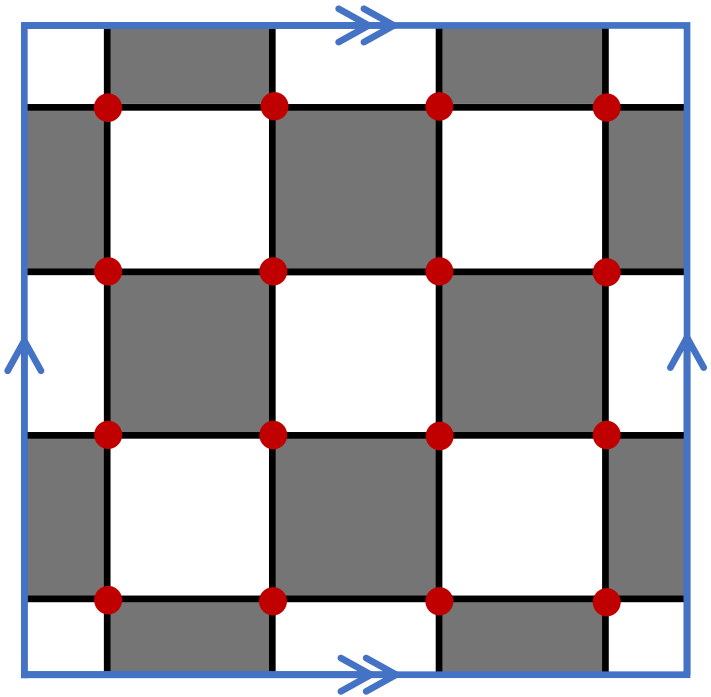


NOT Checkerboardable, $K = 1$

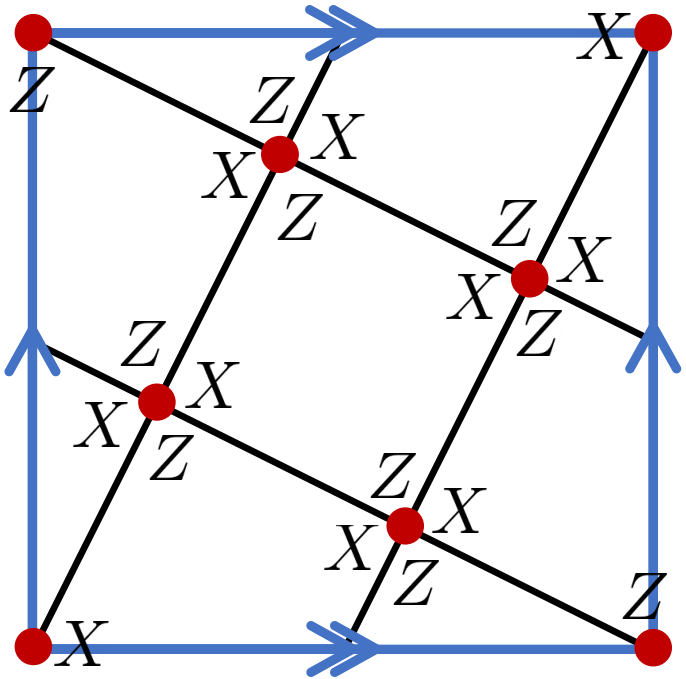


Checkerboardable, $K = 2$

Example: Cyclic Toric Code

- Parameterize by relatively prime, positive integers a, b, with $b > a \geq 1$.
- Draw lines $y = \left(\frac{b}{a}\right)x$ & $y = \left(-\frac{a}{b}\right)x$.

An example on the right with $(a, b) = (1, 2)$.



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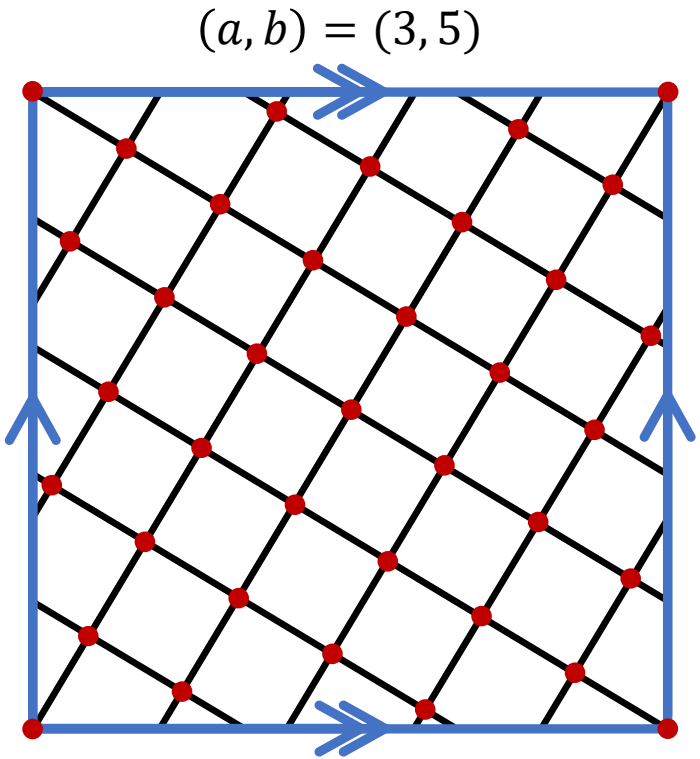
Code Parameters $[\![N, K, D]\!]$

- $N = a^2 + b^2$
- If N is odd, $K = 1$ and $D = a + b$
- If N is even, $K = 2$ and $D = \max(a, b)$

This code achieves $N = KD^2/2$ in two regimes, when $a = b - 1$ or when $a = 1$ and b is odd.

****Proving the distance is non-trivial.**



Checkerboarding examples

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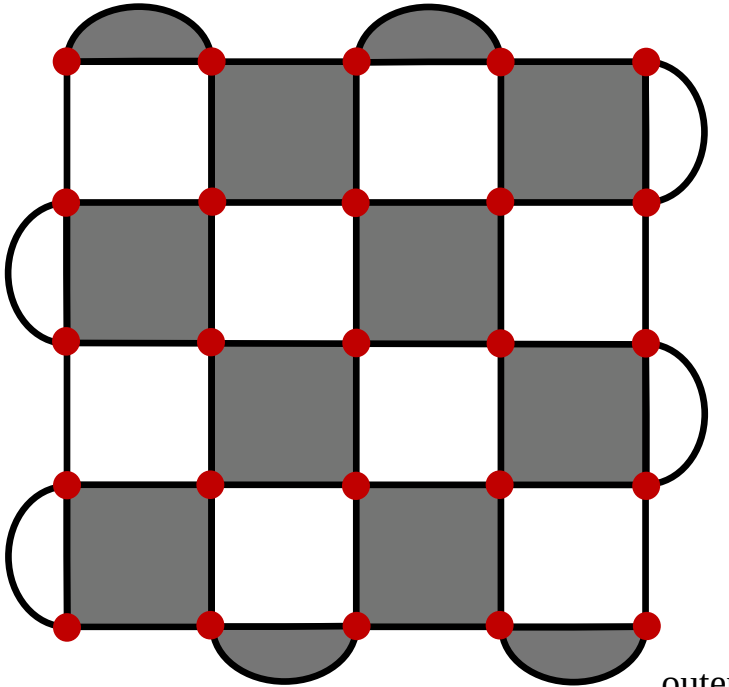
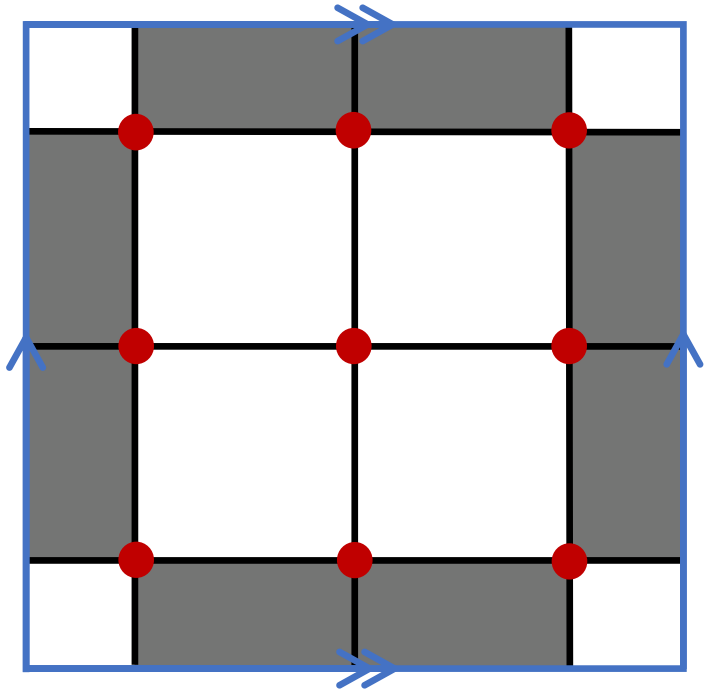


NOT checkerboardable

outer face = ?



