MSE 334 Computation of Equilibria	Lecture 1 - 03/07/2009

Sperner, Brouwer and Nash

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We will start the lecture with the stating the following rather obvious fact: If a directed graph has one vertex with indegree different from outdegree then it must have another one. The above proposition will be the basis for deriving the Sperner's lemma, Brouwer's fixed point theorem and the Existence of Nash and market equilibrium.

Let us start with a few definitions. An *n-simplex* $\Delta(x_0, x_1, \cdots, x_n)$ is the convex hull of $n+1$ independent vectors $x_0, x_1, \cdots, x_n$, i.e., $\Delta(x_0, x_1, \cdots, x_n) = \{v | v = \sum_{i=0}^{n} \alpha_i x_i, \alpha_i \geq 0, \sum_{i=0}^{n} \alpha_i = 1\}$. We usually denote a regular n-simplex $\Delta(e_0, e_1, \cdots, e_n)$ as Δ_n. A *face* of a simplex $\Delta(x_0, x_1, \cdots, x_n)$ is any simplex $\Delta(x_{i1}, x_{i2}, \cdots, x_{ik})$ where $0 \leq i_i < i_2 < \cdots < i_k \leq n$. For $v \in \Delta$, define $\chi(v) = \{i | \alpha_i > 0\}$. *Simplicial subdivision* of Δ is a collection of simplices $\Delta_1, \cdots, \Delta_m$ such that for $\forall i, j$, $\Delta_i \cap \Delta_j$ is either empty or a common face. Let V to be the union of vertices of Δ_i's.

Definition 1. *For a simplex Δ and a simplicial subdivision with vertex set V, a coloring*

$$\lambda : v \to \{0, 1, \cdots, n\}$$

is legal if $\forall v \in V : \lambda(v) \in \chi(v)$.

The following is most well-known as the Sperner's Lemma and is proved in 1928:

Theorem 2. *For a simplex that is properly subdivided and legally colored, the number of simplices in the subdivision that are fully colored is odd.*

Proof We can prove it by induction. The base case $n = 1$ is trivial. It is a segment with two end nodes 0,1. Any interior node must be labeled 0 or 1. Hence, there must be an odd number of edges labeled 0-1 from left to right and an even number of edges labeled 1-0 from left to right.

Suppose the theorem holds for $n-1$ dimension. Consider Δ_n and a legal coloring.

Let C be the set of *Completely* colored simplices of dimension n.

Let B be the set of *Boundary* simplices colored with $\{0, 1, \cdots, n-1\}$ of dimension $n-1$.

Let A be the set of *Almost* completely colored simplices of dimension n, i.e., colored with colors $\{0, 1, \cdots, n-1\}$.

Define graph $G(D, E)$ where $D = A \cup B \cup C$ and two vertices are adjacent if they have a face that is colored with $\{0, 1, \cdots, n-1\}$. A node in C is a completely colored simplex, so it has only one face that is colored with $\{0, 1, \cdots, n-1\}$. A node in B is an $(n-1)$-simplex that is on the boundary of Δ_n and is colored with $\{0, 1, \cdots, n-1\}$, so it is a face on exactly one simplex of dimension n in Δ_n. A node in A has two such faces. Each such face corresponds to one edge in G. Therefore, the degree of a node in C is 1, the degree of a node in B is 1 and the degree of a node in A is 2. So the total degree of the graph G is:

$$\sum_{v \in D} deg(v) = |C| + |B| + 2|A|.$$

By the definition of proper coloring, only one facet of the boundary of Δ_n contains $\{0, 1, \cdots, n-1\}$-simplices and that number is odd by induction, and so $|B|$ is odd. Since $\sum_{v \in D} deg(v)$ is even, $|C|$ must be odd. Hence the theorem holds. □

Theorem 3. *(Brouwer's Fixed Point Theorem) Every continuous function $f : \Delta_n \to \Delta_n$ has a fixed point, i.e., a point $x \in \Delta_n$ such that $f(x) = x$.*

Proof Consider a subdivision and a legal coloring

$$\lambda(v) \in \chi(v) \cap \{i|f_i(v) \le v_i\}.$$

Note that $\chi(v) \cap \{i|f_i(v) \le v_i\}$ is not empty. This is because $f : \Delta_n \to \Delta_n$, so $\sum_{i=0}^n f_i(v) = \sum_{i=0}^n v_i = 1$. $\forall i, f_i(v), v_i \ge 0$. So there must exist some j such that $v_j > 0$ and $f_j(v) \le v_j$. Then $j \in \chi(v) \cap \{i|f_i(v) \le v_i\}$.

By Sperner Lemma, there exists a simplex $(\varepsilon^0, \epsilon^1, \cdots, \epsilon^n)$ that is fully colored, i.e.,

$$\forall i, f_i(\epsilon^i) \le \epsilon^i_i.$$

Now consider a sequence of refinements of the subdivision. By compactness of simplex there exists a subsequence of simplices converging to a point x. By continuity of f : $f_i(x) \le x_i \forall i$ which implies $f(x) = x$. □

Next we will use Brouwer's Fixed Point Theorem to prove the existence of Nash equilibrium.

Definition 4. *A game G is a collection of convex and compact set $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ and a* utility function *for each player i:*

$$u_i : \Sigma_1 \times \cdots \times \Sigma_n \to R.$$

Definition 5. *Nash equilibrium of a game is a joint action $X \in \Sigma_1 \times \cdots \times \Sigma_n$ such that*

$$u_i(y, X_{-i}) \le u_i(X), \forall i, y \in \Sigma_i.$$

Definition 6. *A game is concave if and only if $\forall i, X_{-i}, u_i(y, X_{-i})$ is concave in y.*

Lemma 7. *Given a concave function $f : X \to R$ where X is a convex set, and any $y \in X$, the function $f' = f(x) - ||x - y||^2$ is strictly concave.*

Theorem 8. *Every concave game has a Nash equilibrium.*

Proof For every player i, define $br_i(X) = \arg\max_{y \in \Sigma_i}(u_i(y, X_{-i}) - ||(y, X_{-i}) - X||^2)$. Since u_i is concave, by Lemma 1, the function $u_i(y, X_{-i}) - ||(y, X_{-i}) - X||^2$ is strictly concave and hence has a unique maximum. So $br_i(X)$ is well-defined and is continuous. Define $br(X) = (br_1(X), br_2(X), \cdots, br_n(X))$. Then $br : \Sigma_1 \times \cdots \times \Sigma_n \to \Sigma_1 \times \cdots \times \Sigma_n$ is continuous. By Brouwer's fixed point theorem, it has a fixed point, i.e., a point $X^* \in \Sigma_1 \times \cdots \times \Sigma_n$ such that $br(X^*) = X^*$. By the definition of Nash equilibrium, X^* is a Nash equilibrium. □