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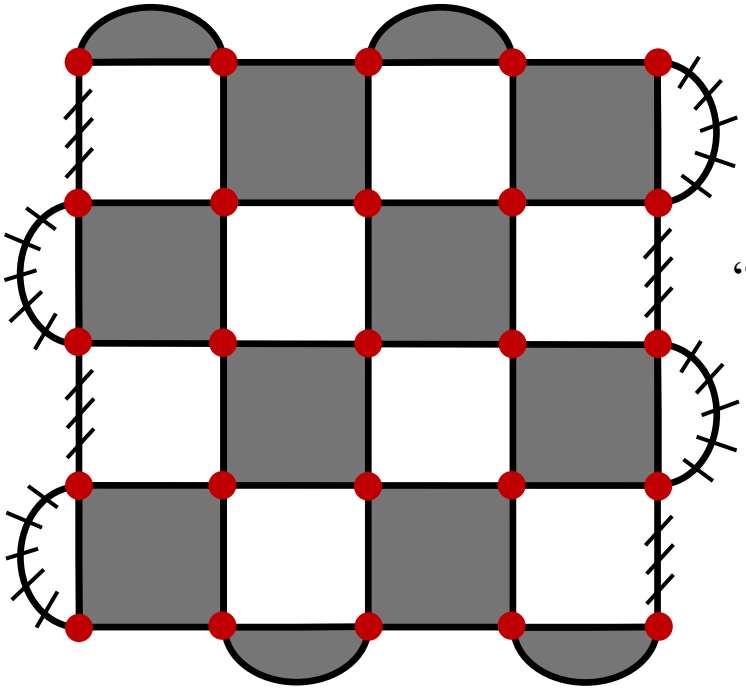

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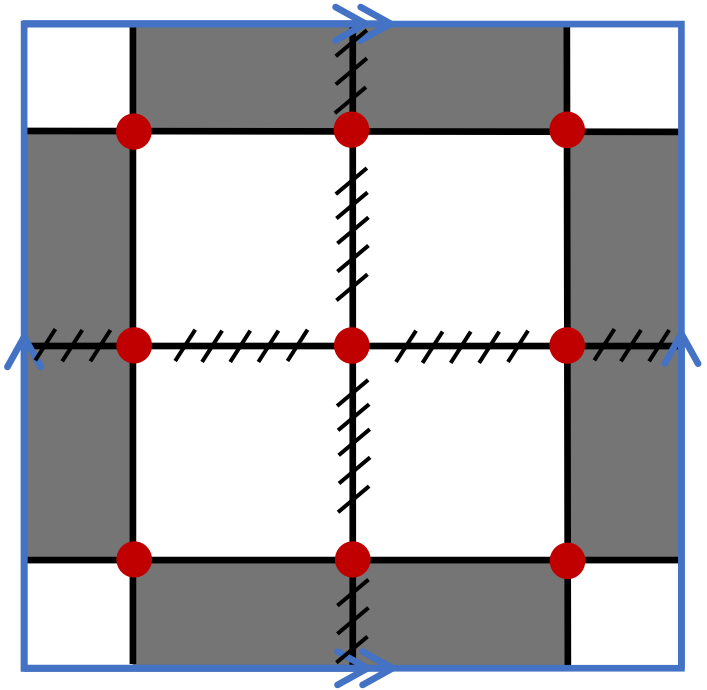
"Defects"



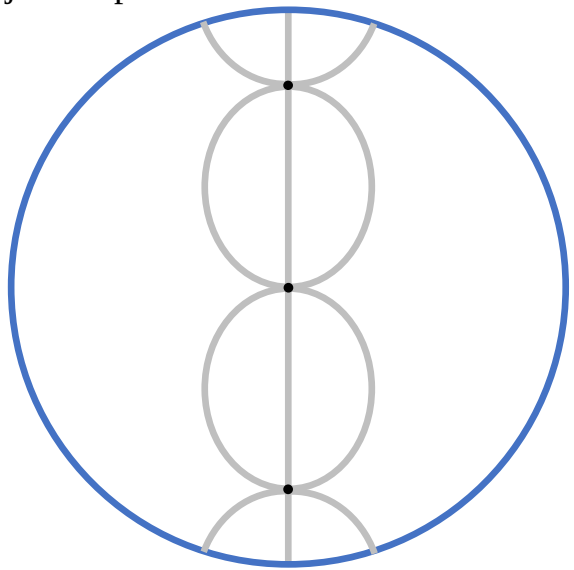
NOT checkerboardable

Relation to Kitaev's model

Kitaev: Given embedded graph G, place qubits on edges, define X-type stabilizers around vertices, and Z-type stabilizers around faces.

In our model, the resulting code is a surface code defined on the **medial graph** of G.

Eg: on a projective plane



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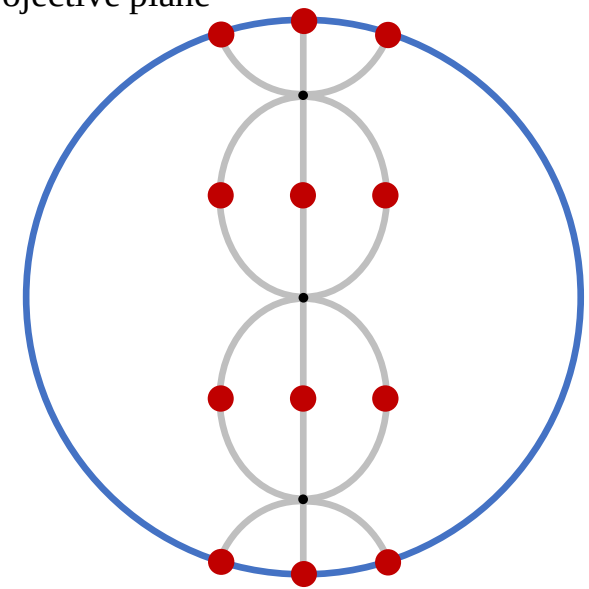

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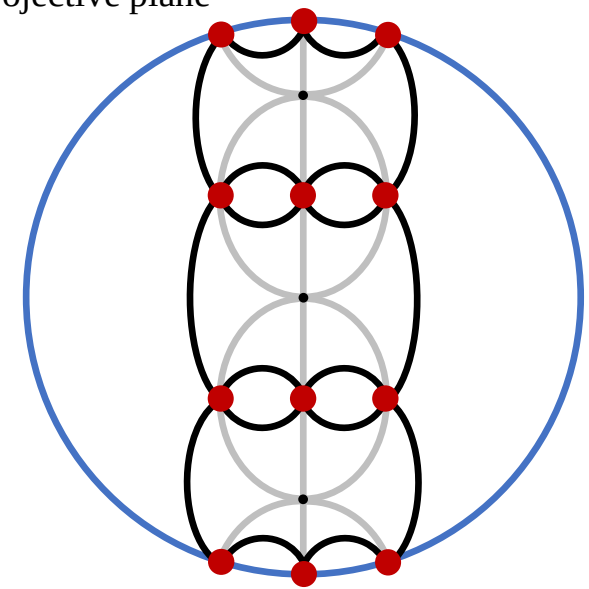
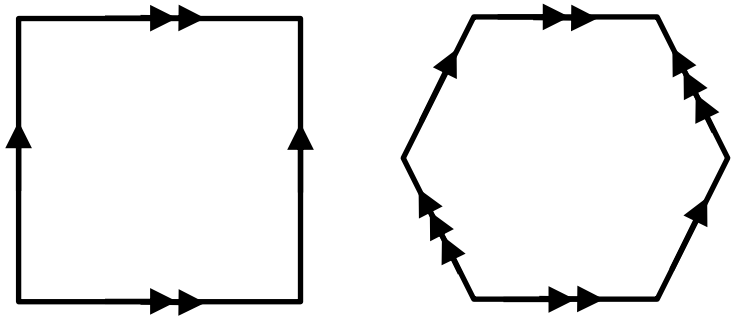


Properties:

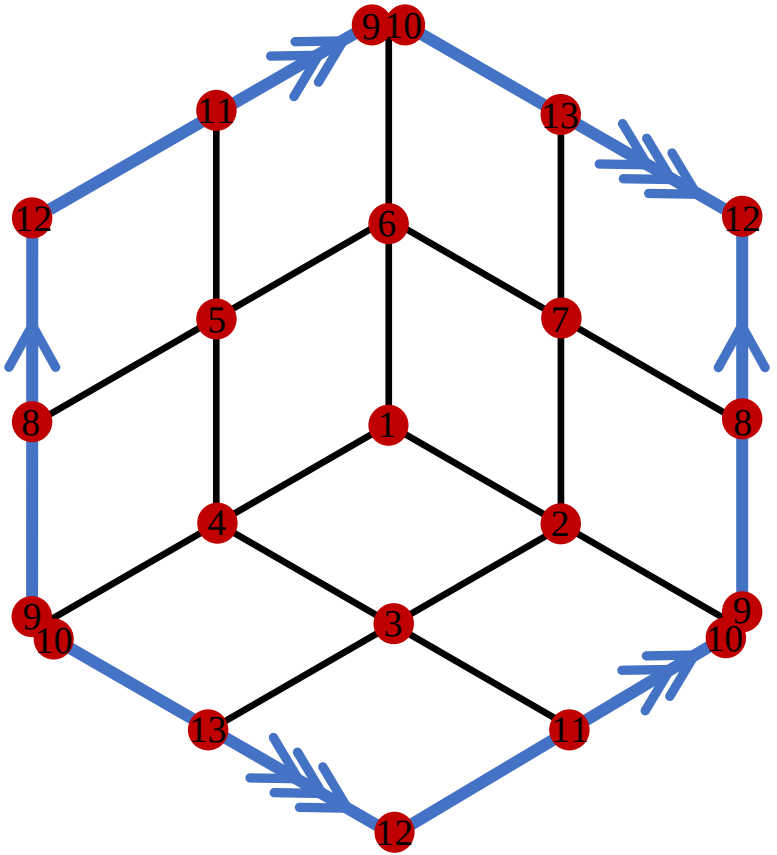
1) Medial graphs are always checkerboardable
2) Medial graphs are always 4-regular

Another example: Codes on higher genus surfaces

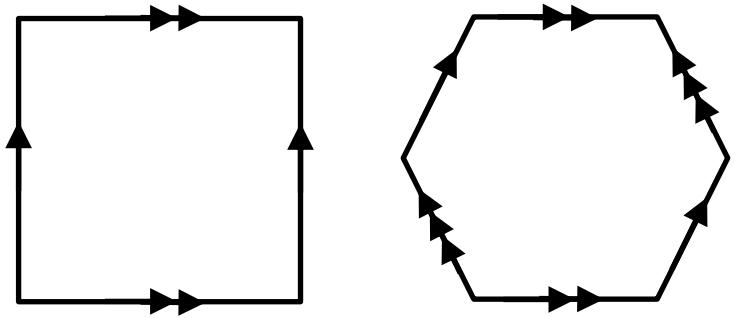


Both represent the torus

Placing the ⟦19,1,5⟧ triangle code on the torus gives a ⟦13,2,4⟧ code.

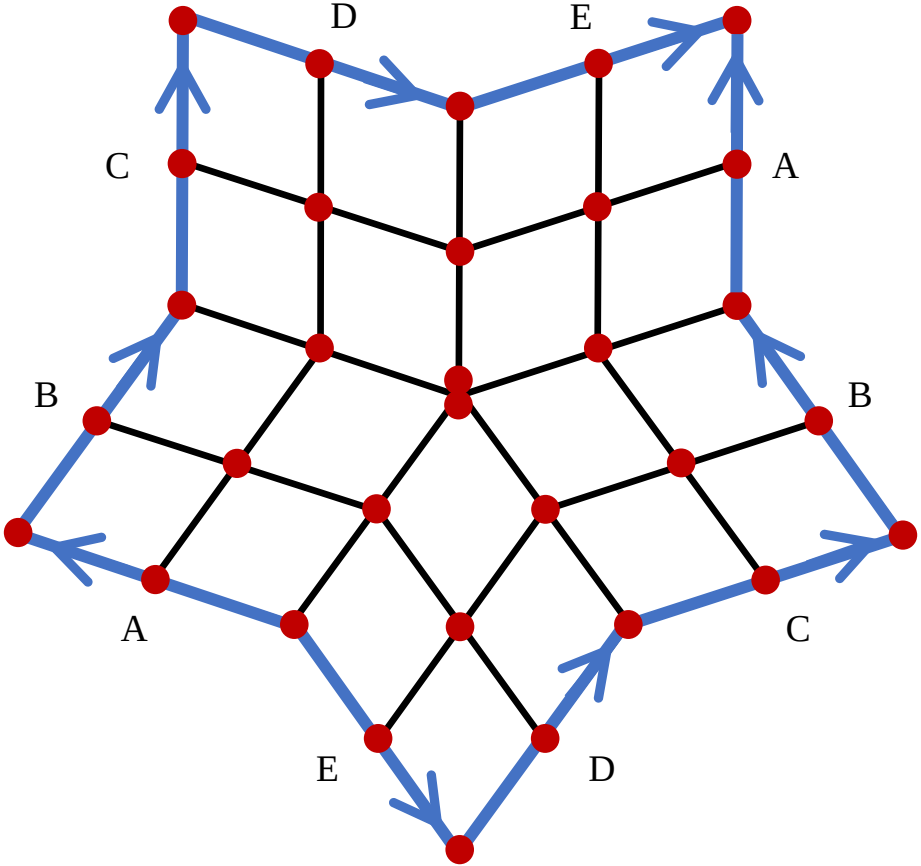


Another Example: Stellated codes on higher genus surfaces



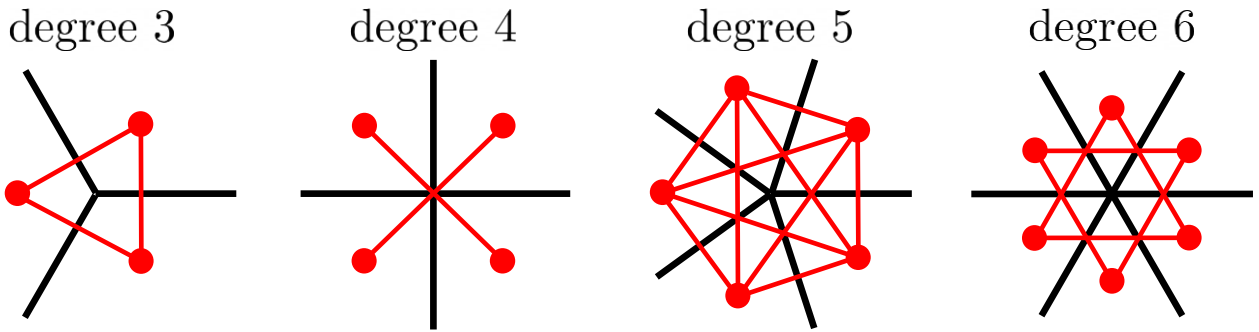
In general, an n-gon represents genus $g = \lfloor n/4 \rfloor$

Placing the ⟦32,2,5⟧ stellated code on the double-torus gives a ⟦23,4,4⟧ code.



General bounds on distance D

We need to use a construction called the decoding graph, obtained from the original graph $G(V, E)$.



Definition 4.2. The decoding graph $G_{\text{dec}} = (V_{\text{dec}}, E_{\text{dec}})$ of a closed 2-cell embedded graph $G = (V, E, F)$ is constructed so that $V_{\text{dec}} = F$, while edges E_{dec} are associated to vertices V in a many-to-one fashion. For each vertex $v \in V$, draw an edge between vertices $v_1', v_2' \in V_{\text{dec}}$ (associated to faces $f_1, f_2 \in F$) if there is a Pauli supported on qubits at v that anticommutes with S_{f_1} and S_{f_2} but commutes with all other stabilizers S_f.

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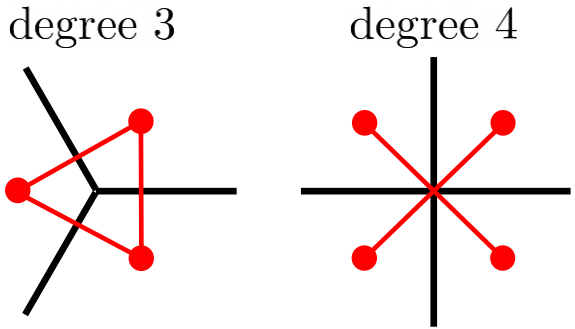

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Without loss of generality, we only need the case for degree 3 and 4 vertices.

Decoding graph properties

- Each edge represents a Pauli that anticommutes with exactly two faces.
- Any Pauli at a vertex can be represented by taking some subset of edges.
- **Logical operators are cycles in the decoding graph!**
- These are consequences of the CAL construction.

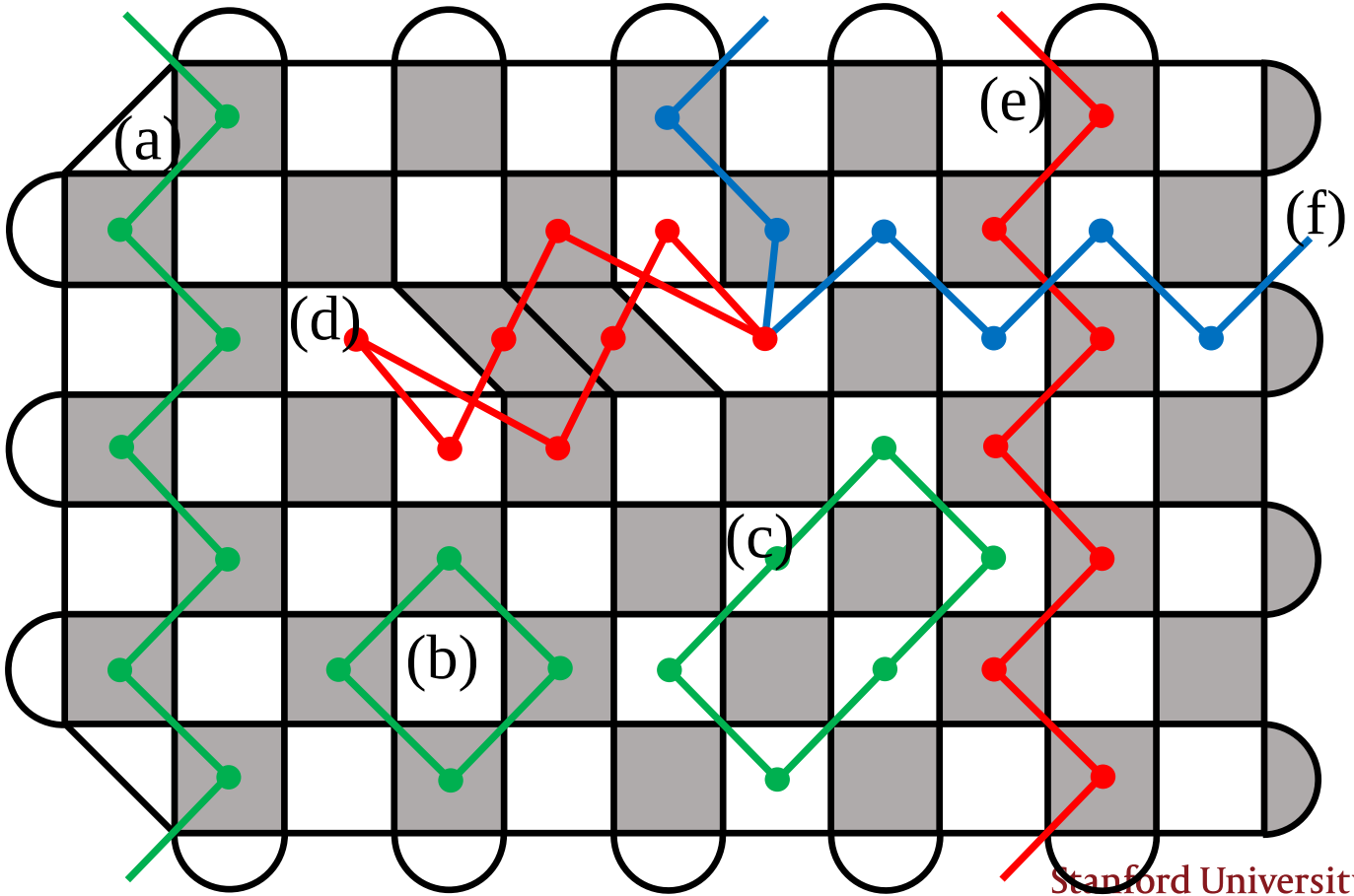
Decoding graph example

Stabilizers: a,b,c

Logical X: d,e

Logical Z: f

<u>Theorem:</u>
Centralizer of S
= cycles in G_{dec}



Efficient algorithm to get distance bounds

1. Given graph G, create its decoding graph G_{dec}.

2. Find a minimum cycle basis (MCB) of G_{dec}. A MCB is a basis of the cycle space $\{c_1, c_2, \ldots, c_b\}$ such that $\Sigma_i |c_i|$ is minimized. This can be done with Horton's algorithm or more efficient, more recent alternatives.

3. Convert each c_i to a Pauli P_i. Find nontrivial c_i, those for which P_i anticommutes with some other P_j.

4. Let W be the length of the shortest nontrivial c_i.

Theorem: If the graph is checkerboardable, $D = W$. If the graph is not checkerboardable, $W/2 \leq D \leq W$.

Doubled graph

Checkerboardable case: G_{dec} has two connected components, both embedding in the manifold.
Non-trivial logical operators are homologically non-trivial cycles in G_{dec}.

Non-checkerboardable case: G_{dec} does not (necessarily) embed in the manifold

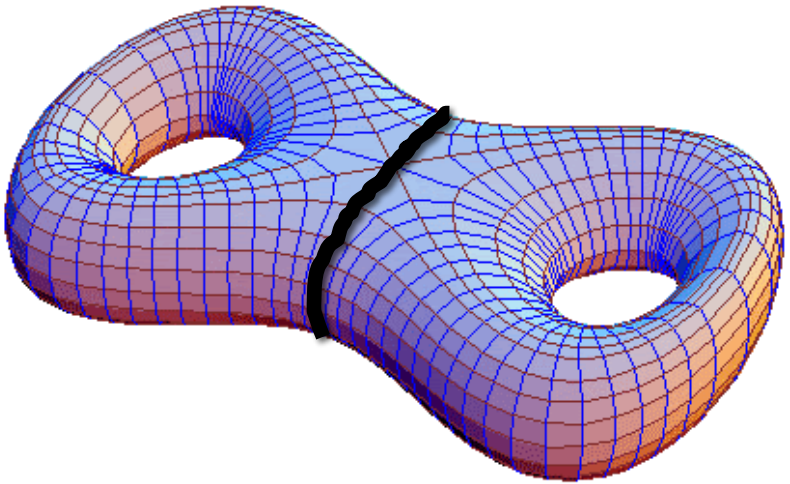
The **doubled graph** G^2 is a checkerboardable graph where either connected component of G_{dec}^2 is isomorphic to G_{dec}
Non-trivial logical operators are homologically non-trivial cycles in G_{dec}^2.

Theorem: Suppose G represents a surface code encoding K qubits.

1) If G is orientable, then G^2 is orientable with genus K.

2) If G is non-orientable, then G^2 is either orientable with genus K or non-orientable with genus $2K$.

Distinction between homological non-triviality and non-contractibility



http://stanwagon.com/wagon/mathimages/htmllinks/mathimages_1.html

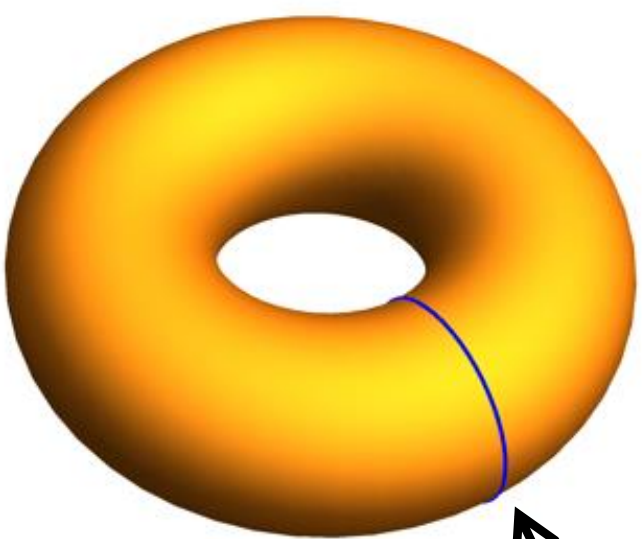
This loop is **homologically trivial**, but is **non-contractible.**

In general:

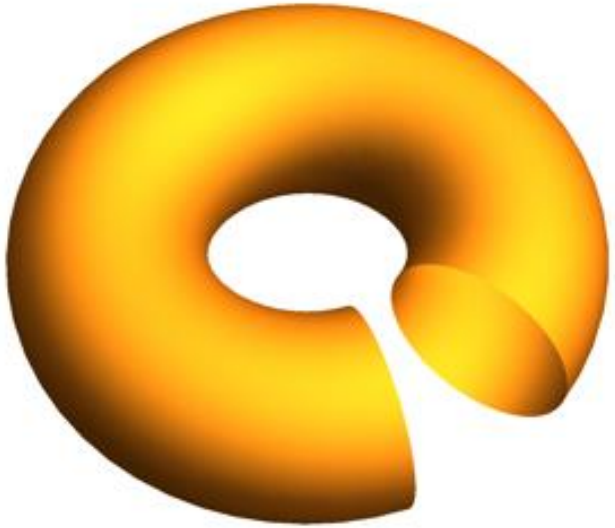
Contractible => homologically trivial

Example 1: Toric code case

“odd x even” square lattice



Defect



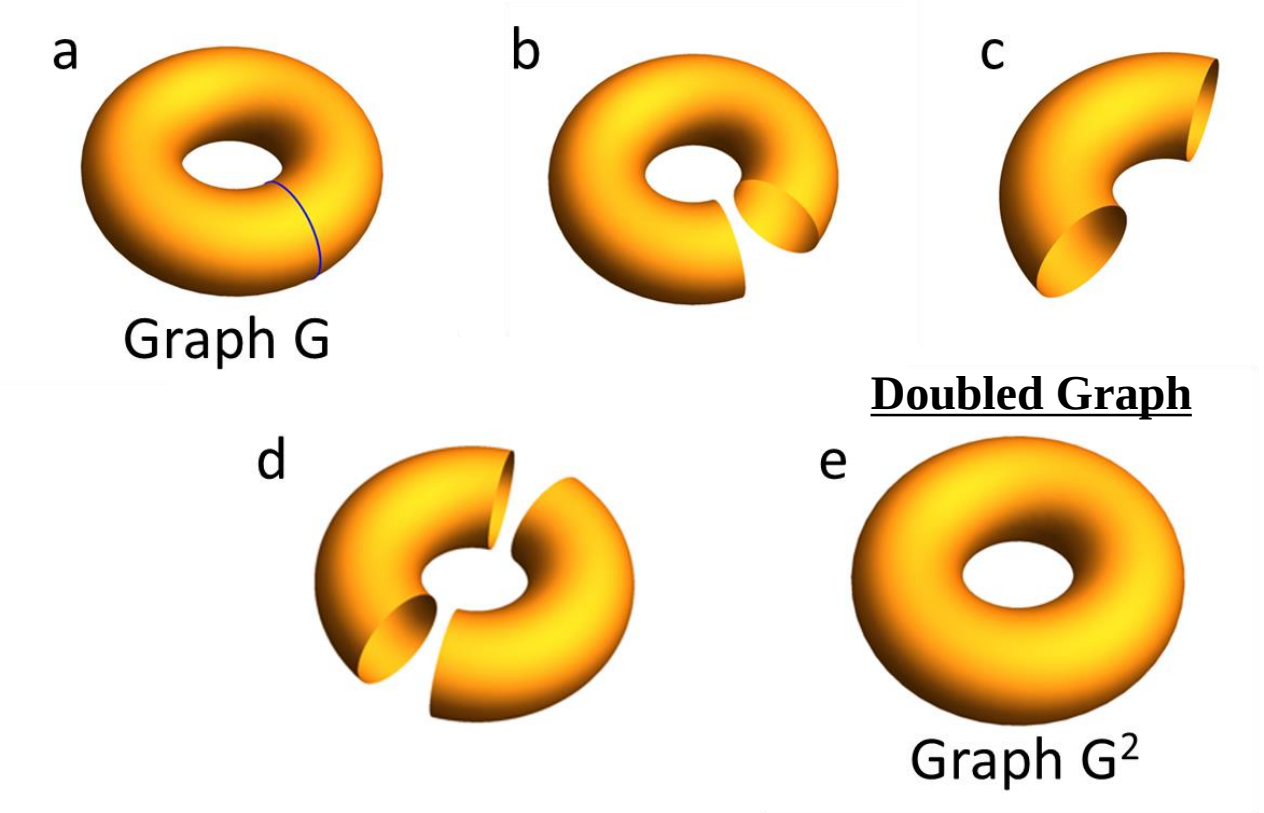
Example 1: Toric code case

Copy A

Copy A

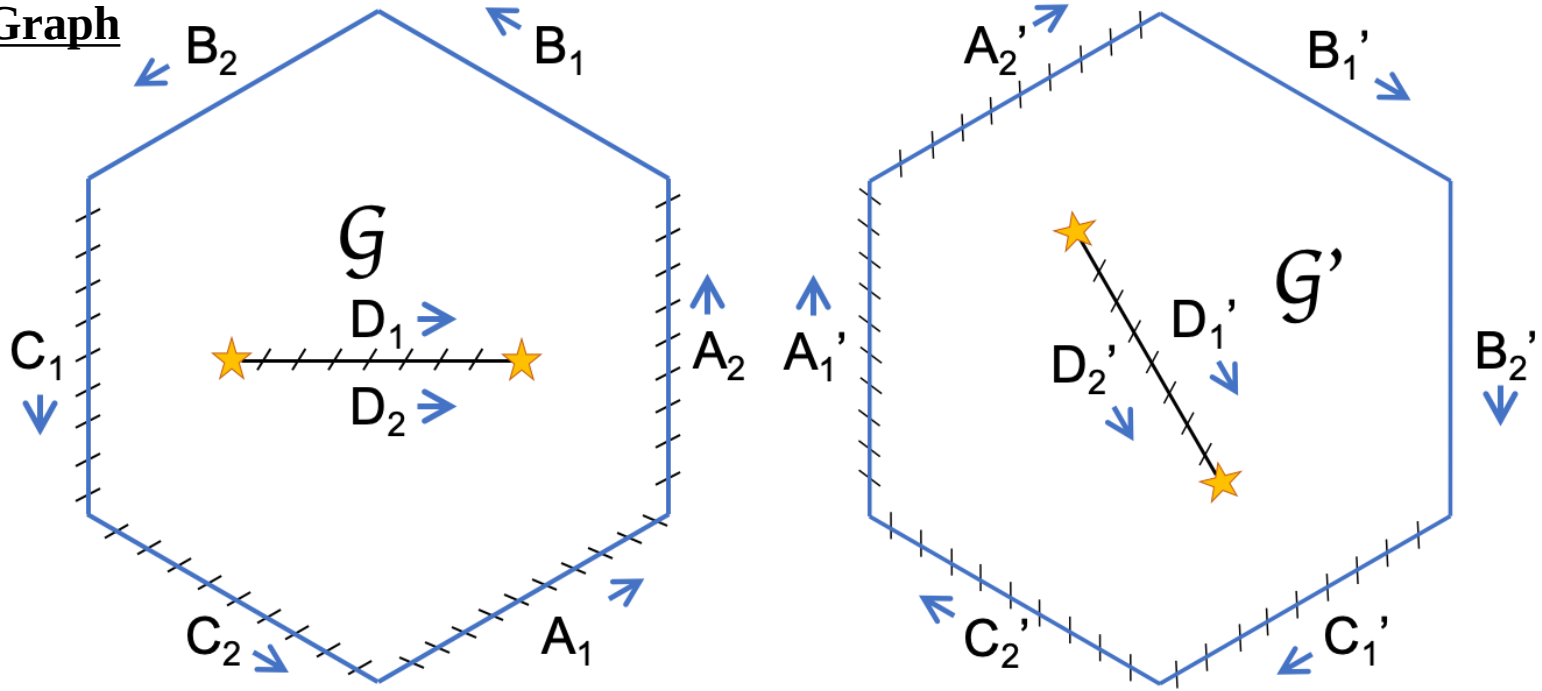
Copy B

Example 1: Toric code case



Example 2: More complicated (non-orientable)

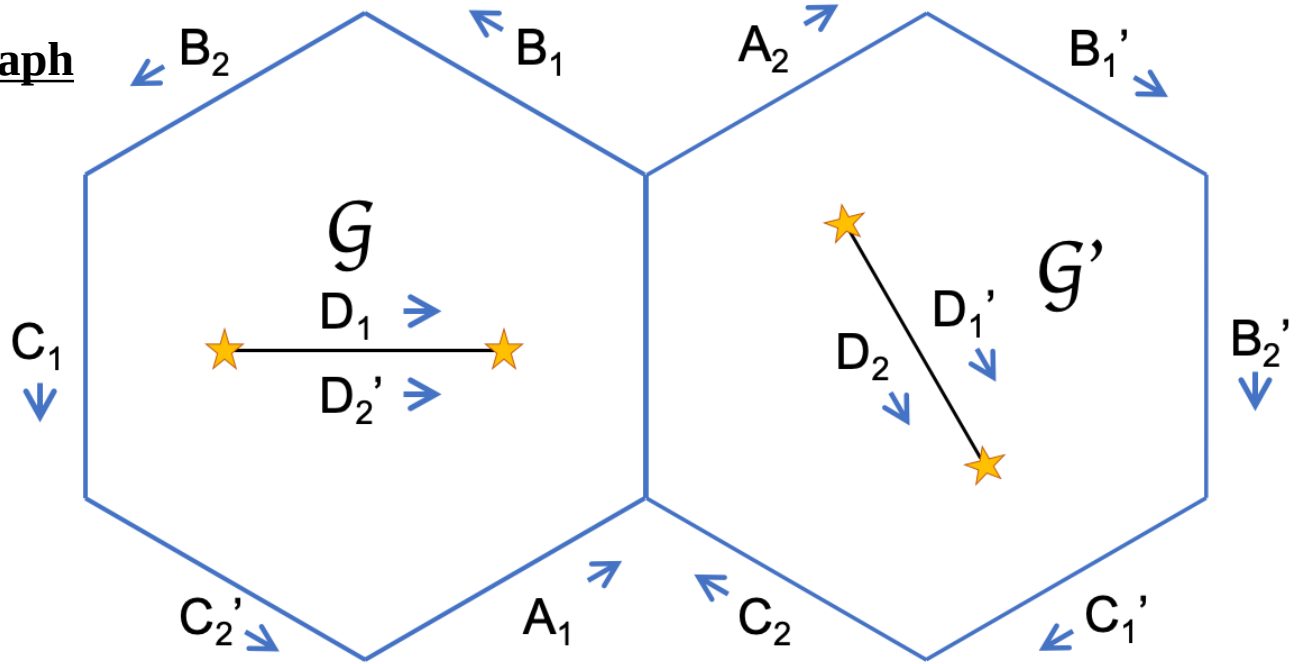
Original Graph



genus = 3, nonorientable
M=2 odd-degree vertices

Example 2: More complicated (non-orientable)

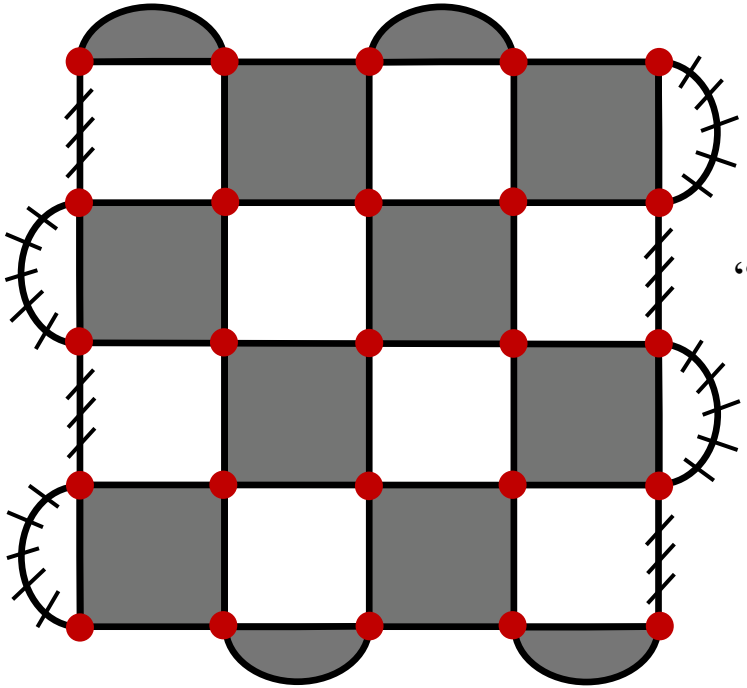
Doubled Graph



genus = 6, nonorientable

Example 3: Rotated toric code

Original Graph



"Defects"

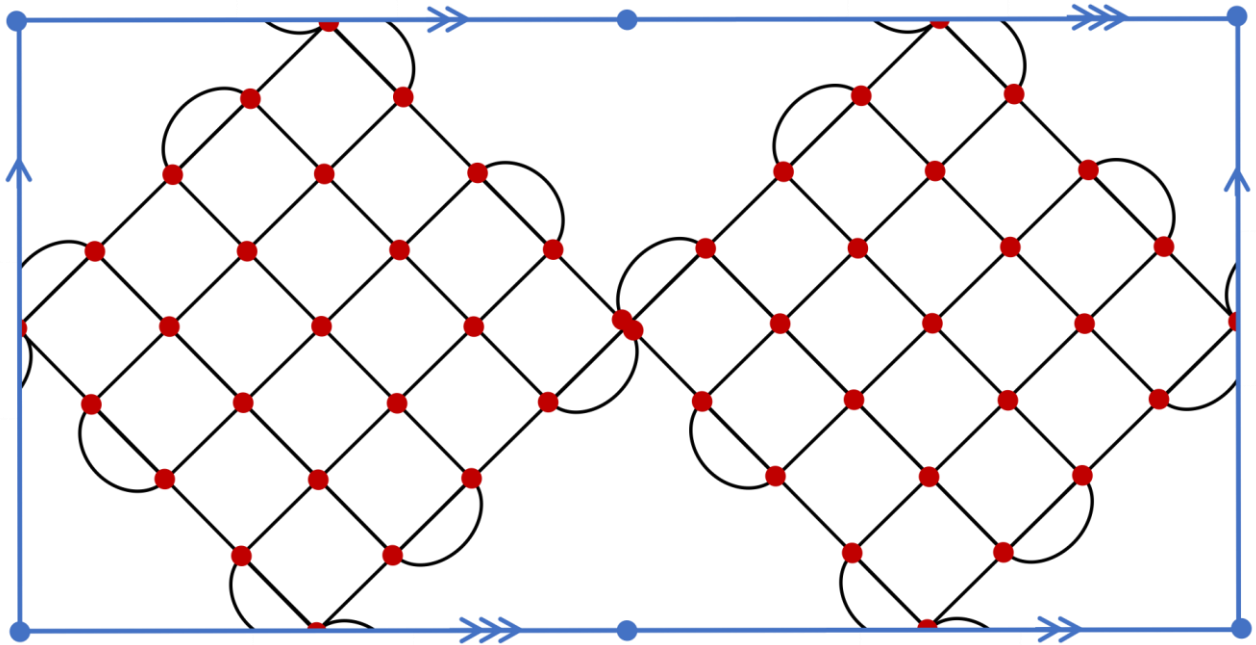
NOT checkerboardable

$K = 1$

genus = 0, sphere

Example 3: Rotated toric code

Doubled Graph

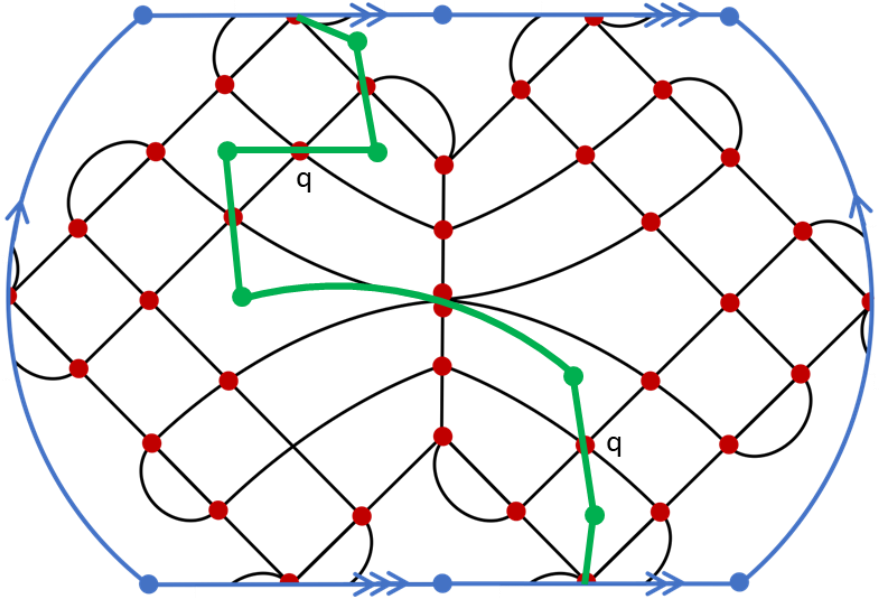


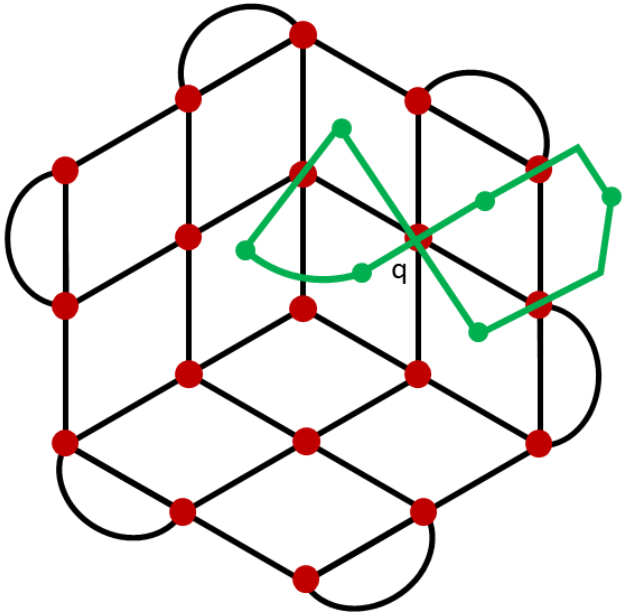
genus = 1, torus

Bounds on distance

Let $csys(G)$ be the length of the shortest homologically nontrivial cycle in G

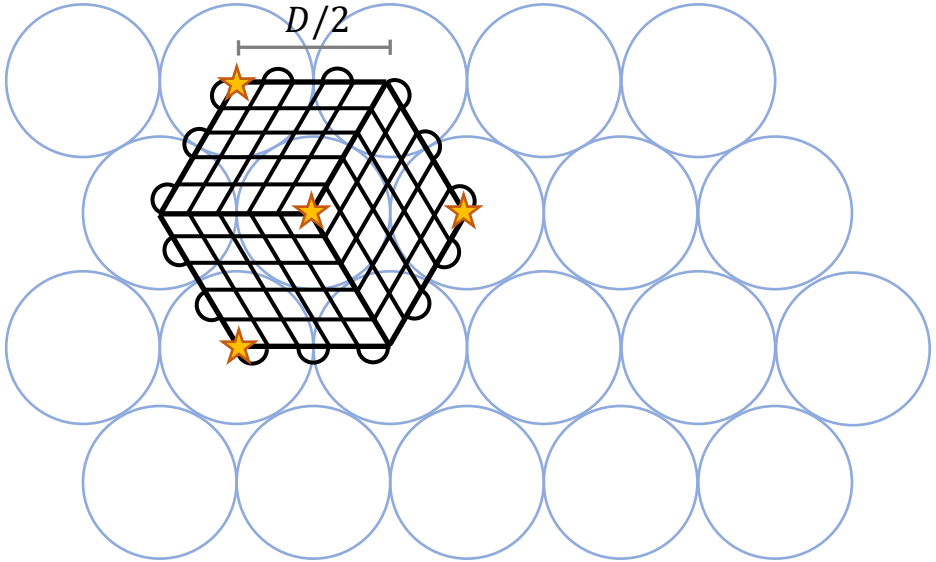
Theorem: If the graph is checkerboardable, $D = csys(G_{dec})$. If the graph is not checkerboardable, $csys(G_{dec}^2)/2 \leq D \leq csys(G_{dec}^2)$.





Planar Packing Codes

The triangle code achieves $N = (3/4)KD^2$



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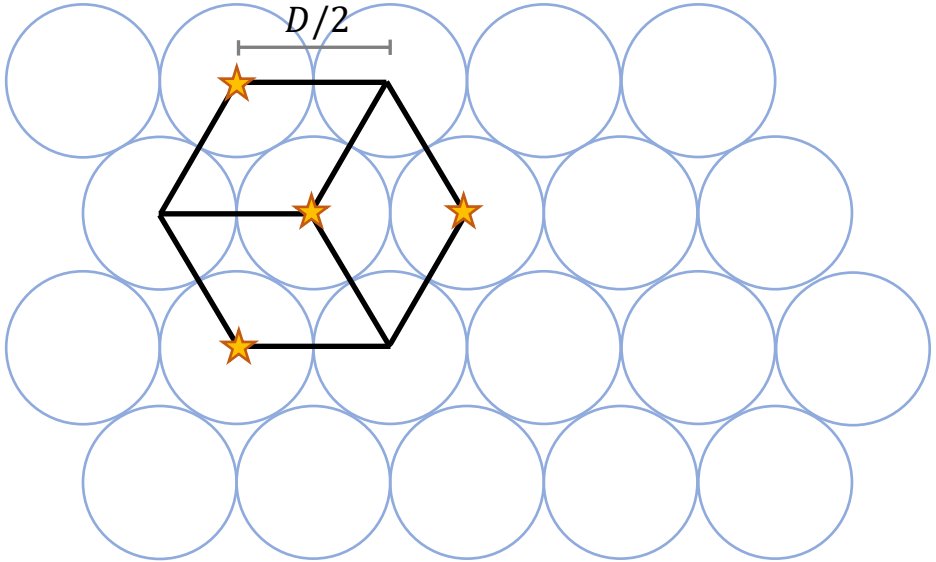

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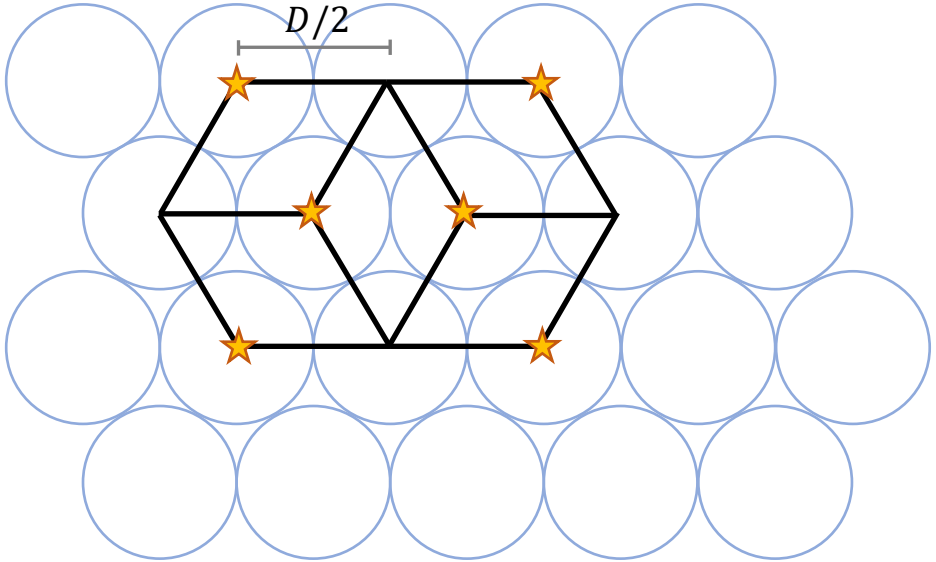
By extending the patch, we get

$$K = 2, \qquad N = \frac{5}{8}KD^2$$



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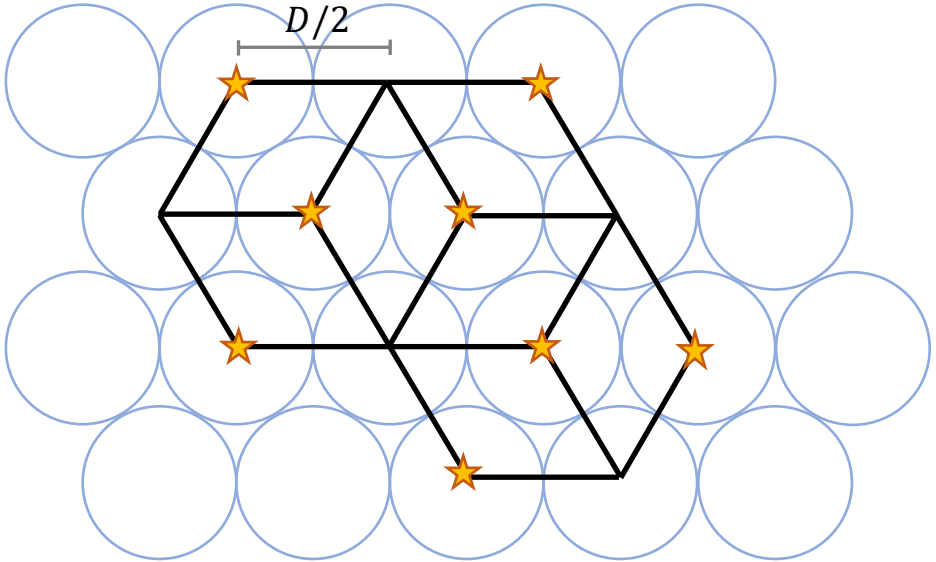

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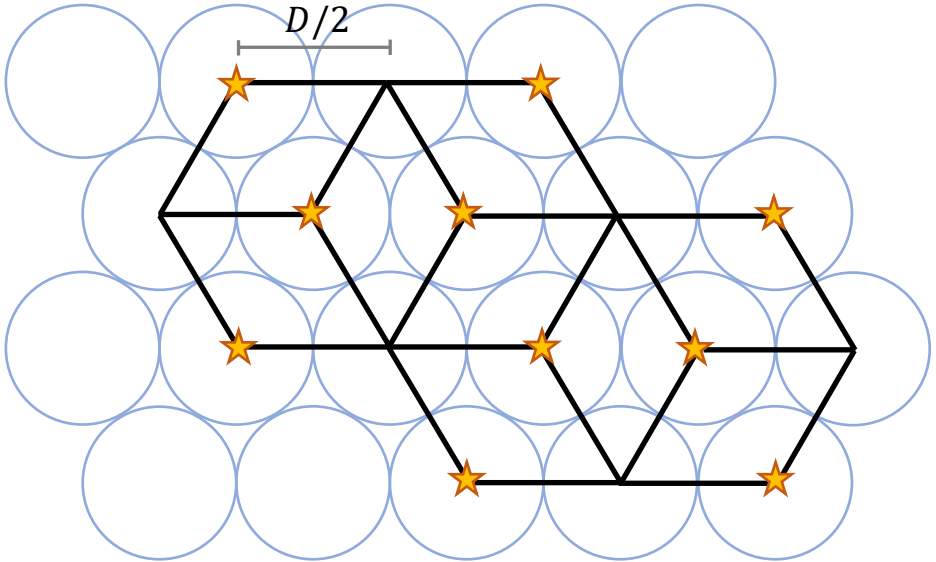
$$K = 4, \qquad N = \frac{9}{16}KD^2$$



Or, generally $$N = \frac{2K+1}{4K}KD^2 \to \frac{1}{2}KD^2$$

Some open problems

- Prove exact distance of some of the qubit surface codes mentioned to in the talk. We have only been able to mathematically prove the distance for the cyclic toric code, rotated surface code.
- Conjectured distances for triangle code, stellated code etc. exist. The planar packing code result is also a conjecture. No rigorous proof exist to the best of my knowledge.
- Does an efficient approximation algorithm (better than ½) exist to calculate the distances.
- How to continue this construction to higher dimensional manifolds (motivation Kitaev's homological codes exist on higher dimensional cell-complexes also)? It is currently not clear how to do this.
- Can the same be done for color codes?

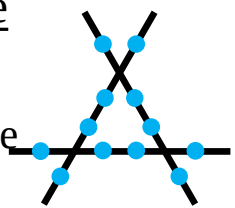
Conclusion

Embedded graph

Embed a graph in a 2d-manifold

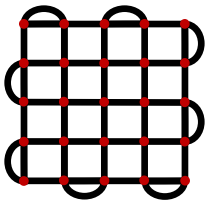
Majorana surface code

Assign Majoranas to half-edges & odd-degree vertices



Qubit surface code

Associate qubits with subsets of Majoranas



Cyclically anticommuting sets of Paulis

For questions, you can email me at rsarkar@stanford.edu

Also included

1) Formula for $K =$ (# encoded qubits)
2) Examples!
   1) Cyclic toric codes
   2) Stellated codes g>0
   3) Planar-packing code
3) Bounds on $D =$ (code distance)